Controllability of nonlinear Schrödinger equations

H. Teismann (Acadia University)

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Abstract

In this talk we will review some recent results on the (local exact) controllability of (certain) nonlinear Schrödinger equations with interior, boundary, and bilinear controls. Some applications to quantum control systems will be discussed.

Long Abstract

The control properties of many PDEs arising in physics and engineering have been studied extensively. For Schrödinger equations, however, the control theory is markedly less developed. For surveys on control results for (linear and nonlinear) Schrödinger equations, see e.g. [1, 2, 3, 4]. The purpose of this talk is to review some recent results on the (local exact) controllability of (certain) nonlinear Schrödinger equations with interior, boundary, and bilinear controls.

Specifically, we consider the the general nonlinear Schrödinger equation with control

$$i\partial_t y - \tfrac{1}{2}\Delta y - \kappa |y|^{2\sigma} y = B(y,u) \qquad (*)$$

where $y = y(t,x) \in \mathbb{C}$ is the state to be controlled, $t \in [0,T]$, $x \in \Omega \subset \mathbb{R}^n$ (to be understood with appropriate boundary conditions), $\kappa, \sigma \in \mathbb{R}$, $\sigma \geq 1$; u is the (vector of) control(s); B is the control operator, such as $[B(y,u)](t,x) := g(x)u(t,x)$, $\mathrm{supp}(g) \subset \omega \subset \Omega$, in the case of *interior* control or a boundary operator in the case of *boundary* control, or

$$[B(y,u)](t,x) := \left(\sum_{k=1}^{K} u_k(t) H_k\right) y(t,x)$$

in the case of *bilinear* control (where H_k are certain given (self-adjoint) operators). The latter case is the relevant one in *quantum-control* applications.

The control problem for nonlinear Schrödinger equations was posed by Zuazua [5, 1]; the following results have since been obtained:

1. [6] local (small-data) controllability for $n = 1$, $\sigma = 1$, $\Omega = (-\pi, \pi)$, periodic boundary conditions, interior control, in the space H^1.

2. [7] local H^1-controllability in the vicinity of the ground state for $n = 1$, $\sigma = 1$, $\Omega = (0, 1)$, homogeneous Dirichlet boundary conditions, interior control.

3. Rosier and Zhang [3]: local (small-data) controllability for $n = 1$, $\sigma = 1$, $\Omega = (-\pi, \pi)$ and periodic boundary conditions as well as $\Omega = (0, \pi)$ and Dirichlet or Neumann boundary conditions, interior control as well as boundary control, in H^s for any $s \geq 0$.

4. Rosier and Zhang [4]: local controllability in the vicinity of an arbitrary smooth solution for arbitrary n, bounded domain $\Omega \subset \mathbb{R}^n$, Dirichlet and Neumann boundary conditions, boundary and interior control in the space $H^s(\Omega)$ if $s > \frac{n}{2}$ or
$$0 \leq s < \tfrac{n}{2}, \quad 1 \leq n < 2 + 2s$$
or $s = 0, 1$ if $n = 2$.

Rosier and Zhang also proved stabilization results. Moreover, Dehman, Gérard, and Lebeau [8] proved "semi-global" controllability of a defocusing nonlinear Schrödinger equation with interior control on a two-dimensional Riemannian manifold without boundary in H^1.

After reviewing the results on interior and boundary control, we will turn to the bilinear control problem. The best results to date on exact[1] controllability of the *linear* ($\kappa = 0$) Schrödinger equation have been obtained by Beauchard and Beauchard and Coron:

1. [10] local controllability in the vicinity of the ground state for $n = 1$, $\Omega = (-\frac{1}{2}, \frac{1}{2})$, homogeneous Dirichlet boundary conditions; here the control operator is given by $[B(y, u)](t, x) = (u(t)x)y$ and controllabiliy is proved in the function space $\{f \in H^7_{(0)}(-\frac{1}{2}, \frac{1}{2}) \mid \|f\|_{L^2(-\frac{1}{2}, \frac{1}{2})} = 1\}$

[1]For a recent result on *approximate* (bilinear) controllability of the *linear* Schrödinger equation, see [9].

2. [11] bound-state-to-bound-state controllability for $n = 1$, $\Omega = (0, 1)$, homogeneous Dirichlet boundary conditions; here the control operator is given by $[B(y, u)](t, x) = (u(t)x^2)y$ and controllabiliy is proved in the function space $\{f \in H^5_{(0)}(0, 1) \mid \|f\|_{L^2(0,1)} = 1\}$

The physical interpretation of these problems is the control of a quantum particle in an infinite square well ("particle-in-a-box") where the control action is given by changing the box's position and varying its length, respectively.

In the remainder of the talk we will report on progress made in proving analogous results for the cubic nonlinear Schrödinger equation (*Gross-Pitaevskii equation*), which corresponds to the "BEC-in-a-box" problem subject to the controls described above. In addition to controllability, we will also briefly discuss adiabaticity for this system, a problem proposed by Band et al. [12]. This part of the talk is based on joint work (in progress) with K. Beauchard (Cachan) and H. Lange (Cologne).

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