Using WKB method for solving the problem of the stability of slowly diverging jet flows

P.S. Landa
Physics Department, Lomonosov Moscow State University, Moscow 119899, Russia
(Dated:)

PACS numbers:
Keywords:

In accordance with our conception [1], turbulent character of jet flows is conditioned by the strong amplification of weak random disturbances which are always present in jets and, mainly, at the jet nozzle exit section. At a not long distance from the nozzle turbulent pulsations are small, and for their calculation we can use a quasi-linear theory, for example, the Krylov–Bogolyubov asymptotic method for distributed systems [2].

For simplicity, we consider a plane jet issuing from a nuzzle of width $2d$. Neglecting compressibility, we may describe the processes in such a jet by the two-dimensional Navier–Stokes equation for the stream function $\Psi(t,x,y)$ [3] that is related to the longitudinal (U) and transverse (V) components of the flow velocity by $U(t,x,y)=\partial\Psi/\partial y$, $V(t,x,y)=-\,\partial\Psi/\partial x$. In dimensionless coordinates $x'=x/d$, $y'=y/d$ and time $t'=U_0t/d$, where U_0 is the mean flow velocity in the nozzle center, equations for the stream function and vorticity are

$$\frac{\partial\tilde{\Omega}'(t',x',y')}{\partial t'}-\frac{\partial\Psi'(t',x',y')}{\partial x'}\,\frac{\partial\tilde{\Omega}'(t',x',y')}{\partial y'}+\frac{\partial\Psi'(t',x',y')}{\partial y'}\,\frac{\partial\tilde{\Omega}'(t',x',y')}{\partial x'}-\frac{2}{\mathsf{Re}}\,\Delta'\tilde{\Omega}'(t',x',y')=0, \tag{1}$$

where Δ' is the Laplacian in terms of x' and y', $\mathsf{Re}=2U_0d/\nu$ is the Reynolds number, ν is cinematic viscosity, and $\Omega'(t',x',y')=\Delta'\Psi'(t',x',y')$ is the vorticity. It can be easily shown that $\Omega'=-\mathrm{rot}\mathbf{U}'$, where $\mathbf{U}'$ is the velocity vector with coordinates $U'=U(t'd/U_0,x'd,y'd)/U_0$ and $V'=V(t'd/U_0,x'd,y'd)/U_0$. From this point onwards the primes will be dropped.

It should be noted that in so deciding on a dimensionless time, the circular frequencies $\omega=2\pi f$ are measured in units of $\mathsf{S}=\omega d/U_0\equiv\pi\mathsf{St}$, where $\mathsf{St}=2fd/U_0$ is the Strouhal number.

The authors of all classical works on turbulence (see [3, 4]), starting with Reynolds, split the solution of initial equations into mean values and small random disturbances. Because of the quadratic nonlinearity in the Navier–Stokes equation the mean values depend on the deviations from them. As a result the problem of closure of these equations appeared. To avoid this problem, we split the solution of Eqs. (1) into dynamical and stochastic constituents [1]. The dynamical constituents are described by stationary Navier-Stokes equations

$$\Omega_{\mathrm{d}}(x,y)=\frac{\partial U_{\mathrm{d}}(x,y)}{\partial y}-\frac{\partial V_{\mathrm{d}}(x,y)}{\partial x},$$
$$\frac{\partial U_{\mathrm{d}}(x,y)}{\partial x}+\frac{\partial V_{\mathrm{d}}(x,y)}{\partial y}=0, \tag{2}$$
$$U_{\mathrm{d}}(x,y)\,\frac{\partial\Omega_{\mathrm{d}}(x,y)}{\partial x}+V_{\mathrm{d}}(x,y)\,\frac{\partial\Omega_{\mathrm{d}}(x,y)}{\partial y}-\frac{2}{\mathsf{Re}}\left(\frac{\partial^2\Omega_{\mathrm{d}}(x,y)}{\partial x^2}+\frac{\partial^2\Omega_{\mathrm{d}}(x,y)}{\partial y^2}\right)=0, \tag{3}$$

and stochastic constituents are described by the equations for deviations from the stationary solutions. Since Eqs. (2), (3) cannot be solved analytically, and even their numerical solving presents insurmountable difficulties, we set the profile of longitudinal velocity component as

$$U_{\mathrm{d}}(x,y)=\frac{1}{1+\tanh(q/\delta_{00}+r_0)}\times\left[1-\tanh\left(q\,\frac{|y|-1}{\delta_0(x)}-r(x)\right)\right], \tag{4}$$

where $\delta_0(x)$, $r(x)$ and $\delta_0(x)$ are unknown functions of x; $\delta_0(x)$ is the boundary layer thickness which is equal to δ_{00} for $x=0$, $r_0=r(0)$.

To calculate the unknown functions, we have used the conservation laws for the fluxes of momentum and energy following approximately from the Navier–Stokes equations. As a result we have found

$$r(x)\approx r_0=0.5,\quad \delta_0(x)=\sqrt{\delta_{00}^2+\frac{32q^2}{3\mathsf{Re}}\,x},$$
$$\delta_1(x)\approx\frac{\delta_0(x)}{3}\,,\ \delta_2(x)\approx\frac{2\delta_0(x)}{3}\,.$$

The value of $\delta_{00}=\delta_0(0)$ depends on the conditions of the nozzle outflow. In the case of the laminar flow, when the boundary layer may be approximately described by the Blasius function [3, 5], δ_{00} is proportional to $1/\sqrt{\mathsf{Re}}$. Further we will set $\delta_{00}=1/\left(b_0\sqrt{\mathsf{Re}}\right)$, where $b_0=0.1$.

The expressions for $V_{\mathrm{d}}(x,y)$ and $\Omega_{\mathrm{d}}(x,y)$ can be found by means of the exact solution of Eqs. (2). Taking into account that $q/\delta_0(x)\gg1$, from the found expressions for V_{d} and Ω_{d} we obtain for these quantities the following approximate expressions at large values of $|y|$:

$$V_{\mathrm{d}}(x,\pm\infty)\approx\mp\frac{16qr_0}{3\delta_0(x)\mathsf{Re}}\,,\quad \Omega_{\mathrm{d}}(x,\pm\infty)\approx\mp\frac{256q^3r_0}{9\delta_0^3(x)\mathsf{Re}^2}\,. \tag{5}$$

We note that the constant transverse velocity component for large $|y|$ directed towards the jet axis accounts for the entrainment of ambient fluid with the jet flow.

Substituting

$$U(t,x,y) = U_{\rm d}(x,y) + \frac{\partial \Psi_{\rm st}(t,x,y)}{\partial y}\,,$$
$$V(t,x,y) = V_{\rm d}(x,y) - \frac{\partial \Psi_{\rm st}(t,x,y)}{\partial x}\,, \qquad (6)$$
$$\Omega(t,x,y) = \Omega_{\rm d}(x,y) + \Omega_{\rm st}(t,x,y)$$

into Eqs. (1) and taking account of (2), (3), we obtain for stochastic constituents $\Psi_{\rm st}(t,x,y)$ and $\Omega_{\rm st}(t,x,y)$ the following equations:

$$\Omega_{\rm st} - \Delta\Psi_{\rm st} = 0, \qquad (7)$$

$$\frac{\partial \Omega_{\rm st}}{\partial t} + U_{\rm d}(x,y)\,\frac{\partial \Omega_{\rm st}}{\partial x} + V_{\rm d}(x,y)\,\frac{\partial \Omega_{\rm st}}{\partial y} - \frac{\partial \Omega_{\rm d}(x,y)}{\partial y}\,\frac{\partial \Psi_{\rm st}}{\partial x} + \frac{\partial \Omega_{\rm d}(x,y)}{\partial x}\,\frac{\partial \Psi_{\rm st}}{\partial y} - \frac{2}{\mathsf{Re}}\,\Delta\Omega_{\rm st} = \frac{\partial \Psi_{\rm st}}{\partial x}\,\frac{\partial \Omega_{\rm st}}{\partial y} - \frac{\partial \Psi_{\rm st}}{\partial y}\,\frac{\partial \Omega_{\rm st}}{\partial x}\,. \qquad (8)$$

Eliminating $\Omega_{\rm st}$ from Eqs. (7), (8) and discarding the nonlinear terms we find the following linear equation for $\Psi_{\rm st}$:

$$\frac{\partial \Delta\Psi_{\rm st}}{\partial t} + U_{\rm d}(x,y)\,\frac{\partial \Delta\Psi_{\rm st}}{\partial x} + V_{\rm d}(x,y)\,\frac{\partial \Delta\Psi_{\rm st}}{\partial y} - \frac{\partial \Omega_{\rm d}(x,y)}{\partial y}\,\frac{\partial \Psi_{\rm st}}{\partial x} + \frac{\partial \Omega_{\rm d}(x,y)}{\partial x}\,\frac{\partial \Psi_{\rm st}}{\partial y} - \frac{2}{\mathsf{Re}}\Delta\Delta\Psi_{\rm st} = 0. \qquad (9)$$

We seek a solution of Eq. (9) in the form of a sum of running waves of frequency S with a slowly varying complex wave number $Q(\mathsf{S},x)$ and the amplitude $f^{(\mathsf{S})}(\mu x,y)$:

$$\Psi_{\rm st}(t,x,y) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} f^{(\mathsf{S})}(x,y) \times \exp\left[i\left(\mathsf{S}t - \int\limits_{0}^{x} Q(\mathsf{S},x)\,dx\right)\right]\,d\mathsf{S}. \qquad (10)$$

The complex wave number $Q(\mathsf{S},x)$ may be represented as $Q(\mathsf{S},x) = \mathsf{S}/v_{\rm ph}(\mathsf{S},x) + i\Gamma(\mathsf{S},x)$, where $v_{\rm ph}(\mathsf{S},x)$ is the wave phase velocity and Γ is the gain factor.

Taking into account that the jet diverges slowly, we can represent the function $f^{(\mathsf{S})}(x,y)$ and the wave number $Q(\mathsf{S},x)$ as series in a conditional small parameter $\mu \sim 1/\sqrt{\mathsf{Re}}$ characterizing the slowness of the jet divergence:

$$f^{(\mathsf{S})}(\mu x,y) = f_0(\mathsf{S},\mu x,y) + \mu f_1(\mathsf{S},\mu x,y) + \dots\,, \qquad (11)$$
$$Q(\mathsf{S},x) = Q_0(\mathsf{S},x) + \mu Q_1(\mathsf{S},x) + \dots\,,$$

where $f_0(\mathsf{S},\mu x,y)$, $f_1(\mathsf{S},\mu x,y)$, ... are unknown functions vanishing, along with their derivatives, at $y = \pm\infty$. Substituting (10), in view of (11), into Eq. (9) and retaining only terms containing first derivations with respect to x we obtain the following equations for $f_0(\mathsf{S},\mu x,y)$ and $f_1(\mathsf{S},\mu x,y)$:

$$L_0(Q_0)f_0(\mathsf{S},\mu x,y) = 0, \qquad (12)$$
$$L_0(Q_0)f_1(\mathsf{S},\mu x,y) = iQ_1L_1(Q_0)f_0(\mathsf{S},\mu x,y) - L_2(Q_0)f_0(\mathsf{S},\mu x,y), \qquad (13)$$

where

$$L_0(Q_0) = i\Big(\mathsf{S} - U_{\rm d}(x,y)Q_0\Big)\left(\frac{\partial^2}{\partial y^2} - Q_0^2\right) + V_{\rm d}(x,y) \times \left(\frac{\partial^3}{\partial y^3} - Q_0^2\,\frac{\partial}{\partial y}\right) + iQ_0\Omega_{{\rm d}y}(x,y) + \Omega_{{\rm d}x}(x,y)\,\frac{\partial}{\partial y} - \frac{2}{\mathsf{Re}}\left(\frac{\partial^4}{\partial y^4} - 2Q_0^2\,\frac{\partial^2}{\partial y^2} + Q_0^4\right), \qquad (14)$$

$$L_1(Q_0) = U_{\rm d}(x,y)\left(\frac{\partial^2}{\partial y^2} - 3Q_0^2\right) + 2\mathsf{S}Q_0 - 2iQ_0 \times V_{\rm d}(x,y)\,\frac{\partial}{\partial y} - \Omega_{{\rm d}y}(x,y) + \frac{8iQ_0}{\mathsf{Re}}\left(\frac{\partial^2}{\partial y^2} - Q_0^2\right), \qquad (15)$$

$$L_2(Q_0) = \mathsf{S}\left(2Q_0\,\frac{\partial}{\partial x} + \frac{\partial Q_0}{\partial x}\right) + U_{\rm d}(x,y)\left[\frac{\partial^3}{\partial x\partial y^2} - 3Q_0\left(Q_0\,\frac{\partial}{\partial x} + \frac{\partial Q_0}{\partial x}\right)\right] - iV_{\rm d}(x,y)\left(2Q_0\,\frac{\partial^2}{\partial x\partial y} + \frac{\partial Q_0}{\partial x}\,\frac{\partial}{\partial y}\right) - \Omega_{{\rm d}y}(x,y)\,\frac{\partial}{\partial x} + \frac{4i}{\mathsf{Re}}\left[2Q_0\,\frac{\partial^3}{\partial x\partial y^2} + \frac{\partial Q_0}{\partial x}\,\frac{\partial^2}{\partial y^2} - Q_0^2\left(2Q_0\,\frac{\partial}{\partial x} + 3\,\frac{\partial Q_0}{\partial x}\right)\right], \qquad (16)$$

where

$$\Omega_{{\rm d}x}(x,y) = \frac{\partial \Omega_{\rm d}(x,y)}{\partial x}\,, \quad \Omega_{{\rm d}y}(x,y) = \frac{\partial \Omega_{\rm d}(x,y)}{\partial y}\,.$$

Equation (12), with the boundary conditions for function f_0 and its derivatives of vanishing at $y = \pm\infty$, describes a non-self-adjoint boundary-value problem, where Q_0 plays the role of a complex eigenvalue. It should be noted that similar boundary-value problems are studied by mathematician very poorly.

Consistent with Fredholm's theorem on alternative [6], the linear boundary-value problem described by an inhomogeneous equation is solvable if the right-hand member of this equation is orthogonal to each of eigenfunctions $\chi(\mathsf{S},x,y)$ of the Hermitian-conjugate boundary-value problem. The orthogonality condition is

$$iQ_1\int\limits_{-\infty}^{\infty} \overline{\chi}(\mathsf{S},x,y)L_1(Q_0)f_0(\mathsf{S},x,y)\,dy - \int\limits_{-\infty}^{\infty} \overline{\chi}(\mathsf{S},x,y)L_2(Q_0)f_0(\mathsf{S},x,y)\,dy = 0, \qquad (17)$$

where $\overline{\chi}(\mathsf{S},x,y)$ is a function complex conjugate with $\chi(\mathsf{S},x,y)$. Condition (17) allows us to find the correction $Q_1(\mathsf{S},x)$ to the eigenvalue $Q_0(\mathsf{S},x)$.

It is known from experiments that in a plane jet near its nuzzle the velocity pulsations are even functions of the transverse coordinate y. That is why we can seek the stream function as an odd function of y. In this case we can consider only positive values of y. Because for $y \le y_1$, where $y_1(x)$ is the internal boundary of the boundary layer, $U_\mathrm{d}(x,y) \approx 1$, $V_\mathrm{d}(x,y) \approx 0$, $\Omega_{\mathrm{d}x}(x,y) \approx \Omega_{\mathrm{d}y}(x,y) \approx 0$, the odd general solution of Eq. (14) over this region of y is

$$f_0(x,y) = C_1 \sinh\Big(B_{11}(x)y\Big) + C_2 \sinh\Big(B_{12}(x),y\Big), \quad (18)$$

where

$$B_{11}(x) = Q_0(x),\ B_{12}(x) = \sqrt{Q_0^2(x) + \frac{i\Big(\mathsf{S} - Q_0(x)\Big)\mathsf{Re}}{2}} \quad (19)$$

are the roots of the characteristic equation corresponding to Eq. (14), and C_1, C_2 are arbitrary functions of x. It follows from here that for any y smaller then a certain value of $y = y_0(x)$ corresponding to the so called turn-point[?], the solution of Eq. (14) can be represented as

$$f_0(x,y) = C_1 f_{01}(x,y) + C_2 f_{02}(x,y), \quad (20)$$

where $f_{01}(x,y)$ and $f_{02}(x,y)$ are two linearly independent partial solutions of Eq. (14) with initial conditions

$$f_{01}(x,0) = 0,\ \left.\frac{\partial f_{01}}{\partial y}\right|_{y=0} = Q_0(x),\ \left.\frac{\partial^2 f_{01}}{\partial y^2}\right|_{y=0} = Q_0^2, \quad (21)$$
$$\left.\frac{\partial^3 f_{01}}{\partial y^3}\right|_{y=0} = Q_0^3(x),\ f_{02}(x,0) = 0,\ \left.\frac{\partial f_{02}}{\partial y}\right|_{y=0} = B_{12}(x),$$
$$\left.\frac{\partial^2 f_{02}}{\partial y^2}\right|_{y=0} = B_{12}^2(x),\ \left.\frac{\partial^3 f_{02}}{\partial y^3}\right|_{y=0} = B_{12}^3(x).$$

To find a solution for $y \ge y_0$, we note that for $y \ge y_2$, where $y_2 > 1 + \delta_2(x)$, $U_\mathrm{d}(x,y) \approx 0$, $V_\mathrm{d}(x,y) \approx V_\mathrm{d}(x,\infty)$, $\Omega_{\mathrm{d}x}(x,y) \approx \Omega_{\mathrm{d}x}(x,\infty)$. Hence, over this region the coefficients of Eq. (12) are also independent of y. Its solution vanishing at $y \to \infty$ for $y \ge y_2$ can be written as

$$f_0(x,y) = C_3 \exp\Big(B_{21}(x)(y-y_2)\Big) + C_4 \exp\Big(B_{22}(x)(y-y_2)\Big), \quad (22)$$

where C_3 and C_4 are arbitrary functions of x, $B_{21}(x)$ and $B_{22}(x)$ are the roots of the characteristic equation

$$B^4 - \frac{V_\mathrm{d}(x,\infty)\mathsf{Re}}{2} B^3 - \left(2Q_0^2 + \frac{i\mathsf{S}\mathsf{Re}}{2}\right) B^2 - \frac{\mathsf{Re}}{2} \times \left(\frac{\partial \Omega_\mathrm{d}(x,\infty)}{\partial x} - Q_0^2 V_\mathrm{d}(x,\infty)\right) B + Q_0^4 + \frac{i\mathsf{S}Q_0^2\mathsf{Re}}{2} = 0 \quad (23)$$

with negative real parts. It follows from (22) that the solution of Eq. (12) vanishing at $y \to \infty$ for any $y \ge y_0$ may be represented as the following linear combination of two linearly independent partial solutions:

$$f_0(x,y) = C_3 f_{03}(x,y) + C_4 f_{04}(x,y), \quad (24)$$

where $f_{03}(x,y)$ and $f_{04}(x,y)$ are partial solutions of Eq. (12) with boundary conditions

$$f_{03}(x,y_2) = 1,\quad \left.\frac{\partial f_{03}}{\partial y}\right|_{y=y_2} = B_{21}(x),\quad \left.\frac{\partial^2 f_{03}}{\partial y^2}\right|_{y=y_2} = B_{21}^2(x),\quad \left.\frac{\partial^3 f_{03}}{\partial y^3}\right|_{y=y_2} = B_{21}^3(x),$$
$$(25)$$
$$f_{04}(x,y_2) = 1,\quad \left.\frac{\partial f_{04}}{\partial y}\right|_{y=y_2} = B_{22}(x),\quad \left.\frac{\partial^2 f_{04}}{\partial y^2}\right|_{y=y_2} = B_{22}^2(x),\quad \left.\frac{\partial^3 f_{04}}{\partial y^3}\right|_{y=y_2} = B_{22}^3(x).$$

The functions C_j ($j = 1,2,3,4$) and value of $Q_0(x)$ should be taken so that for $y = y_0$ solution (20) graded into (24). This value of $Q_0(x)$ is just an eigenvalue in the approximation under consideration.

Equating (20) and (24), along with their derivations, at point y_0 we obtain the system of equations for C_j:

$$\sum_{j=1}^{4} C_j f_{0j}(x,y_0) = 0,\quad \sum_{j=1}^{4} C_j \frac{\partial f_{0j}(x,y_0)}{\partial y} = 0,$$
$$(26)$$
$$\sum_{j=1}^{4} C_j \frac{\partial^2 f_{0j}(x,y_0)}{\partial y^2} = 0,\ \sum_{j=1}^{4} C_j \frac{\partial^3 f_{0j}(x,y_0)}{\partial y^3} = 0.$$

Eigenvalues of $Q_0(x)$ should be found from the condition that the determinant of system of equations (26) is equal zero. This determinant may be written as

$$\begin{aligned} D(Q_0,x,y_0) &= r_{23}(x,y_0)q_{01}(x,y_0) - r_{13}(x,y_0)q_{02}(x,y_0) \\ &+ r_{12}(x,y_0)q_{03}(x,y_0) + r_{03}(x,y_0)q_{12}(x,y_0) \\ &- r_{02}(x,y_0)q_{13}(x,y_0) + r_{01}(x,y_0)q_{23}(x,y_0), \end{aligned} \quad (27)$$

where

$$q_{ij}(x,y) = f_{01}^{(i)}(x,y)f_{02}^{(j)}(x,y) - f_{01}^{(j)}(x,y)f_{02}^{(i)}(x,y), \quad (28)$$
$$r_{ij}(x,y) = f_{03}^{(i)}(x,y)f_{04}^{(j)}(x,y) - f_{03}^{(j)}(x,y)f_{04}^{(i)}(x,y),$$

i =0–2, j=1–3, $f_{0k}^{(j)}(x,y) = \dfrac{\partial^j f_{0k}(x,y)}{\partial y^j}$, k=1–4.

It should be noted that solutions $f_{01}(x,y)$ found numerically are unstable. For some values of y they merge with $f_{02}(x,y)$. These numerical difficulties result in that we cannot use directly formulas (27) and (28) to calculate eigenvalues and eigenfunctions. One of the ways of calculating these quantities lies in the approximate solving Eq. (12) by using a method similar to WKB method. This method goes back to H. Poincaré and G. Birkhoff [7, 8] and then it evolves by Ya.D. Tamarkin [9, 10]. Concerning method that now is called the WKB method see [11, 12]. However, in these books the method is set out only for a second order differential equation. Here we give the generalization of this method to the fourth order equation.

For using the WKB method we take into account that $V_{\rm d}(x,y)/U_{\rm d}(x,y) \sim 1/\lambda$, where $\lambda = \sqrt{\mathsf{Re}}$. Therefore it is convenient to rewrite Eq. (12) in terms of $v_{\rm d}(x,y) = V_{\rm d}(x,y)\lambda$ and resolve it with respect to higher derivative. As a result, we find

$$\frac{\partial^4 f_0(x,y)}{\partial y^4} - 2Q_0^2\frac{\partial^2 f_0(x,y)}{\partial y^2} + Q_0^4 f_0(x,y) -\lambda\frac{v_{\rm d}(x,y)}{2}\left(\frac{\partial^3 f_0(x,y)}{\partial y^3} - Q_0^2\frac{\partial f_0(x,y)}{\partial y}\right) - \frac{\lambda^2}{2} \times\Bigg[i\Big(\mathsf{S} - U_{\rm d}(x,y)Q_0\Big)\left(\frac{\partial^2 f_0}{\partial y^2} - Q_0^2 f_0\right) + \Omega_{{\rm d}x}(x,y) \times\frac{\partial f_0(x,y)}{\partial y} + iQ_0\Omega_{{\rm d}y}(x,y)f_0(x,y)\Bigg] = 0. \tag{29}$$

This equation contain a large parameter λ.

According to the main idea of WKB method, a partial solution of Eq. (29) may be sought as

$$f_0(x,y) = C\exp\Big(\lambda G(x,y,\lambda)\Big), \tag{30}$$

where $G(x,y,\lambda)$ is a unknown function that may be represented as a series in the small parameter λ^{-1}:

$$G(x,y,\lambda) = G_0(x,y) + \lambda^{-1}G_1(x,y) + \dots . \tag{31}$$

It is convenient to set

$$G_0(x,y) = \int g_0(x,y)\,dy,\ G_1(x,y) = \int g_1(x,y)\,dy,\ \dots \tag{32}$$

Substituting (30), in view of (31), into Eq. (29), restricting expansion (31) to the term of order λ^{-1} and equating the coefficients of the same powers of λ, we obtain for $g_0(x,y)$ and $g_1(x,y)$ the following equations:

$$g_0^2(x,y)\Big(2g_0^2(x,y) - v_{\rm d}(x,y)g_0(x,y) -i\Big(\mathsf{S} - U_{\rm d}(x,y)Q_0\Big)\Big) = 0, \tag{33}$$

$$\Big(8g_0^2(x,y) - 3v_{\rm d}(x,y)g_0(x,y) - 2i\Big(\mathsf{S} - U_{\rm d}(x,y)Q_0\Big)\Big) \times g_0(x,y)g_1(x,y) + \Big(12g_0^2(x,y) - 3v_{\rm d}(x,y)g_0(x,y) - i\Big(\mathsf{S} -U_{\rm d}(x,y)Q_0\Big)\Big)\frac{\partial g_0(x,y)}{\partial y} - \Omega_{{\rm d}x}(x,y)g_0(x,y) = 0, \tag{34}$$

$$\Big(i\Big(\mathsf{S} - U_{\rm d}(x,y)Q_0\Big) + 3v_{\rm d}(x,y)g_0(x,y) - 12g_0^2(x,y)\Big)\left(\frac{\partial g_1(x,y)}{\partial y} + g_1^2(x,y)\right) + \Big(\Omega_{{\rm d}x}(x,y) + 3v_{\rm d}(x,y)\frac{\partial g_0(x,y)}{\partial y} -24g_0(x,y)\frac{\partial g_0(x,y)}{\partial y}\Big)g_1(x,y) - i\Big(\mathsf{S} - U_{\rm d}(x,y)Q_0\Big)Q_0^2 +iQ_0\Omega_{{\rm d}y}(x,y) + 4g_0^2(x,y)Q_0^2 - 6\left(\frac{\partial g_0(x,y)}{\partial y}\right)^2 - 8g_0(x,y) \times\frac{\partial^2 g_0(x,y)}{\partial y^2} + v_{\rm d}(x,y)\left(\frac{\partial^2 g_0(x,y)}{\partial y^2} - Q_0^2 g_0(x,y)\right) = 0. \tag{35}$$

Equation (33) has three roots: $g_{01}(x,y) = 0$ and

$$g_{02,03}(x,y) = \frac{v_{\rm d}(x,y) \mp \sqrt{v_{\rm d}^2(x,y) + 8i\Big(\mathsf{S} - U_{\rm d}(x,y)Q_0\Big)}}{4}. \tag{36}$$

It follows from (36) that outside the boundary layer $g_{02,03}(x,y)$ is equal to

$$g_{02,03}(x,y) = \begin{cases} \mp\dfrac{1+i}{2}\sqrt{\mathsf{S} - Q_0(x)} & y \le y_1, \\ \dfrac{v_{\rm d}(x,\infty) \mp \sqrt{v_{\rm d}^2(x,\infty) + 8i\mathsf{S}}}{4} & y \ge y_2. \end{cases} \tag{37}$$

For the first root Eq. (34) is identical, and the differential equation for $g_{11}(x,y)$ corresponding to the first root follows from Eq. (35):

$$i\Big(\mathsf{S} - U_{\rm d}(x,y)Q_0\Big)\left[\frac{\partial g_{11}(x,y)}{\partial y} + g_{11}^2(x,y) - Q_0^2\right] +\Omega_{{\rm d}x}(x,y)g_{11}(x,y) + iQ_0\Omega_{{\rm d}y}(x,y) = 0. \tag{38}$$

Eq. (38) is a Riccati equation. It is known that by the change of variable $g_{11}(x,y) = \dfrac{\partial\Big(\ln f_{01}(x,y)\Big)}{\partial y}$ Eq. (38) reduces to the linear second order equation

$$i\Big(\mathsf{S} - U_{\rm d}(x,y)Q_0\Big)\left(\frac{\partial^2 f_{01}(x,y)}{\partial y^2} - Q_0^2 f_{01}(x,y)\right) +\Omega_{{\rm d}x}(x,y)\frac{\partial f_{01}(x,y)}{\partial y} + iQ_0\Omega_{{\rm d}y}(x,y)f_{01}(x,y) = 0, \tag{39}$$

that coincides with Eq. (12) for $\mathsf{Re} = \infty$ and $V_{\rm d}(x,y) \equiv 0$.

For $y \le y_1$, when $\Omega_{{\rm d}x}(x,y) \approx 0$ and $\Omega_{{\rm d}y}(x,y) \approx 0$, from Eq. (38) we find

$$g_{111,112}(x,0) \approx \pm Q_0(x) \tag{40}$$

These expressions coincide with the roots of the characteristic equation corresponding to Eq. (12) for $\mathsf{Re} = \infty$, $V_\mathrm{d}(x, y) \equiv 0$ and $y \le y_1$.

For $y \ge y_2$, when $U_\mathrm{d}(x, y) = 0$, $\Omega_{\mathrm{d}x}(x, y) \approx \Omega_{\mathrm{d}x}(x, \infty)$ and $\Omega_{\mathrm{d}y}(x, y) \approx 0$, we find from (38)

$$g_{111,112}(x,\infty) \approx \frac{i\Omega_{\mathrm{d}x}(x,\infty)}{2\mathsf{S}} \mp \sqrt{Q_0^2 - \frac{\Omega^2_{\mathrm{d}x}(x,\infty)}{4\mathsf{S}^2}}\,. \tag{41}$$

It can be easily shown that only $g_{111}(x, \infty)$ has a negative real part and close to the root B_{21} of characteristic equation (23).

It should be noted that steady state solutions of Eq. (38) with negative real parts are unstable with respect to small perturbations. Therefore for their finding special appropriate measures are necessary.

For the second and third roots of Eq. (33) we find from Eq. (34) $g_{12,13}(x, y)$:

$$g_{12,13}(x,y) = \Bigg[-\Omega_{\mathrm{d}x}(x,y) + \Bigg(12g_{02,03}(x,y) - 3v_\mathrm{d}(x,y) - \frac{i\Big(\mathsf{S} - U_\mathrm{d}(x,y)Q_0\Big)}{g_{02,03}(x,y)}\Bigg)\frac{\partial g_{02,03}(x,y)}{\partial y}\Bigg]\Bigg(\Big(3v_\mathrm{d}(x,y) - 8g_{02,03}(x,y)\Big)g_{02,03}(x,y) + 2i\Big(\mathsf{S} - U_\mathrm{d}(x,y)Q_0\Big)\Bigg)^{-1} \tag{42}$$

Expression (42) allows us to calculate the additions to $g_{02}(x, y)$ and $g_{03}(x, y)$ of order $1/\lambda$.

Within the regions $y \le y_1$ and $y \ge y_2$, where $\partial g_{02,03}(x, y)/\partial y \approx 0$, as following from (42)

$$g_{12,13}(x,y) = \begin{cases} 0 & y \le y_1, \\ g_{12,13}(x,\infty) & y \ge y_2, \end{cases} \tag{43}$$

where

$$g_{12,13}(x,\infty) = -\Omega_{\mathrm{d}x}(x,\infty)\Bigg(\Big(3v_\mathrm{d}(x,\infty) - 8g_{02,03}(x,\infty)\Big)g_{02,03}(x,\infty) + 2i\mathsf{S}\Bigg)^{-1}.$$

It may be shown that only $g_{02}(x, \infty)$ and $g_{12}(x, \infty)$ have negative real parts.

It follows from (37) and (43) that the second partial solution of Eq. (38), being a rapidly varying function of y, in the region $y \le y_0(x)$ is

$$f_{02}(x,y) = \exp\Big(G_3(x,y)\Big) - \exp\Big(G_4(x,y)\Big), \tag{44}$$

where

$$G_3(x,y) = \int_0^y g_3(x,u)\,du, \quad G_4(x,y) = \int_0^y g_4(x,u)\,du,$$
$$g_3(x,y) = \sqrt{\mathsf{Re}}\,g_{02}(x,y) + g_{12}(x,y),$$
$$g_4(x,y) = \sqrt{\mathsf{Re}}\,g_{03}(x,y) + g_{13}(x,y).$$

In the region $y \ge y_0(x)$ the second partial solution of Eq. (38) vanishing at $y \to \infty$ may be represented as

$$f_{04}(x,y) = \exp\left(\int_{y_2}^{y} g_3(x,y)\,dy\right). \tag{45}$$

Sewing the solution found for $y \le y_0$ with that found for $y \ge y_0$ we obtain the system of equations of the form (26). Its determinant is described by (27).

To find the eigenfunctions we need to solve Eqs. (26) for the found value of Q_0. As a result, we express three from unknown functions C_j in terms of the fourth one that may be set equal 1.

To find the addition $Q_1(\mathsf{S}, x)$ to eigenvalue $Q_0(\mathsf{S}, x)$ we must calculate the adjoint eigenfunctions with the condition of vanishing at $|y| \to \infty$. The value of $Q_1(\mathsf{S}, x)$ can be found from formula (17). Our calculations have shown that $Q_1(\mathsf{S}, x)$ are negligible for large Reynolds numbers.

[1] P.S. Landa, and P.V.E. McClintock Phys. Rep. **397**, 1–62 (2004).

[2] Landa P.S. Self-oscillations in distributed systems. Moscow, Nauka, 1983 (in Russian.
Stability of slowly diverging jet flow. J. Fluid Mech. 77 (1976) 397–413. J. Fluid Mech. 92 (1979) 209–215. Aerosp. Sci. 21 (1984) 159–199. non-parallel correction, J. Fluid Mech. 424 (2000) 367–377.

[3] L.D. Landau and E.M. Lifshitz, Hydrodynamics (Nauka, Moscow, 1986) (in Russian).

[4] A.S. Monin and A.M. Yaglom, Statistical Gydromechanics, vol. 1, 2 (Gidrometeoizdat, Sankt-Petersburg, 1992) (in Russian).

[5] H. Schlichting, Grenzschicht–Theorie (Verlag G. Braun, Karlsruhe, 1965).

[6] R. Courant and D. Hilbert, Methods Of Mathematical Physics, Vol. 2, (John Wiley, New York–London, 1962).

[7] Poincaré H. Sur les courbes dfinies par les équations différentielles. (Gauthier–Villars, Paris,1886).

[8] Birkhoff G.D. On the asymptotic character of the solutions of certain linear differential equations containing a parameter. Trans. Amer. Math. Soc., 1908, vol. 9, p. 219.

[9] Tamarkin Ya.D. On some general problems of the theory of differential equations and expansion of functiond in

series. Petrograd, 1917 (in Russian).

[10] Tamarkin Ya.D. Some general problems of the theory of ordinary linear differential equations and expansion of an arbitrary function in series of fundamental functions. Math. Z., 1927, Bd. 27, No 1, S. 1.

[11] Heading J. An Introduction to Phase-Integral Methods, (John Wiley, London–Methuen–New York, 1962)

[12] Fröman N. and Fröman P.O., JWKB Approximation, (North-Holland Publ., Amsterdam, 1965).

[] We call the turn-point the value of y for which the absolute value of $f_0(x,y)$ becomes decay in place of growing.