QUANTUM THEORY OF OPTIMAL CONTROL AND QUANTUM NONLINEAR FILTERING

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A brief account of quantum theory of optimal control and filtering is given. The dynamical programming method for optimization of quantum control with convex constraints and concave cost functions of quantum conditional state is presented. Consideration is given to open loop control corresponding to deterministic conditionally-Markov dynamics of a quantum unstable system. The results are demonstrated on the example of quantum unstable controlled dynamics admitting continuous demolition measurement. A single controlled decaying qubit conditioned by its survival is considered as an example.
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1. Introduction

Quantum Cybernetics (QC) will be dealing with self-organizing nanomechanical, nanoelectrical or nanobiological systems described by *quantum laws* of physics, probability and information. Like the classical cybernetics is essentially the systems theory of classical bits with input controls, output observations and feedback in the description of Norbert Wiener, QC can be defined as quantum systems theory of open to observation and feedback control qubit systems. However, unlike in the classical case, quantum observed systems must decohere and may become even unstable under the demolition measurements. The aim of QC is therefore the dynamical optimization of quantum observation and feedback control on the purpose described by costs and bequest functionals of quantum states during the quantum information processing. The main ingredients of such theory is the theory of optimal quantum filtering and quantum feedback control theory based on the quantum stochastic innovation dynamics which was developed by the author since [1] in a serious cited in the review paper [2].

Here we give a brief account of this theory on an example of a decaying qubit under the continuous watching of the decreasing event of its survival.

Such systems can be described as sub-Markov quantum processes governed by a class of nonlinear nonstochastic master equation specified in this paper.

2. Some facts and notations

Let $\mathbb{A} = \mathfrak{B}(\mathfrak{h})$ be the operator algebra on a Hilbert space $\mathfrak{h}$. The predual space $\mathbb{A}_\star$ can be realized by the densities $\varrho \in \mathbb{A}$, $\mathrm{tr}|\varrho| < \infty$ with respect to the trace tr such that the linear functional $\rho(\mathtt{A}) = \mathrm{tr}[\varrho\mathtt{A}] \equiv \langle \varrho, \mathtt{A}\rangle$ describes a quantum state defining the expectations of all $\mathtt{A} \in \mathbb{A}$ by a Hermitian-positive trace-one operator ϱ.

Let $\mathcal{S} \subset \mathbb{A}_\star$ denote the state space realized by such $\varrho \geq 0$, $\langle \varrho, \mathtt{I}\rangle = 1$ with the tangent space $\mathcal{T}_0 = \left\{v = v^\dagger \in \mathbb{A}_\star : \langle v, \mathtt{I}\rangle = 0\right\}$ and the cotangent space $\mathcal{T}_0^\star = \bar{\mathbb{A}}^{\mathrm{h}}/\mathbb{R}\mathtt{I}$. Every state $\varrho \in \mathcal{S}$ can be parametrized as $\varrho(\mathtt{q}) = \varrho_0 - \mathtt{q}$ by a *tangent element* $\mathtt{q} \in \mathcal{T}_0$. Cotangent elements $\mathtt{p} \in \mathcal{T}_0^\star$ are the equivalence classes

$$\mathtt{p}(\mathtt{X}) = \{\mathtt{A} \in \mathbb{A} : \mathtt{A} = \mathtt{X} + \lambda\mathtt{I}, \text{ for some } \lambda \in \mathbb{R}\}.$$

Example: A single quantum bit is described by the algebra of (2×2)-matrices $\mathtt{A} = \alpha\mathtt{I} + \sigma_{\vec{p}}$, where $\sigma_{\vec{p}} \in \mathcal{T}_0$ is decomposed into Pauli matrices

$$\sigma_{\vec{a}} = a_x\sigma_x + a_y\sigma_y + a_z\sigma_z \equiv \vec{a}\cdot\vec{\sigma}, \qquad \vec{a} \in \mathbb{C}^3.$$

The normalized trace $\langle \mathtt{I}, \mathtt{A}\rangle = \mathrm{tr}\{\mathtt{A}\} = \alpha$ defines the standard pairing $\langle \varrho, \mathtt{X}\rangle = -\vec{q}\cdot\vec{p}$ of $\mathtt{X} = \sigma_{\vec{p}}$ with quantum bit states $\varrho = 1 - \sigma_{\vec{q}}$ given by the coordinate $\mathtt{q} = \sigma_{\vec{q}}$ parametrized by real vector $\vec{q} \in \mathbb{R}^3$ from the unit ball $|\vec{q}| \leq 1$ with respect to the central state $\varrho_0 = \mathtt{I}$ and $\vec{p}(\mathtt{X}) \in \mathbb{R}^3$ identified with $\mathtt{p}(\mathtt{X}) \ni \mathtt{X}$ such that $\langle \mathtt{q}, \mathtt{p}\rangle = \vec{q}\cdot\vec{p}$.

2.1. *Affine and concave costs*

An affine functional $\mathsf{G}(u, \varrho) = \langle \varrho, G(u)\rangle$ of $\varrho \in \mathcal{S}$ given by a function $u \mapsto G(u)$ in Hermitian operators $G(u) \vdash \mathbb{A}$ on a measurable space $\mathcal{U}$ is called *expected cost* of the control $u \in \mathcal{U}$. The *minimal expected cost* $\mathsf{S}[\varrho] = \inf\{\langle \varrho, G(u)\rangle : u \in \mathcal{U}\}$ is not affine but *concave* functional into $\mathbb{R}_\wedge = [-\infty, \infty[$ on the convex set $\mathcal{S}$ in the sense

$$\mathsf{S}[\lambda\varrho_0 + (1-\lambda)\varrho_1] \geq \lambda\mathsf{S}[\varrho_0] + (1-\lambda)\mathsf{S}[\varrho_1]$$

for any $\lambda \in [0,1]$ and $\varrho_0, \varrho_1 \in \mathcal{S}$. It can have value $-\infty$ for an unbounded from below function $G(u)$ and is continuous in the lower topology on $\mathbb{R}_\wedge$. Such lower continuous concave function will be called *proper concave* if $\mathsf{S}[\varrho] > -\infty$ at least for one ϱ.

Example: $\mathsf{S}[\varrho] = \inf_{\vec{u}\in\mathbb{R}^3}\{\langle\varrho,\sigma_{\vec{u}}\rangle + |\vec{u}|\} = O^-_{\mathcal{B}_1}(\vec{q})$, where $O^-_{\mathcal{B}_1}(\vec{q}) := \{^{0,\ \vec{q}\in\mathcal{B}_1}_{-\infty,\ \vec{q}\notin\mathcal{B}_1} \equiv -O^+_{\mathcal{B}_1}(\vec{q})$ is the max-plus indicator function of for the ball $\mathcal{B}_1 = \left\{\vec{q}\in\mathbb{R}^3 : |\vec{q}| \le 1\right\}$.

2.2. *Legendre-Fenchel transform*

The above example for an affine G corresponding to $G(u) = |\vec{u}|\,\mathtt{I} + \sigma_{\vec{u}}$ is a special case of the Legendre-Fenchel transform

$$\mathsf{S}[\varrho] = \inf_{\mathtt{p}\in\mathcal{T}_0^\star}\{g(\mathtt{p}) - \langle\mathtt{q}(\varrho),\mathtt{p}\rangle\} = \inf_{\mathtt{X}=\mathtt{X}^\dagger}\{\langle\varrho,\mathtt{X}\rangle + \mathsf{G}[\mathtt{X}]\}$$

of the proper convex function $g(\mathtt{p}) = |\vec{p}|$ for $\mathtt{X} = \sigma_{\vec{p}} - \lambda\mathtt{I}$ with $\mathsf{G}[\mathtt{X}] = g(\mathtt{p}) + \lambda$. Every proper concave functional $\mathsf{S}[\varrho]$ on $\mathcal{S}$ can be obtained as optimization $\inf_{u\in\mathcal{U}}\{\mathsf{G}(u,\varrho)\}$ of a proper affine function $\varrho\mapsto\mathsf{G}(u,\varrho)$, e.g. as the LF transform corresponding to $u = \mathtt{p}$ and

$$\mathsf{G}(\mathtt{p}(\mathtt{X}),\varrho) = \mathsf{G}[\mathtt{X}] + \langle\varrho,\mathtt{X}\rangle = g(\mathtt{p}(\mathtt{X})) - \langle\mathtt{q}(\varrho),\mathtt{p}(\mathtt{X})\rangle\,.$$

The functional G, uniquely defining $g(\mathtt{p}) = \mathsf{G}(\mathtt{p},\varrho_0)$ on $\mathcal{T}_0^\star$ by the property $\mathsf{G}[\mathtt{X}] = \mathsf{G}[\mathtt{X}+\lambda\mathtt{I}] - \lambda$, can be found as a *proper convex* functional on the representatives $\mathtt{X}$ of $\mathtt{p}(\mathtt{X})$ into $\mathbb{R}_\vee =]-\infty,\infty]$ by the inverse Legendre transform $\mathsf{G}[\mathtt{X}] = \sup\{\langle\varrho,\mathtt{X}\rangle + \mathsf{S}[\varrho]\}$ of the functional $\mathsf{S}[\varrho]$ extended as $-\infty$ outside of $\mathcal{S}$. Thus, the *relative entropies* $\mathsf{S}[\varrho:\mu] = -\mathrm{tr}\left\{\varrho\ln\frac{\varrho}{\mu}\right\}$ of (a) $\ln\frac{\varrho}{\mu} = \ln\varrho - \ln\mu$ and (b) $\ln\frac{\varrho}{\mu} = \ln\left(\mu^{-1}\varrho\right)$ types are the transforms of some proper convex functionals G_a and G_b, and the *thermodynamic relative entropy* is defined as the transform

$$\mathsf{S}_\mathrm{t}[\varrho:\mu] = \inf\left\{\langle\varrho,\mathtt{X}\rangle + \ln\left\langle\mu,\mathrm{e}^{-\mathtt{X}}\right\rangle : \mathtt{X} = \mathtt{X}^\dagger\right\}.$$

3. Quantum sub-Markov dynamics

In general quantum controlled systems are governed by sub-Markov dynamics described for each controlling process $y = [u(r)\,|r\in\mathbb{R}_+]$ in terms of a hemigroup $[\tau_r^y(t)\,|t > r\in\mathbb{R}_+]$ of normal completely positive contracting maps $\tau_r^y(t) : \mathbb{A}\to\mathbb{A}$ such that

$$\tau_r^y(t)\circ\tau_t^y(s) = \tau_r^y(s) \quad \forall r < t < s.$$

The dependence on y is assumed to be causal, $\tau_r^{y_1}(t) = \tau_r^{y_2}(t)$ whenever $y_1^t = [u(r)\,|r < t] = y_2^t$, and usually $\tau_t^y(s)$ are assumed to be independent also on the past y^t of the admissible controls y defined by the cadlag trajectories $u(t) = u(t^+)$ as right continuous in $\mathbb{R}^{n_t}$ having the left limits $u(t^-)$ at each t.

3.1. *Quantum controlled generator*

The continuous feedback controlled sub-Markov dynamics $\{\tau_r^{\mathrm{y}}(t)\,|t > r \in \mathbb{R}_+\}$ is usually determined by its generator

$$\mathfrak{L}(u(t), \mathtt{X}) = \lim_{s \searrow 0} \frac{1}{s}\left(\tau_t^{y}(t+s, \mathtt{X}) - \mathtt{X}\right)$$

given by a completely dissipative map $\mathfrak{L}(u) : \mathbb{A} \to \mathbb{A}$ controlled by the values $u(t) = \left(u^1, \ldots, u^n\right)$ of some parameters $u^j \in \mathbb{R}$. Such dynamics is described by the controlled semi-Lindblad generators $\mathfrak{L}(u, \mathtt{X}) = \mathrm{i}\,[H(u), \mathtt{X}] + \mathfrak{L}^m(u, \mathtt{X})$ with the dissipation part

$$\mathfrak{L}^m(\mathtt{X}) = \sum_j \mathtt{L}_j \mathtt{X} \mathtt{L}_j^\dagger - \frac{1}{2}(\mathtt{MX} + \mathtt{XM}), \quad \mathtt{M} \geq \sum_j \mathtt{L}_j \mathtt{L}_j^\dagger$$

prepared for a continuous indirect measurement, say, of the observables $\mathtt{L}_j + \mathtt{L}_j^\dagger$ affiliated to $\mathbb{A}$. Here we assumed that the dynamics is controlled only by the Hamiltonian $H(u) = H(u)^\dagger$ and $\mathtt{M} \vdash \mathbb{A}$ (affiliated to $\mathbb{A}$) determines the decay operator $\mathtt{D} = \mathtt{M} - \sum_j \mathtt{L}_j \mathtt{L}_j^\dagger = -\mathfrak{L}(\mathtt{I})$.

3.2. *Deterministic quantum Master equations*

The density operator $\varrho^t \in \mathbb{A}_\star$ of a quantum state evolves from a normalized $\varrho^0 = \varrho_0$ by resolving the *controlled Master equation* $\frac{\mathrm{d}}{\mathrm{d}t}\varrho^t = \mathfrak{L}_\star(u, \varrho^t)$ with the predual generator $\langle \mathfrak{L}_\star(\varrho), \mathtt{A}\rangle = \langle \varrho, \mathfrak{L}(\mathtt{A})\rangle$. It is normalized on the decaying probability of the survival $\pi^t = \langle \varrho^t, \mathtt{I}\rangle$. The renormalized $\varrho_t = \varrho^t/\pi^t$ describes the quantum state conditioned by the survival effect up to time t. Its evolution is described by the velocity $\upsilon(\varrho) = \frac{\mathrm{d}}{\mathrm{d}t}\mathtt{q}(\varrho) \in \mathcal{T}_0$ defining the *deterministic nonlinear filtering equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\varrho_t = \mathfrak{L}_\star(\varrho_t) - \langle \varrho_t, \mathfrak{L}(\mathtt{I})\rangle \varrho_t \equiv -\upsilon(\varrho_t).$$

Example: An unstable quantum bit is described by the Hamiltonian $H(\vec{u}) = \frac{1}{2}\sigma_{\vec{u}}$, $\mathtt{L} = \frac{1}{2}\lambda\sigma_z = \mathtt{L}^\dagger$, $\mathtt{M} = \mu\mathtt{I} + \sigma_{\vec{m}}$ with $\mu \geq |\vec{m}| + \lambda^2/4$. Then $\upsilon(\varrho) = \sigma_{\vec{u}\times\vec{q}} + \upsilon_m(\varrho)$ where

$$\upsilon_m(\varrho) = \left(\vec{m} - \frac{\lambda^2}{2}\vec{q}_z^{\perp} - (\vec{m}\cdot\vec{q})\,\vec{q}\right)\cdot\vec{\sigma} \equiv \vec{\upsilon}_m \cdot \vec{\sigma}.$$

4. Quantum dynamical programming

The *cost to go* of a control $u(t)$ conditioned by survival is

$$\mathsf{J}[\{u(\cdot)\}; t, \varrho_t] = \int_t^\tau \mathsf{C}(u(r), \varrho_r)\,\mathrm{d}r + \mathsf{G}(u(\tau), \varrho_\tau).$$

Due to the statistical interpretation of quantum states,

$$\mathsf{C}(u,\varrho)=\langle\varrho,C(u)\rangle\,,\ \mathsf{G}(u,\varrho)=\langle\varrho,G(u)\rangle\,.$$

The optimal average cost on the interval $(t,\tau]$ to be $\mathsf{S}(t,\varrho):=\inf_{\{u(r)\}}\left[\mathsf{J}[u(\cdot);t,\varrho]\right]$ with $\mathsf{S}(\tau,\varrho)=\inf\{\langle\varrho,G(u)\rangle:u\in\mathbb{R}^{n_\tau}\}\equiv\mathsf{S}[\varrho]$ given by any concave functional $\mathsf{S}:\varrho\mapsto\mathsf{S}[\varrho]\in\mathbb{R}_\wedge$ of the terminal state $\varrho=\varrho_\tau$.

4.1. *Quantum Hamilton-Jacobi equation*

Since $\mathsf{J}[\{u(\cdot)\};t,\varrho_t]=\int_t^{t'}\mathsf{C}(u(r),\varrho_r)\,\mathrm{d}r+\mathsf{J}[\{u\};t',\varrho_t]$ at the times $t<t'\le\tau$, one has

$$\mathsf{S}(t,\varrho)=\inf_{\{u\}}\left\{\int_t^{t'}\mathsf{C}(u(r),\varrho_r)\,\mathrm{d}r+\mathsf{J}[u(\cdot);t',\varrho_{t'}]\right\}.$$

Suppose that $\{u^{\mathrm{o}}(r,\varrho):r>t\}$ is an optimal control when starting in state ϱ at the time t, and denote by $\{\varrho_r:r\in(t,\tau]\}$ the corresponding state trajectory starting at a state ϱ at t. Bellman's optimality principle observes that

$$-\frac{\partial}{\partial t}\mathsf{S}(t,\varrho)=\inf_{u\in\mathcal{U}}\{\mathsf{C}(u,\varrho)-\langle\upsilon(u,\varrho),\nabla_\varrho\mathsf{S}(t,\varrho)\rangle\}\,,$$

The equation is then to be solved subject to

$$\mathsf{S}(\tau,\varrho)=\mathsf{G}(u^{\mathrm{o}}(\tau_+,\varrho),\varrho)\equiv\mathsf{S}[\varrho]\,.$$

4.2. *Pontryagin's maximum principle*

We may rewrite this as the *Hamilton-Jacobi* equation

$$-\frac{\partial}{\partial t}\mathsf{S}(t,\varrho(\mathtt{q}))+\mathsf{H}_\upsilon(\mathtt{q},\mathtt{p}(\nabla_\varrho\mathsf{S}(t,\varrho))(\mathtt{q}))=0$$

introducing the Pontryagin's Hamiltonian as the transform

$$\mathsf{H}_\upsilon(\mathtt{q},\mathtt{p})=\sup_{u\in\mathcal{U}}\{\langle\upsilon(u),\mathtt{p}\rangle-\mathsf{C}(u)\}(\mathtt{q})$$

which is affine in $\mathtt{p}\in\mathcal{T}_0^\star$. This leads to the Hamiltonian boundary value problem

$$\begin{cases}\mathtt{q}_t-\nabla_\mathtt{p}\mathsf{H}_\upsilon(\mathtt{q}_t,\mathtt{p}_t)=0,\ \mathtt{q}_0=\mathtt{a}\\ \mathtt{p}_t+\nabla_\mathtt{q}\mathsf{H}_\upsilon(\mathtt{q}_t,\mathtt{p}_t)=0,\ \mathtt{p}_\tau=\mathtt{b}\end{cases}$$

as the *Hamilton-Pontryagin problem* with the solutions defining for $\mathtt{a}=\varrho_0-\varrho^0$, $\mathtt{b}=\nabla_\varrho\mathsf{S}[\varrho]$ the minimal costs by the path integral in

$$\mathsf{S}(t_0,\varrho_0)=\int_{t_0}^{\tau}[\langle\mathtt{q}_t,\mathtt{p}_t\rangle-\mathsf{H}(\mathtt{q}_t,\mathtt{p}_t)]\,\mathrm{d}r+\mathsf{S}[\varrho(\mathtt{q}_\tau)]\,.$$

4.3. *Optimal qubit decay control*

Let $\mathtt{L}_0$ be $\mathtt{L} = \frac{\lambda}{2}\sigma_z = \mathtt{L}^\dagger$ $(\lambda = \overline{\lambda})$, and let us ignore the effect of environment by taking $\mathtt{L}_j = 0$ for $j \neq 0$. We may also take cost $c(\vec{u}) = O^+_{\mathcal{B}_1}(\vec{u})$ of constraint $\mathcal{B}_1 = \{\vec{u} : |\vec{u}| \leq 1\}$ and any concave bequest functions $\mathsf{S}[\varrho]$, the qubit entropy say, as conditional entropy of the survival $\mathsf{S}[\varrho_\tau] = \frac{1}{\pi^\tau}\mathsf{S}[\varrho^\tau : 2\pi^\tau \mathtt{I}]$:

$$\mathsf{S}[\mathtt{I} + \sigma_{\vec{q}}] = q \ln \sqrt{\frac{1-q}{1+q}} - \ln \sqrt{\frac{1-q^2}{4}}, \quad q = |\vec{q}| \ .$$

We have $\langle \upsilon(u, \varrho), \mathtt{p}\rangle = \vec{v}(\vec{u}, \vec{q}) \cdot \vec{p}$ linear in $\vec{u}$:

$$\vec{v}(\vec{u}, \vec{q}) = \vec{v}_m + \vec{u} \times \vec{q}, \quad \vec{v}_m = \vec{m} - (\vec{m} \cdot \vec{q})\,\vec{q} - \frac{\lambda^2}{2}\vec{q}_z^{\perp}$$

which is maximized by the unit $\vec{u}^{\mathrm{o}} = \vec{q} \times \vec{p} / |\vec{q} \times \vec{p}|$ under the constraint $|\vec{u}| \leq 1$. This leads to the Hamiltonian function

$$\mathsf{H}_\upsilon(\vec{p}, \vec{q}) = |\vec{q} \times \vec{p}| + \vec{v}_m \cdot \vec{p}.$$

The HJ equation for optimal qubit control under this constraint is

$$-\frac{\partial \mathsf{S}}{\partial t} + \left|\vec{p} \times \vec{\nabla}\mathsf{S}\right| + \left(\vec{m} - (\vec{m} \cdot \vec{r})\,\vec{r} + \frac{\lambda^2}{2}\vec{r}_z^{\perp}\right)\vec{\nabla}\mathsf{S} = 0.$$

The solution $\mathsf{S}(t_0, \varrho_0)$ of this backward evolution equation corresponding to the entropy $\mathsf{S}(\tau, \varrho_\tau) = \mathsf{S}[\varrho_\tau]$ gives the minimal entropic cost of the controlled unstable qubit under the condition of its survival up to a time $\tau \geq t_0$.

5. Discussion

Our analysis is based on the fact that quantum state is a sufficient coordinate not only for closed, but also for open unstable quantum systems under the Markov approximation. However we have to deal with differential equation of high or infinite dimensionality if $\dim \mathfrak{h} = \infty$. Nevertheless, the Bellman principle can then be applied in much the same spirit as for classical states and we are able to derive the corresponding HJB theory for a wider class of cost functionals than traditionally considered.

References

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