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OBSERVATIONS' CONTROL FOR STATISTICALLY UNCERTAIN SYSTEMS

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Abstract

Estimation problems for so-called statistically uncertain systems are considered where along with stochastic disturbances there exist ones with no statistical description. In this paper, a controller uses uncertain disturbances in the system as control actions to produce worst signals for an observer, or, along with this task, to achieve his own aims unknown for the observer. On the other hand, the observer applies a minimax state estimation algorithm and does not know the aims of the controller. Such problems arise, for example, in aviation, when the plane must do some work to go unnoticed. Besides, there are other examples in economics, financial mathematics, and biology.

Key words

statistically uncertain systems, information sets, observations' control

1 Introduction

In many estimation problems from mechanics, economics, biology, and financial mathematics, there are both stochastic disturbances in the system and the observation's channel and uncertain ones with unknown statistics. In particular, the stochastic part may be absent in special case of set-membership description of uncertainty, [Schweppe(1973); Bertsecas and Rhodes(1971); Kurzhanski and Vályi(1996)]. Some problems of observations' control for determinate systems were considered in [Ananyev(2011)]. Estimation problems, where uncertain and stochastic disturbances act simultaneously, were examined in [Katz and Kurzhanski(1975); Ananiev(2010); Ananiev(2007)]. Here we consider the observations' control problems for statistically uncertain systems in which the controller can choose the disturbances and other parameters in order to counteract the observer. First, the linear multistage systems with unknown additive disturbances are examined. In these systems, the variances of random noises may be also used by the controller for his own purposes unknown to the observer. Next, the observations' control for some nonlinear multistage systems and estimation algorithms is studied. After that we consider problems for time-continuous systems. In last section some examples are given.

2 Problems for multistage systems

In this section, we consider observations' control problem for multistage systems. Cases of linear and nonlinear systems are examined separately.

2.1 Linear case

Consider an n-dimensional multistage linear system with m-dimensional observation:

$$\begin{aligned} x_i &= A_i x_{i-1} + v_i + B_i\xi_i, \quad i \in 1:N, \\ y_i &= G_i x_{i-1} + w_i + C_i\eta_i, \end{aligned} \tag{1}$$

where ξ_i and η_i are independent sequences of mutually independent standard Gaussian vectors; $v_i \in \mathbf{V}$ and $w_i \in \mathbf{W}$ are disturbances that are contained in convex compact sets. Suppose that disturbances $v_i,\ w_i$ may depend on observations $y_{1:i-1}$, the sets are symmetric with respect to zero, and B_i and C_i are known matrices.

We assume that the observer knows the coefficients of equations (1), the sets $\mathbf{V},\ \mathbf{W}$, the parameters $\bar{x}_0 = \mathrm{E}x_0,\ P_0 = \mathrm{var}(x_0)$ of initial Gaussian vector x_0, where E is the expectation, but does not know the realizations of v_i and w_i. To estimate the state vector x_i the observer uses algorithm from [Katz and Kurzhanski(1975)], where at any instant t he solves the problem:

$$\sup_{v_{1:t}, w_{1:t}} \mathrm{E}\{\|x_t - \psi_t(y_{1:t})\|^2 \mid y_{1:t}\} \to \inf_{\psi_t(\cdot)}. \tag{2}$$

Hereafter, the symbol $\|\cdot\|$ means Euclidean norm. The conditional expectation in (2) is equal to $\mathrm{tr}P_t + \|\hat{x}_t -$

$\psi_t(y_{1:t})\|^2$, where (see [Liptser and Shiryayev(2000)])

$$\begin{aligned} P_i &= \mathbb{A}_i P_{i-1} A_i' + B_i B_i', \ \ i \in 1:t, \\ \hat{x}_i &= \mathbb{A}_i \hat{x}_{i-1} + v_i + \mathbb{B}_i(y_i - w_i), \ \ \hat{x}_0 = \bar{x}_0, \\ \mathbb{A}_i &= A_i - \mathbb{B}_i G_i, \ \ \mathbb{B}_i = A_i P_{i-1} G_i'(C_i C_i' \\ &\quad + G_i P_{i-1} G_i')^+. \end{aligned} \tag{3}$$

Here the pseudoinverse matrix is denoted by A^+, the symbol $'$ means the transposition, and tr$A$ is the trace of the matrix A. In view of symmetry of the sets $\mathbf{V}, \ \mathbf{W}$ the optimal solution $\psi_t = \hat{x}_t^0$ of problem (2) is defined by the equation

$$\hat{x}_i^0 = \mathbb{A}_i \hat{x}_{i-1}^0 + \mathbb{B}_i y_i, \ \ \hat{x}_0^0 = \bar{x}_0.$$

The *point estimation* $\hat{x}_i^0$ is used by the observer, but actually the estimation obeys equation (3). Therefore, we come to the following problem for the controller.

Problem 1. *Choose the disturbances* $v_{1:N}, \ w_{1:N}$ *in* (3) *such that they maximize the value* $\|\hat{x}_N^0 - \hat{x}_N\|^2$ *at the end of process.*

Let $e_i = \hat{x}_i - \hat{x}_i^0$. The value of e_i is known to the controller, but not to the observer. We have

$$e_i = \mathbb{A}_i e_{i-1} + v_i - \mathbb{B}_i w_i, \ \ e_0 = 0. \tag{4}$$

Using the dynamic programming method we come to the following sufficient condition of optimality.

Theorem 1. *Let the sequences* $v_{1:N}, \ w_{1:N}$ *give the maximum in relations* $V_{i-1}(e_{i-1}) = \max_{v_i, w_i} V_i(e_i), \ i \in 1:N, \ V_N(e_N) = \|e_N\|^2$. *Then they are optimal in Problem* 1*, where the optimal cost value is equal to* $V_0(0)$.

Remark 1. *It follows from Theorem* 1 *that the optimal disturbances* $v_i, \ w_i$ *for the controller are positionally defined and may depend on* e_{i-1}. *Thus, they may depend on* $y_{1:i-1}$. *This fact does not contradict the generalized Kalman filter equation* (3).

On the other hand, the necessary and sufficient condition is formulated as follows. Denote by $\mathbb{A}_{N,k}$ the product $\mathbb{A}_N \cdots \mathbb{A}_k$.

Theorem 2. *The sequences* $v_i(y_{1:i-1}), w_i(y_{1:i-1}), i \in 1:N$*, are optimal in Problem* 1 *iff there exists a vector* $l^0 \neq 0$ *such that*

$$\begin{aligned} & l^{0\prime} \sum_{i=1}^N \mathbb{A}_{N,i+1}(v_i - \mathbb{B}_i w_i) \\ &= \max_{\|l\|=1} \sum_{i=1}^N \Big\{ \rho(l' \mathbb{A}_{N,i+1} \mid \mathbf{V}) \\ & + \rho(l' \mathbb{A}_{N,i+1} \mathbb{B}_i \mid \mathbf{W}) \Big\} = \sqrt{V_0(0)}, \ \ \mathbb{A}_{N,N+1} = \mathbb{I}. \end{aligned}$$

Here the value $\max_{v \in \mathbf{V}} l'v$ *is denoted by* $\rho(l' \mid \mathbf{V})$.

Now suppose that matrices $B_i, \ C_i$ in (1) are also undefined, and we have inclusions $B_i \in \mathbf{B}, \ C_i \in \mathbf{C}$, where $\mathbf{B}, \ \mathbf{C}$ are compact sets in corresponding matrix spaces. In this case, the minimax problem of type (2) for the observer becomes more complicated. Assume that the observer uses some unknown for the controller estimation algorithm. We will not solve the problem for the observer, but consider a step-by-step control procedure constructing the disturbances recursively.

Problem 2. *Let sequences* $v_{1:t-1}$, $w_{1:t-1}$, $B_{1:t-1}$, *and* $C_{1:t-1}$ *satisfying the inclusions be already selected at the instant* t. *Choose elements* $v_t, \ w_t, \ B_t, \ C_t$ *such that* $\mathrm{tr} P_t + \|e_t\|^2 \to \max_{v_t, w_t, B_t, C_t}$, *where matrix* P_t *and vector* e_t *are defined from equations* (3) *and* (4).

Similarly to Problem 1 the controller can maximize the final value $\mathrm{tr} P_N + \|e_N\|^2$. Let us call this situation as **Problem 3**. In this case, we obtain the generalization of Theorem 1:

Theorem 3. *Let the sequences* $v_{1:N}$, $w_{1:N}$, $B_{1:N}$, *and* $C_{1:N}$ *give the maximum in relations*

$$V_{i-1}(P_{i-1}, e_{i-1}) = \max_{v_i, w_i, B_i, C_i} V_i(P_i, e_i),$$
$$i \in 1:N, \ \ V_N(P_N, e_N) = \mathrm{tr} P_N + \|e_N\|^2.$$

Then they are optimal in Problem 3*, where the optimal cost value is equal to* $V_0(P_0, 0)$.

2.2 Nonlinear case

First, we consider the nonlinear case in set-membership formulation. At the end of the section we explain how to extend the results for stochastic systems. Given is the multistage system

$$x_i = f_i(x_{i-1}, v_i), \ \ y_i = g_i(x_{i-1}) + w_i, \tag{5}$$

where $i \in 1:T$, $x_i \in R^n$ is the state vector, $y_i \in R^m$ is the observed vector, v_i and w_i are disturbances. In addition, we believe that the first equation in (5) may be written in the equivalent inverse form

$$x_{i-1} = F_i(x_i, v_i), \ \ i \in 1:T,$$

and all the functions in these equations are continuous. Suppose that the observer knows equations (5) and a priori constraints

$$h_0(x_0) + \sum_{i=1}^N h_i(v_i, w_i) \leq 1, \tag{6}$$

where functions $h_i \geq 0$ are lower semicontinuous (l.s.c.) and such that the level sets $\{v, w : h_i(v, w) \leq \alpha\}$ are bounded and nonempty for $\alpha = 0$.

The observer tries to know the state vector of system (5) using a set-membership estimation algorithm.

Definition. The set $\mathcal{X}_t(y)$ is said to be *informational one* if it consists of all state vectors $\{x_t\}$, for which there exist sequences $v_{1:t}$, $w_{1:t}$ and initial state x_0 such that system (5) realizes the observed signal $y_{1:t}$ and inequality (6) is fulfilled.

Let $H_t(x, v, y) = h_t(v, y - g_t(x)),\ t \geq 1$.

Theorem 4. *The inclusion $x \in \mathcal{X}_t(y)$ is equivalent to the inequality*

$$W_t(x, y) = \min_{v_{1:t}} \Big\{ \sum_{i=1}^{t} h_0(x_0) \\ + H_i(F_i(x_i, v_i), v_i, y_i) \Big\} \leq 1, \ \ x_t = x.$$

The functions $W_t(x, y)$, $t \in 1 : T$, are l.s.c. and satisfy the recurrent relations

$$W_t(x, y) = \min_{v_t} \Big\{ W_{t-1}(F_t(x, v_t), y) \\ + H_t(F_t(x, v_t), v_t, y_t) \Big\}, \ \ W_0(x, y) = h_0(x), \tag{7}$$

for all $t \in 1 : N$.

According to Theorem 4 the observer calculates functions (7) and constructs the informational sets $\mathcal{X}_t(y) = \{x : W_t(x, y) \leq 1\}$. These sets are compact and contain the real state vector x_t.

The problem for the controller is to maximize the estimation error. The disturbances may be constrained also by additional relations not known to the observer. Let the estimation error be given by the functional

$$I = \sum_{i=0}^{N} f_0^i(x_i, W_i(\cdot, y)). \tag{8}$$

The controller maximizes functional over all the parameters $x_0, v_{1:N}, w_{1:N}$ constrained by (6).

The functions f_0^i in (8) may be, for example, defined as

$$f_0^i(x_i, W_i(\cdot, y)) = \max_{z \in \mathcal{X}_i(y)} \|x_i - z\|, \tag{9}$$

or as

$$f_0^i(x, W(\cdot)) = \|x - x^0\|, \\ x^0 = \operatorname*{argmin}_{x \in R^n} \max_{\{z : W(z) \leq 1\}} \|x - z\|, \tag{10}$$

where x^0 is the Chebyshev center of the informational set. Other choices for functions $f_0^i(x, W(\cdot))$ are

$$\operatorname{diam}\{z : W(z) \leq 1\}, \\ \min_{x \in R^n} \max_{\{z : W(z) \leq 1\}} \|x - z\|, \\ \operatorname{meas}\{z : W(z) \leq 1\}, \quad \max_{\{z : W(z) \leq 1\}} \|z\|, \tag{11}$$

where $meas$ means the Lebesgue measure of the set in R^n. In any case from (11), the functional I depends only on the size of the informational set. In some cases the functional I may be absent, and the controller does not counteract the observer, but tries to achieve his own purpose not known to the observer.

We can see from (7) that the function $W_t(\cdot, y)$ is built from signal y_t and previous $W_{t-1}(\cdot, y)$. Therefore, we can consider the generalized dynamic system

$$W_i(\cdot, y) = \Gamma_i(x_{i-1}, W_{i-1}(\cdot, y), w_i), \ \ W_0(\cdot, y) \\ = h_0(\cdot), \ \ x_i = f_i(x_{i-1}, v_i), \tag{12}$$

where the operator $\Gamma_i(x_{i-1}, W_{i-1}(\cdot, y), w_i)$ is defined by (7). A pair $\{x_i, W_i(\cdot, y)\}$ is called a *state* of system (12). By definition of informational sets, we have the inequality $W_i(x_i, y) \leq 1$ for all $i \in 0 : N$. Note that the pair $\{x_i, W_i(\cdot, y)\}$ depends on parameters $x_0, v_{1:i}, w_{1:i}$.

Now we are able to define recursively the Bellman functions

$$\Lambda_{i-1}(x, W(\cdot)) = \max_{v,w} \big\{ \Lambda_i(f_i(x, v), \Gamma_i(x, \\ W(\cdot), w)) : h_i(v, w) \leq 1 - W(x) \big\} \\ + f_0^{i-1}(x, W(\cdot)), \ \ i \in 1 : N, \tag{13}$$

where $\Lambda_N(x, W(\cdot)) = f_0^N(x, W(\cdot))$.

There are some modifications of functions (13). For example, the sequence $v_{1:N}$ is used by the controller for his own purposes. Then the maximization in (13) should be carried out over parameters $w_{1:N}$. At last, the maximization may be absent.

By induction, we obtain the theorem.

Theorem 5. *Let $f_0^i(x, W(\cdot))$ be one of the functions from* (9) – (11)*. Then functional* (8) *is upper semicontinuous with respect to variables x_0, $v_{1:N}$, $w_{1:N}$, and the maximum in* (13) *is reached. The optimal value of functional* (8) *is equal to $I^* = \max\{\Lambda_0(x, h_0(\cdot)) : h_0(x) \leq 1\}$. The optimal parameters x_0^*, $v_{1:N}^*$, $w_{1:N}^*$ are defined recursively (multiple valued) and are maximizers in relations* (13)*.*

In one particular case, the detailed proof of the Theorem 5 is given in [Ananyev(2011)].

The optimal parameters defined in the Theorem 5 generate explicitly the trajectory $x_{1:N}^*$ and the signal $y_{1:N}^*$ of system (5). This signal may be called *the worst* for observation in sense of functional (8).

We may suggest a simpler step-by-step procedure. To this end for stage $i \in 1:N$ we choose the elements

$$
\begin{aligned}
\{v_i, w_i\} \in \operatorname*{Argmax}_{v,w} f_0^i(f_i(\bar{x}_{i-1}, v), \\
\Gamma_i(\bar{x}_{i-1}, W_{i-1}(\cdot, \bar{y}), w)),
\end{aligned} \tag{14}
$$

where $\bar{x}_0 \in \operatorname{Argmax}_{\{x: h_0(x)\le 1\}} f_0^0(x, h_0(\cdot))$. Elements $\{v_i, w_i\}$ exist here.

Now, let us explain how to extend some results for nonlinear stochastic systems. Consider the system

$$
\begin{aligned}
x_i(\omega) = f_i(\omega, x_{i-1}(\omega), z_i) + z_i(\omega), \\
i \in 1:N, \;\; x_N(\omega) = \xi(\omega), \\
y_i(\omega) = g_i(\omega, x_{i-1}(\omega)) + \eta_i(\omega).
\end{aligned} \tag{15}
$$

The solution of system (15) is looked for as a sequence of pairs $[x_{i-1}; z_i] \in R^{2n}$, $i \in 1:N$, of random values, which are given on filtered random space $(\Omega, \mathcal{F}, \{\mathcal{F}_i\}, P)$ with $\mathcal{F} = \mathcal{F}_N$, where σ-algebras $\mathcal{F}_{i-1} \subset \mathcal{F}_i$, $i \in 1:N$. Besides, x_i, $z_i \in L_2^n(\mathcal{F}_i)$, where $L_2^n(\mathcal{F}_i)$ is a space of square integrable vector-functions that are measurable w.r.t. σ-algebra $\mathcal{F}_i$.

The stochastic sequence $\eta_i(\omega)$ in (15) is supposed to comply with constraints

$$
h_0(x_0) + \mathrm{E}\sum_{i=1}^{N} h_i(\eta_i, z_i) \le 1. \tag{16}
$$

Note that for any $\xi \in L_2^n(\mathcal{F})$ backward stochastic difference equation in (15) has a unique solution (see [Cohen and Elliott(2010)]) under following

Assumption.

1. The functions $f_i(\cdot, x_{i-1}(\cdot), z_i) \in L_2^n(\mathcal{F}_{i-1})$ for all $z_i \in \mathcal{M}_i^n$, $i \in 1:N$ and $x_{i-1}(\cdot) \in L_2^n(\mathcal{F}_{i-1})$.
2. The mapping $f_i(\omega, \cdot, z_i) : R^n \to R^n$ is a bijection with inverse mapping $f_i^{-1}(\omega, \cdot; z_i)$, which also satisfies the condition 1.

Here by $\mathcal{M}_i^n$ we denote the space $\{z \in L_2^n(\mathcal{F}_i) \mid \mathrm{E}(z|\mathcal{F}_{i-1}) = 0\}$, $i \in 1:N$, of martingale-differences. The problem for the observer may be formulated as follows. Let the observed process in (15) be realized under $\xi^* \in L_2^n(\mathcal{F})$ and $\eta_i^* \in L_2^m(\mathcal{F}_{i-1})$. One needs to determine the random informational set (RIS) $\mathcal{X}_N(y)$ consisting of all vectors $\xi \in L_2^n(\mathcal{F})$, for which there exist disturbances $\eta_i \in L_2^m(\mathcal{F}_{i-1})$ such that inequality (16) and equations (15) are hold P-a.s. for any $i \in 1:N$. Of course, the RIS $\mathcal{X}_t(y)$ may be constructed at any moment $t \in 1:N$.

The problem for the controller is to maximize the estimation error as above. For example, consider the step-by-step procedure, where

$$
f_0^i(x_i, \mathcal{X}_i(y)) = \max_{z\in\mathcal{X}_i(y)} \mathrm{E}\|x_i - z\|^2.
$$

Then for stage $i \in 1:N$ the controller chooses the element $\{\eta_i\} \in \operatorname{Argmax}_{\eta_i} f_0^i(x_i, \mathcal{X}_i(y))$, provided that the elements $\bar{\eta}_{1:i-1}$ already selected. For the initial stage, we have $\bar{x}_0 \in \operatorname{Argmax}_{x\in\mathcal{X}_0} \max_{z\in\mathcal{X}_0} \|x - z\|^2$.

3 Problems for time-continuous systems

We restrict ourselves by the set-membership formulation. Let the nonlinear system

$$
\dot{x} = f(t, x, v), \quad t \in [0, T], \tag{17}
$$

be given, where $x \in R^n$ is an unobserved state vector, $v \in R^p$ is an uncertain disturbance. The equation of observation is of the form

$$
y = g(t, x) + w, \quad y \in R^m. \tag{18}
$$

The unknown functions and initial data satisfy the constraints

$$
h_0(x_0) + \int_0^T h(t, v(t), w(t))dt \le 1. \tag{19}
$$

Let the standard conditions

$$
\begin{aligned}
\|f(t, x_1, v) - f(t, x_2, v)\| \le \lambda \|x_1 - x_2\|, \\
\|f(t, x, v)\| \le \kappa(1 + \|x\| + \|v\|), \;\; x \in R^n, \;\; v \in R^p,
\end{aligned}
$$

be fulfilled. The informational set $\mathcal{X}(t, y)$ for continuous case is defined in the same manner as above. Introduce the Bellman function $V(t, x) = \min_{v(\cdot)} J(t, x, v)$, where the functional J is of the form

$$
\begin{aligned}
J(t, x, v) = h_0(x_0) + \int_0^t h(\tau, v, y(\tau) - g(\tau, x))d\tau, \\
x(t) = x.
\end{aligned}
$$

The Bellman equation for $V(t, x)$ can be written as:

$$
\begin{aligned}
V_t = \min_v \Big\{ -f'(t, x, v)V_x + h(t, v, y(t) \\
-g(t, x))\Big\}, \;\; V(0, x) = h_0(x),
\end{aligned} \tag{20}
$$

If the solution of (20) is found, we have $\mathcal{X}(t, y) = \{x : V(t, x) \le 1\}$. To solve (20) the observer can use any known methods. In the special case, when $f(t, x, v) = f(t, x) + B(t)v$, $h(t, v, w) = h(t, w) + v'Q(t)v$, he obtains

$$
\begin{aligned}
V_t = -f'(t, x)V_x + h(t, y(t) - g(t, x)) \\
-V_x' B Q^{-1} B' V_x / 4, \\
V(0, x) = h_0(x), \;\; v_0(t, x) = Q^{-1}B'V_x/2.
\end{aligned} \tag{21}
$$

In this equation the minimization is absent, but there is a nonlinear summand. Instead of inequality (19) we consider the following one:

$$h_0(x_0)+\int_0^T\{f_0(t,w)+v'(t)Q(t)v(t) \\ +v_0'(t)Q(t)v_0(t)\}dt\le 1, \tag{22}$$

and the linear Lyapunov eouation

$$\mathcal{V}_t=-f'(t,x)\mathcal{V}_x+h(t,y(t)-g(t,x)), \\ \mathcal{V}(0,x)=h_0(x),\;\; v_0(t,x)=Q^{-1}B'\mathcal{V}_x/2. \tag{23}$$

In connection with equations (22), (23), consider the set $\mathbf{V}(t,y)=\{x:\mathcal{V}(t,x)\le 1\}$, and suppose that the observer uses it as some approximation of $\mathcal{X}(t,y)$ in case (21). On the other hand, the controller can use the method of characteristics to compute the function $\mathcal{V}(t,x)$ and the set $\mathbf{V}(t,y)$ for various disturbances. Note that the method of characteristics is a powerful one that allows to reduce any first-order linear PDE to an ODE, which can be subsequently solved using ODE techniques.

So, the problem for controller may consist in maximization of the function like (9)–(11) with $W(\cdot)=\mathcal{V}(T,\cdot)$ and with the final state $x=x(T)$ of equation (17).

4 Examples

In this section, we consider some examples.

Example 1. Given the one-stage stochastic system with measurements and constraints of type (15):

$$x=x_0+\mathrm{E}zz'b+z,\;\; x=\xi,\;\; y=x_0+\eta, \\ b\in R^n,\;\; p_0\|x_0\|^2+q_1\|\eta\|^2+q_2\mathrm{E}\|z\|^2\le 1,$$

the controller seeks for the determinate values η and x_0 to maximize $f_0^1(x,\mathcal{X}_1(y))=\max_{\xi\in\mathcal{X}_1(y)}\mathrm{E}\|\xi\|^2$. Here constants $p_0,\ q_1,\ q_2>0$ are known. As z is a martingale-difference, then $z=\xi-\mathrm{E}\xi$ and $x_0=\mathrm{E}\xi-K_{\xi\xi}b$, where $K_{\xi\xi}=\mathrm{var}(\xi,\xi)$ is the covariance matrix of vector ξ. From constraints we obtain the inequality

$$p_0\|\mathrm{E}\xi-K_{\xi\xi}b\|^2+q_1\|y-\mathrm{E}\xi+K_{\xi\xi}b\|^2+q_2\mathrm{tr}K_{\xi\xi}\le 1.$$

This inequality defines the set $\mathcal{X}_1(y)$. Let $\nu^2=1-q_2\mathrm{tr}K_{\xi\xi}$. Then the optimal cost value is equal to

$$\max_y\left\{\left\|K_{\xi\xi}b+\frac{q_1y}{p_0+q_1}\right\|^2\right. \\ \left.+\frac{\sqrt{\nu^2(p_0+q_1)-p_0q_1\|y\|^2}}{p_0+q_1}\left\|K_{\xi\xi}b+\frac{q_1y}{p_0+q_1}\right\|\right\}.$$

Here nor the observer, nor the controller can influence the matrix $K_{\xi\xi}$.

Example 2. Consider equations of Euler approximation for perturbed linear oscillator with observation of velocity:

$$x_i=\begin{bmatrix}1 & \Delta\\ -\Delta & 1\end{bmatrix}x_{i-1}+v_i+\begin{bmatrix}0\\ b\end{bmatrix}\xi_i, \\ y_i=\begin{bmatrix}0 & 1\end{bmatrix}x_{i-1}+w_i+c\eta_i,$$

where constraints for uncertain parameters are of the form $v_i^1=0$, $|v_i^2|\le 1$, $|w_i|\le 1$, $b\in\{0,1\}$, $c\in\{0,1\}$. Let the controller solve step-by-step Problem 2. Let the initial state be equal zero. At first stage we have $b_1=1$, $v_1^2=\pm 1$. At the subsequent stages we have $b_i=1$, $c_i=1$, $v_i=1$, $w_i=-1$. If $\Delta=0.1$, then the maximal value of $\mathrm{tr}P_i+\|e_i\|^2$ tends to 2.6351.

Example 3. Two companies are engaged in wild animals monitoring. Certain quantity of the animal units belonging to these companies from different populations is placed in one area, and further they develop and compete among themselves according to the equations

$$\dot{x}_1=x_1(a-bx_1-cx_2),\;\; \dot{x}_2=x_2(v-dx_1-ex_2), \tag{24}$$

where x_i is a quantity of animal units in i-th population, $i=1,2$; a,b,c,d,e are known positive constants. Representatives of 1-st company aspire to specify number of animals agree to the data arriving from sensors

$$y_1^i=x_1(t_{i-1})+w_1^i,\;\; y_2^i=x_2(t_{i-1})+w_2^i, \tag{25}$$

in discrete instants $0=t_0<t_1<,\dots,$ where $w^i=[w_1^i;w_2^i]$ is a vector of measurement disturbances. They consider the parameter v in (24) uncertain keeping a constant value on a half-interval $[t_{i-1},t_i)$. Uncertain parameters of system (24), (25) are restricted by geometrical constraints $\alpha\le v\le\beta,\quad |w_j^i|\le\gamma$. 1-st company uses a minimax algorithm of an estimation which is reduced to the following. Let $\mathcal{D}_i(X)$ be the attainability domain of system (24) at the instant t_i from the set X at the instant t_{i-1}. Information sets $\mathcal{X}_i$ are formed by the following rule

$$\mathcal{X}_i=\hat{\mathcal{D}}_i(\mathcal{X}_{i-1}\cap Y_i),\quad Y_i=\{x:|x_1-y_1^i| \\ \vee|x_2-y_2^i|\le\gamma\},\;\; \mathcal{X}_0=X_0.$$

Representatives of the second company know about intentions of the first and do not hinder with it. But unlike the first company they have continuous access to the sensor $y_2(t)$ and form parameter $v=v(y_2)$ by a principle of feedback for the purpose of maintenance of number of the animals at certain level. Function $v(y)$ is defined by the formula $v(y)=\begin{cases}\beta,\text{if } y<=\overline{x}_2\\ \alpha,\text{if } y>\overline{x}_2\end{cases}$, where

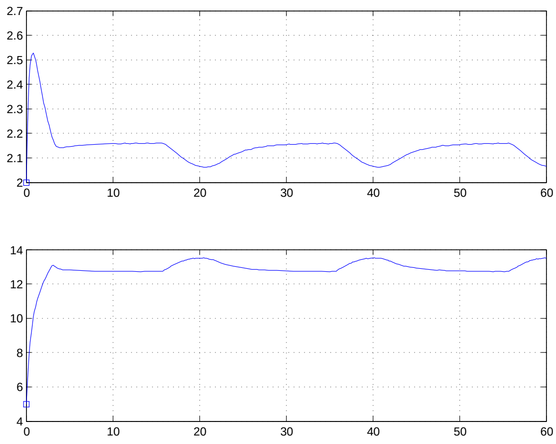

Figure 1. Real trajectories.

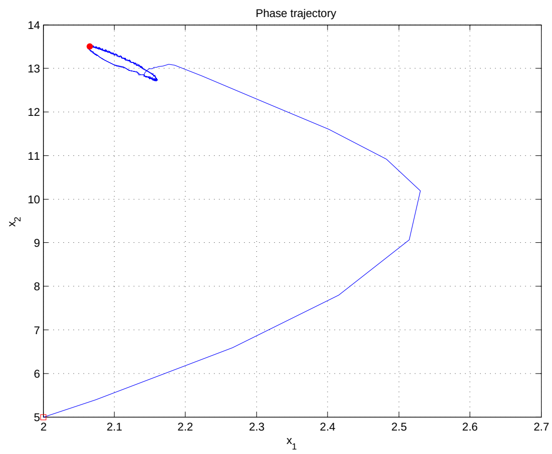

Figure 2. State trajectory.

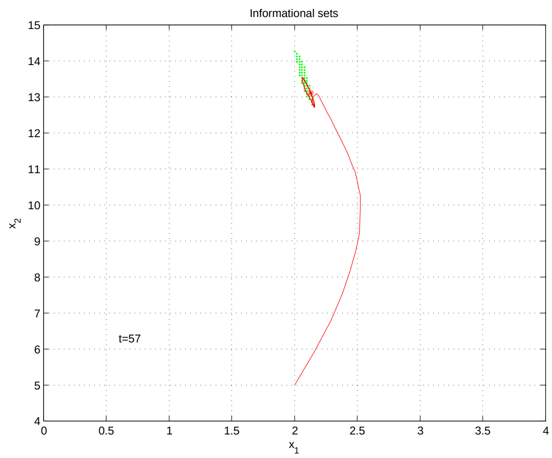

Figure 3. Information set, t=57.

$\overline{x}_2$ is a threshold value. The time in equation (24) is measured in months, coordinates are in tens pieces.

Numerical parameters are: a=3; b=0.8; c=0.1; d=1.1; e=0.2; $\alpha = 4.92$; $\beta = 5.08$; $\gamma = 1$; $\overline{x}_2 = 12.5$. Initial set is: $X_0 = \{x : 1.5 <= x_1 <= 2.3;\ 4 <= x_2 <= 5\}$. Modeling was done for the initial data $x_1^0 = 2$, $x_2^0 = 5$ and disturbances $w_1 = \sin(t\pi/10)$; $w_2 = -\cos(t\pi/10)$ on the segment [0,60]. Real trajectories are of the form depicted on fig. 1. The state trajectory is represented on fig. 2. Information set for 57 is shown on fig. 3. The asterisk indicates a position of the real path.

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