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WHAT IS A PARAMETRIC EXCITATION IN STRUCTURAL DYNAMICS?

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Abstract
It is commonly stated that a system is parametrically excited if the excitation appears as time-varying coefficients of the equations of motion. It is shown that this statement is contradicted in those structural problems with unconstrained motions whose excitation terms, either boundary forces or displacements, appear as inhomogeneities in the boundary conditions. Yet, these excitations, under pertinent conditions, may cause parametric Hill-type instabilities. It is only when suitable coordinate transformations are introduced or a constrained version of the motions is sought (e.g., via a Bubnov-Galerkin approach) that the parametric nature of the excitation is brought out explicitly and unambiguously.

Key words
Parametric resonance, geometrically exact approach, cable, rod, shell, Mathieu-Hill instability.

1 Introduction

The phenomenon of parametric instability is frequently encountered in mechanics as well as in various areas of physics.

Faraday [Faraday, 1831] was one of the first to observe the phenomenon of parametric resonance noting that surface waves in a fluid-filled cylinder under vertical excitation exhibited twice the period of the excitation.

Melde [Melde, 1859] was the first to observe the phenomenon in structural dynamics. He tied a string between a rigid support and the extremity of the prong of a massive tuning fork. He observed that the string could oscillate laterally, although the driving force was longitudinal, at one half the frequency of the fork under a number of critical conditions. Two decades later, Lord Rayleigh [Strutt, 1883] provided a theoretical basis for interpreting the parametric resonance of strings and conducted further experiments.

The parametric resonance is not necessarily an instability since a small parametric-resonance load can also stabilize a system which is unstable or can be exploited for vibration suppression via autoparametric transfer of energy. Stephenson [Stephenson, 1908] made the remarkable observation that a column subject to an axial periodic load was stable even though the steady value of the load was twice that of the Euler load.

Later, Belayev [Belayev, 1924] analyzed the response of a straight elastic hinged-hinged column subject to an axial load of the form $p(t) = p_0 + P\cos\Omega t$. He obtained a Mathieu equation for the dynamic response of the column and determined the principal parametric resonance frequency. He showed that the column could be made to oscillate with the frequency $\frac{1}{2}\Omega$ if it is close to one of the natural frequencies of the lateral motion even at load magnitudes below the static buckling load. Einaudi [Einaudi, 1936] was the first to study the parametric resonance caused in plates by pulsating pressures on the lateral surfaces. These investigations were furthered in [Chelomei, 1939; Bolotin, 1964].

The cited references are only a small part of an extremely rich literature, including several books and monographs. Some of them are mostly devoted to the theory of parametrically excited linear discrete systems [Yakubovich, 1975; Nayfeh, 1979; Cartmell, 1990].

A good deal of efforts has been directed towards methods for constructing the instability regions of parametrically excited systems [Nayfeh, 1979; Seyranian, 2001].

An extensively large number of works has addressed the principal parametric resonance of rods and its cancellation [Zavodney, 1989; Yabuno, 2003; Lacarbonara, 2007].

In all of the studies conducted on the phenomenon of the parametric resonance, the fundamental question of what a parametric excitation is and how it can be recognized a priori in a structural system has often been left in the background. It is a common belief that a system is parametrically excited when the excitation appears as time-varying coefficients of the governing equations of motion. The aim of this paper is to show that in a multitude of cases a parametric excitation can be masked in a way that one would not be led to assert that it acts as a parametric excitation. We discuss ways to disclose the parametric nature of the excitations.

2 Parametric resonances

The parametric excitation mechanism differs physically and mathematically with respect to a direct excitation mechanism. The distinguished physical difference is in that a small excitation cannot produce a large response in directly excited systems unless the driving frequency is close to one of the natural frequencies (primary resonance). Conversely, a small parametric excitation (in principle, infinitesimal provided that the system dissipation be negligible) can produce a large response when the driving frequency is close to twice one of the natural frequencies of the system (principal parametric resonance). Moreover, in linear parametrically excited systems, the amplitude of the unstable solution grows exponentially unbounded in spite of the presence of viscous dissipation contrary to directly excited systems where the resonance can be bounded by the damping. The nonlinearities, under suitable conditions, act to saturate the parametric instabilities. The saturation is caused by the fact that, due to the dependence of the eigenfrequency on the motion, the growth of the parametrically excited oscillations causes the eigenfrequency to be shifted out of resonance.

In the following sections, we will discuss the parametric excitation of naturally discrete systems, first, and then, of distributed-parameter systems, in particular, cables, rods, and shells. Throughout the paper, we employ Gibbs notation for vectors and tensors. Euclidean vectors and vector-valued functions are denoted by lower-case, italic, bold-face symbols. The dot product and cross product of (vectors) $\boldsymbol{u}$ and $\boldsymbol{v}$ are denoted $\boldsymbol{u} \cdot \boldsymbol{v}$ and $\boldsymbol{u} \times \boldsymbol{v}$, respectively. Tensors are denoted by upper-case, italic, bold-face symbols.

2.1 The simple pendulum and the autoparametric transfer of energy

The simplest example of a parametrically excited system is the pendulum subject to a vertical motion of its suspension point. We denote θ the angle that the pendulum arm makes with the downward vertical line, assumed positive in the counter-clockwise direction and $y(t)$ the motion of the pivot. The equation of motion is

$$ml^2\theta_{tt} + ml\,(g + y_{tt}) \sin\theta = 0 \tag{1}$$

In the pendulum problem, the term $y_{tt} \sin\theta = y_{tt}\theta + O(\theta^3)$ is the parametric excitation term that causes the instability when y is either harmonic or periodic with appropriate frequencies.

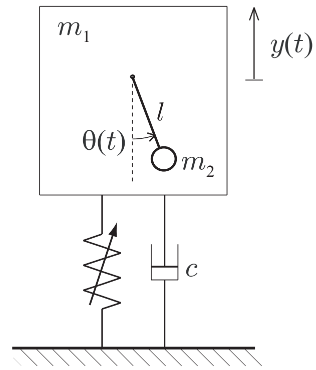

Figure 1. The autoparametric vibration absorber.

The phenomenon of parametric resonance is not necessarily harmful as it can be beneficially exploited to transfer energy from the directly excited system to a parametrically coupled substructure, acting as an autoparametric vibration absorber [Cartmell, 1990]. Within the context of an autoparametric transfer of energy between a primary structure and an attached pendulum, the primary structure consists of a mass m_1, whose motion is denoted $y(t)$, a nonlinearly elastic spring, whose constitutive law is $N(t) = \hat{N}(y)$, and a dashpot of viscous coefficient c. The mass is subject to a direct force $F(t)$. Attached to the mass m_1, there is a pendulum of mass m_2 and length l whose angle with respect to the downward vertical line is denoted θ. The equations of motion are (1) (with $m = m_2$) and

$$\begin{aligned}(m_1 + m_2)y_{tt} + m_2 l(\theta_{tt} \sin\theta + \theta_t^2 \cos\theta)\\ + cy_t + \hat{N}(y) = F(t) - m_2 g\end{aligned} \tag{2}$$

By letting $\omega_1^2 := k/m_1$ and $\omega_2^2 := g/l$ be the frequencies of the structure (k is the linear elastic constant appearing in the linearization of the spring constitutive law) and the pendulum, respectively, the autoparametric transfer of energy may occur when $\omega_1 \approx 2\omega_2$. The motion of the structure excites parametrically the pendulum and can cause its resonance with large-amplitude pendulations.

It is clear that, in discrete parametrically excited systems, the parametric input enters the equations of motion as periodic time-varying coefficients. Further, another distinguished feature is that the parametrically excited motion is orthogonal to the direction of the excitation.

In the next sections, we discuss cables, rods and shells subject to parametric excitations.

3 Cables subject to support motions

We consider cables resisting only tension forces and refer them to the fixed Cartesian frame $(O, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$ shown in Fig. 2. The cable is considered stress-free in the configuration $\mathcal{B}$ that represents any reference line whose length is L. We let the arclength along this line,

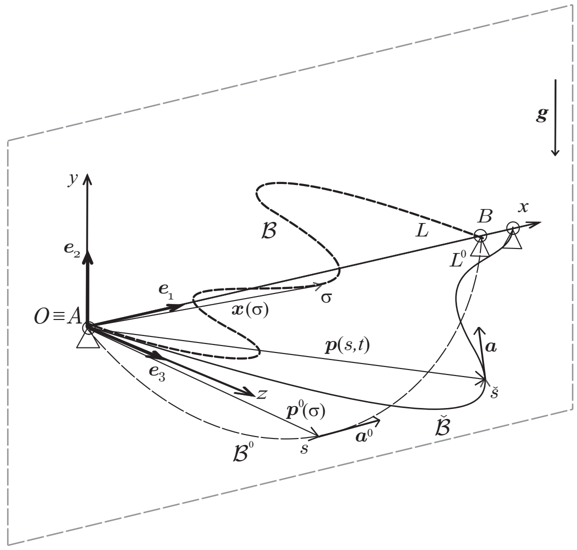

Figure 2. Stress-free configuration $\mathcal{B}$, pre-stressed configuration $\mathcal{B}^0$ and actual configuration $\breve{\mathcal{B}}$.

denoted σ, be the coordinate identifying the material sections of the cable, with $\sigma \in [0,\, L]$.

When the cable ends are fixed to two points, say A and B, and the cable is let free to hang under the action of gravity, the cable occupies an equilibrium configuration, here denoted $\mathcal{B}^0$, lying in the vertical plane ($\boldsymbol{e}_1$, $\boldsymbol{e}_2$), the well-known catenary. This configuration is typically taken inextensible due to the negligible elastic effects in sufficiently rigid cables under their own weight. The arclength along the configuration $\mathcal{B}^0$ is denoted s and the position vector is $\boldsymbol{p}^0 = x(\sigma)\boldsymbol{e}_1 + y(\sigma)\boldsymbol{e}_2$.

Under the prescribed motions of the supports, the cable suffers a change of configuration from $\mathcal{B}^0$ to $\breve{\mathcal{B}}$ whose position vector is $\boldsymbol{p}(s,t) = \boldsymbol{p}^0(s) + \boldsymbol{u}(s,t)$ with $\boldsymbol{u}(s,t) = u(s,t)\boldsymbol{e}_1 + v(s,t)\boldsymbol{e}_2 + w(s,t)\boldsymbol{e}_3$. The gradient of the position vector, $\boldsymbol{p}_s = \boldsymbol{a}^0 + \boldsymbol{u}_s = \nu\boldsymbol{a}$ yields the cable incremental stretch

$$\nu = \sqrt{(\cos\theta^0 + u_s)^2 + (\sin\theta^0 + v_s)^2 + w_s^2} \tag{3}$$

where the Cartesian representation of the unit tangent vector in $\mathcal{B}^0$ is $\boldsymbol{a}^0 = \cos\theta^0\boldsymbol{e}_1 + \sin\theta^0\boldsymbol{e}_2$ (θ^0 is the angle between $\boldsymbol{a}^0$ and $\boldsymbol{e}_1$); the unit vector in the current tangential direction is obtained as $\boldsymbol{a} = \boldsymbol{p}_s/\nu = (\boldsymbol{a}^0 + \boldsymbol{u}_s)/\nu$ which, in componential form with respect to the fixed basis, becomes

$$\boldsymbol{a} = \frac{(\cos\theta^0 + u_s)\boldsymbol{e}_1 + (\sin\theta^0 + v_s)\boldsymbol{e}_2 + w_s\boldsymbol{e}_3}{\sqrt{(\cos\theta^0 + u_s)^2 + (\sin\theta^0 + v_s)^2 + w_s^2}} \tag{4}$$

Enforcing the balance of linear and angular momentum leads to the following equation of motion:

$$\left[\breve{N}(s,t)\boldsymbol{a}(s,t)\right]_s = \rho A(s)\boldsymbol{p}_{tt}(s,t) \tag{5}$$

where ρA is the mass per unite reference length and $\boldsymbol{n}(s,t) = \breve{N}(s,t)\boldsymbol{a}(s,t)$ is the current tension. The incremental form of the equation of motion is obtained once we let $\breve{N}(s,t) = N^0(s) + N(s,t)$ where $N^0(s)$ denotes the tension in the catenary configuration and $N(s,t)$ is the incremental dynamic tension. For a nonlinearly visco-elastic material, the constitutive law for the incremental tension may be expressed as $N(s,t) = \hat{N}(\nu, \nu_t, s)$. The equation of motion becomes

$$\begin{aligned}[\hat{N}(\nu,\nu_t,s)\boldsymbol{a}(s,t)]_s + \left[N^0(s)\left(\boldsymbol{a}(s,t) - \boldsymbol{a}^0(s)\right)\right]_s \\ = \rho A(s)\boldsymbol{p}_{tt}(s,t)\end{aligned} \tag{6}$$

For cables suspended from points at the same level and with the right support subject to a horizontal motion $u_B(t)\boldsymbol{e}_1$, the boundary conditions are

$$\boldsymbol{p}(0,t) = \boldsymbol{o}, \quad \boldsymbol{p}(L,t) = (l + u_B(t))\boldsymbol{e}_1$$

whereas, for inclined cables, they are $\boldsymbol{p}(0,t) = \boldsymbol{o}$, $\boldsymbol{p}(L,t) = (l + u_B(t))\boldsymbol{e}_1 + h\boldsymbol{e}_2$ with l and h indicating the span between the supports A and B and the difference in levels, respectively.

Clearly, in the given unconstrained version of the visco-elastic motions of the cable, the excitation merely appears as inhomogeneity of the geometric boundary conditions. However, it is theoretically and experimentally known that a cable, subject to a horizontal support motion with frequency nearly twice the frequency of one of its transverse modes, may suffer a principal parametric instability.

The parametric nature of the excitation can be identified in different ways. One way is to introduce a coordinate transformation such that the inhomogeneous boundary conditions are transformed into homogeneous boundary conditions. To this end, for the horizontal cable, we let $\boldsymbol{p}(s,t) = \bar{\boldsymbol{p}}(s,t) + \boldsymbol{q}(s,t)$ with $\boldsymbol{q}(\boldsymbol{o},t) = \boldsymbol{o}$ and $\boldsymbol{q}(L^0,t) = \boldsymbol{o}$. Consequently, $\bar{\boldsymbol{p}}(0,t) = \boldsymbol{o}$ and $\bar{\boldsymbol{p}}(L^0,t) = (l + u_B(t))\boldsymbol{e}_1$. One possible choice is $\bar{\boldsymbol{p}}(s,t) = s/L^0(l + u_B(t))\boldsymbol{e}_1$. By employing the newly introduced position vector $\boldsymbol{q}(s,t) = q_1\boldsymbol{a}_1 + q_2\boldsymbol{a}_2 + q_3\boldsymbol{a}_3$ as kinematic descriptor, and by exploiting $\boldsymbol{p}_s = \nu\boldsymbol{a}$, we obtain

$$\boldsymbol{a}(q_j, u_B, s) = \left(\frac{l + u_B(t)}{L^0}\boldsymbol{e}_1 + \boldsymbol{q}_s\right)/\nu \tag{7}$$

$$\nu(q_j, u_B, s) = \sqrt{\left[\frac{l + u_B(t)}{L^0} + q_{1s}\right]^2 + q_{2s}^2 + q_{3s}^2}$$

Further, the inertia force becomes $\rho A\boldsymbol{p}_{tt} = \rho A u_{Btt}(t)s/L^0\,\boldsymbol{e}_1 + \rho A\boldsymbol{q}_{tt}$. Hence, the ensuing equation of motion (6) with the new coordinates will now exhibit seemingly parametric-type forcing terms in $[\hat{N}(\nu,\nu_t,s)\boldsymbol{a}(s,t)]_s$ and direct excitation terms in $\left[N^0(s)\boldsymbol{a}(s,t)\right]_s$ and $\rho A u_{Btt}(t)s/L^0\,\boldsymbol{e}_1$. Clearly, to ascertain that the seemingly parametric-type forcing terms do not appear at higher order, a linearization of the equation of motion would reveal that there are terms of the form $u_{Btt}(t)q_j(s,t)$.

Next, we shall show how, in a constrained model, as the Irvine's model of shallow cables, the support motions become, as a consequence of the geometric and material constraint assumptions, time-varying coefficients of the constrained equations of motion.

3.1 Irvine's model of shallow cables

Shallow/taut cables are such that $\cos\theta^0 \approx 1$ and $\sin\theta^0 \approx \theta^0$. Moreover, according to Irvine [Irvine, 1974], $|u_x| << 1$ where x is the horizontal coordinate along $\boldsymbol{e}_1$; further, the horizontal acceleration u_{tt} may be neglected. The catenary equilibrium tends to the parabolic equilibrium where $H^0 = N^0\cos\theta^0$ denotes the horizontal projection of the gravity-induced tension. We let $\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 \approx \cos\theta^0(\boldsymbol{e}_1 + (y_x+v_x)\boldsymbol{e}_2 + w_x\boldsymbol{e}_3)$. For the equilibrium in the horizontal direction, under the prevailing assumption of negligible horizontal inertia, $\breve{N}a_1 = \breve{H} =$ const. However, $\breve{N}\cos\theta^0(1+u_x)/\nu \approx \breve{N}\cos\theta^0 = (N^0 + \hat{N})\cos\theta^0 = H^0 + \hat{H}$ where $\hat{H} = \hat{N}\cos\theta^0$ is the approximate horizontal projection of the incremental tension. The balance equations in the $(\boldsymbol{e}_2,\, \boldsymbol{e}_3)-$directions become, respectively,

$$
\begin{aligned}
\rho A v_{tt} - H^0 v_{xx} - \hat{H}(y_{xx}+v_{xx}) = 0 \\
\rho A w_{tt} - (H^0 + \hat{H}) w_{xx} = 0
\end{aligned}
\tag{8}
$$

The horizontal projection of the incremental tension is obtained via a linearly elastic constitutive law and through expansion of the incremental stretch ν up to second-order terms of the displacement gradients, namely,

$$
\hat{H} = \frac{EA}{L^{\mathrm{e}}}\left[u_B(t) + \int_0^l y_x v_x dx + \frac{1}{2}\int_0^l \left(v_x^2 + w_x^2\right) dx\right]
\tag{9}
$$

where $L^{\mathrm{e}} = \int_0^l (\sec\theta^0)^3 dx \approx l$. Consequently, the equations of motion become

$$
\begin{aligned}
&\rho A v_{tt} - H^0 v_{xx} - \frac{EA}{L^{\mathrm{e}}} u_B(t)(y_{xx}+v_{xx}) \\
&\quad - \frac{EA}{L^{\mathrm{e}}}(y_{xx}+v_{xx})\int_0^l y_x v_x dx \\
&\quad - \frac{1}{2}\frac{EA}{L^{\mathrm{e}}}(y_{xx}+v_{xx})\int_0^l \left(v_x^2+w_x^2\right) dx = 0 \\
&\rho A w_{tt} - H^0 w_{xx} - \frac{EA}{L^{\mathrm{e}}} u_B(t) w_{xx} \\
&\quad - \frac{EA}{L^{\mathrm{e}}} w_{xx}\int_0^l y_x v_x dx \\
&\quad - \frac{1}{2}\frac{EA}{L^{\mathrm{e}}} w_{xx}\int_0^l \left(v_x^2+w_x^2\right) dx = 0
\end{aligned}
\tag{10}
$$

In these widely used equations of motion of shallow cables, the support motion $u_B(t)$ appears as a direct forcing term in $u_B(t)y_{xx}$ and as a time-varying (parametric) coefficient in $u_B(t)v_{xx}$ or $u_B(t)w_{xx}$.

3.2 The simply supported straight rod subject to a pulsating end thrust

The planar equations of motion of a straight rod, with a compact closed section, subject to an end thrust (see Fig. 3) are

$$
\begin{aligned}
\boldsymbol{n}_s(s,t) &= \rho A \boldsymbol{p}_{tt}(s,t) \\
\boldsymbol{m}_s(s,t) + \boldsymbol{p}_s(s,t)\times\boldsymbol{n}(s,t) &= \rho J \theta_{tt}\boldsymbol{b}_3
\end{aligned}
\tag{11}
$$

where $\boldsymbol{n}(s,t)$ and $\boldsymbol{m}(s,t)$ denote the contact force and contact couple at the rod section s, respectively. The rod strains are $\boldsymbol{p}_s(s,t) = \nu\boldsymbol{b}_1 + \eta\boldsymbol{b}_2$ where $(\boldsymbol{b}_1,\, \boldsymbol{b}_2)$ are the rod section-fixed directors in the normal and transverse directions, respectively. Once we let $\boldsymbol{n}(s,t) = N(s,t)\boldsymbol{b}_1(s,t) + H(s,t)\boldsymbol{b}_2(s,t)$, the equations of motion, in componential form, are

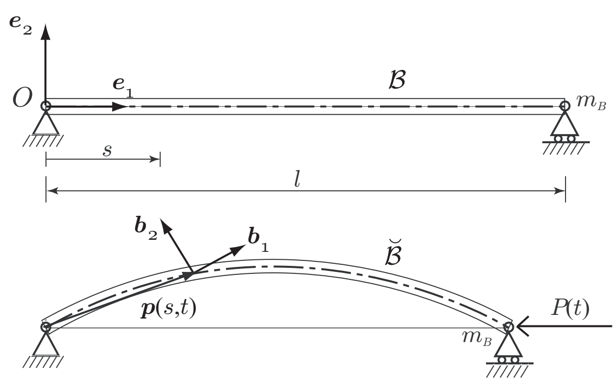

Figure 3. Stress-free configuration $\mathcal{B}$ of the rod and actual configuration $\breve{\mathcal{B}}$ under the end thrust $P(t)$.

$$
N_s - \theta_s H = \rho A \boldsymbol{p}_{tt}\cdot\boldsymbol{b}_1 \tag{12}
$$

$$
H_s + \theta_s N = \rho A \boldsymbol{p}_{tt}\cdot\boldsymbol{b}_2 \tag{13}
$$

$$
M_s + \nu H - \eta N = \rho J \theta_{tt} \tag{14}
$$

The boundary conditions, for the simply supported rod with a lumped mass m_B at B, are

$$
\begin{aligned}
&\boldsymbol{p}(0,t) = \boldsymbol{o},\ \boldsymbol{p}(l,t)\cdot\boldsymbol{e}_2 = 0, \\
&M(0,t) = M(l,t) = 0, \\
&(-\boldsymbol{n}(l,t) - P(t)\boldsymbol{e}_1 + V(t)\boldsymbol{e}_2)\cdot\boldsymbol{e}_1 = m_B\boldsymbol{p}_{tt}(l,t)\cdot\boldsymbol{e}_1
\end{aligned}
\tag{15}
$$

The above given mechanical boundary condition is rewritten as

$$
\begin{aligned}
m_B\boldsymbol{p}_{tt}(l,t)\cdot\boldsymbol{e}_1 + N(l,t)\cos\theta(l,t) \\
- H(l,t)\sin\theta(l,t) + P(t) = 0
\end{aligned}
\tag{16}
$$

For such unconstrained extensional/flexural/shearing motions, the governing equations of motion do not exhibit time-varying coefficients and the time-varying force $P(t)$ appears as a direct excitation term in the boundary condition.

On the other hand, were we to enforce the inextensibility and unshearability by prescribing the internal kinematic constraints $\nu = 1$ and $\eta = 0$, the ensuing equation of motion would exhibit parametric time-varying coefficients. We express the displacement $\boldsymbol{u}$ in the basis $(\boldsymbol{e}_1, \boldsymbol{e}_2)$. Consequently,

$$\begin{aligned} \nu &= \sqrt{(1+u_s)^2 + v_s^2}, \\ \eta &= -(1+u_s)\sin\theta + v_s\cos\theta \end{aligned} \tag{17}$$

Solving both constraints, $\nu = 1$ and $\eta = 0$, with respect to u_s and θ, and calculating the flexural curvature yields

$$\begin{aligned} &u_s = -1 + \sqrt{1-v_s^2}, \quad \theta = \tan^{-1}\left(\frac{v_s}{\sqrt{1-v_s^2}}\right) \\ &\mu = \theta_s = \frac{v_{ss}}{\sqrt{1-v_s^2}} \end{aligned} \tag{18}$$

We solve (14) with respect to the shear force and get $H = -M_s + \rho J\theta_{tt}$. Substituting it into (12) and (13) yields

$$N_s + \theta_s M_s - \theta_s \rho J\theta_{tt} = \rho A\boldsymbol{p}_{tt}\cdot\boldsymbol{b}_1 \tag{19}$$
$$(\rho J\theta_{tt})_s - M_{ss} + \theta_s N = \rho A\boldsymbol{p}_{tt}\cdot\boldsymbol{b}_2 \tag{20}$$

Next, solving (16) for $N(l,t)$ yields

$$\begin{aligned} N(l,t) &= \rho J\theta_{tt}(l,t)\tan\theta(l,t) - M_s(l,t)\tan\theta(l,t) \\ &\quad - m_B u_{tt}(l,t)\sec\theta(l,t) - P(t)\sec\theta(l,t) \end{aligned} \tag{21}$$

Integrating (19) and using (21) delivers the tension as

$$\begin{aligned} &N(s,t) = N(l,t) \\ &\quad + \int_l^s \left(\theta_s\rho J\theta_{tt} - \theta_s M_s + \rho A\boldsymbol{p}_{tt}\cdot\boldsymbol{b}_1\right) d\xi \end{aligned} \tag{22}$$

Substituting it into (20) yields the governing equation of motion

$$\begin{aligned} &\rho A\boldsymbol{p}_{tt}\cdot\boldsymbol{b}_2 - (\rho J\theta_{tt})_s + \hat{M}_{ss} \\ &\quad + \theta_s\left[m_B u_{tt}(l,t)\sec\theta(l,t) - \rho J\theta_{tt}(l,t)\tan\theta(l,t)\right] \\ &\quad + \theta_s\left[\hat{M}_s(l,t)\tan\theta(l,t)\right] \\ &\quad - \theta_s\int_l^s\left(\theta_s\rho J\theta_{tt} - \theta_s\hat{M}_s + \rho A\boldsymbol{p}_{tt}\cdot\boldsymbol{b}_1\right) d\xi \\ &\quad + P(t)\theta_s\sec\theta(l,t) = 0 \end{aligned} \tag{23}$$

where a nonlinear visco-elastic law has been introduced in the form $M(s,t) = \hat{M}(\mu, \mu_t, s)$. The horizontal motion u is obtained integrating $(18)_1$ as

$$u(s,t) = -s + \int_0^s \sqrt{1-v_s^2}\, d\xi \tag{24}$$

By differentiating (24) and $(18)_2$ twice with respect to time, the inertia force in (23) can be expressed in terms of v, and, substituted into (23), yields an integro-partial-differential equation with time-varying coefficients.

An alternative derivation may be further considered. Equation (15), without dotting it with $\boldsymbol{e}_1$, is solved for $\boldsymbol{n}(l,t)$ that, by exploiting the kinematic boundary condition at B, delivers

$$\boldsymbol{n}(l,t) = -[P(t) + m_B u_{tt}(l,t)]\boldsymbol{e}_1 + V(t)\boldsymbol{e}_2 \tag{25}$$

The integration of the linear momentum equation yields

$$\begin{aligned} \boldsymbol{n}(s,t) &= \boldsymbol{n}(l,t) + \int_l^s \rho A\boldsymbol{p}_{tt} d\xi = \\ &- [P(t) + m_B u_{tt}(l,t)]\boldsymbol{e}_1 + V(t)\boldsymbol{e}_2 \\ &+ \int_l^s \rho A\boldsymbol{p}_{tt} d\xi \end{aligned} \tag{26}$$

In turn, the contact force delivers the shear force H. The inextensibility and unshearability constraints deliver $\boldsymbol{p}_s = \boldsymbol{b}_1 = \cos\theta\boldsymbol{e}_1 + \sin\theta\boldsymbol{e}_2$. Hence, the balance of angular momentum becomes

$$\begin{aligned} &M_s + [P(t) + m_B u_{tt}(l,t) - \int_l^s \rho A u_{tt} d\xi]\sin\theta \\ &\quad + [V(t) + \int_l^s \rho A v_{tt} d\xi]\cos\theta = \rho J\theta_{tt} \end{aligned} \tag{27}$$

To determine the reaction force $V(t)$, we impose the balance of angular momentum of the entire beam with respect to O, that is,

$$\boldsymbol{p}(l,t)\times V(t)\boldsymbol{e}_2 = \int_0^l \rho A\boldsymbol{p}\times\boldsymbol{p}_{tt}\cdot\boldsymbol{e}_3 ds \tag{28}$$

Therefore, the final equation of motion is

$$\begin{aligned} &\hat{M}_s(\mu,\mu_t,s) + \frac{\int_0^l \rho A\left[(x+u)v_{tt} - v u_{tt}\right] ds}{\int_0^l \sqrt{1-v_s^2}\, ds}\cos\theta \\ &\quad + [P(t) + m_B u_{tt}(l,t) - \int_l^s \rho A u_{tt} d\xi]\sin\theta \\ &\quad + \cos\theta\int_l^s \rho A v_{tt} d\xi = \rho J\theta_{tt} \end{aligned} \tag{29}$$

This equation, when the time derivatives are neglected and a linealry elastic constitutive law is considered, reduces to the well-known elastica equation, $(EJ\theta_s)_s + P\sin\theta = 0$.

3.3 Mettler's equation of motion

Mettler [Mettler, 1962] assumed a linearly elastic unshearable, extensible and flexible rod undergoing small rotations and subject to a prescribed horizontal motion of the support. Therefore, by further neglecting rotatory inertia and considering $\theta_s M_s$ of higher order in (19), the equations of motion (19) and (20) become

$$N_s = 0,\ -M_{ss} + \theta_s N = \rho A v_{tt} \qquad (30)$$

By taking only the first-order term in the curvature and expressing ν in a Mac Laurin series gives

$$\theta_s = v_{ss}, \quad \nu = 1 + u_s + \frac{1}{2}v_s^2 + O(u_s^2) \qquad (31)$$

Linearly elastic constitutive laws are introduced

$$\begin{aligned} N &= EA(\nu - 1) = EA\left(u_s + \frac{1}{2}v_s^2\right) \\ M &= EJ\mu = EJv_{ss} \end{aligned} \qquad (32)$$

Equation $(30)_1$ entails that the tension is constant; consequently, using $(32)_1$ yields

$$\begin{aligned} N &= \frac{1}{l}\int_0^l EA\left(u_s + \frac{1}{2}v_s^2\right)ds \\ &= \frac{EA}{l}u_B(t) + \frac{EA}{2l}\int_0^l v_s^2 ds \end{aligned} \qquad (33)$$

where $u_B(t)$ is the prescribed support motion. The final governing equation of motion is the following integro-partial-differential equation with periodic time-varying coefficients:

$$\rho A v_{tt} + EJv_{ssss} - \frac{EA}{l}u_B(t)v_{ss} - \frac{EA}{2l}v_{ss}\int_0^l v_s^2 ds = 0 \qquad (34)$$

This approximate equation has been extensively employed for studies about the parametric resonance of straight rods; a similar version exists for shallow arches [Mettler, 1962].

4 The cantilivered rod subject to a vertical motion

We consider a cantilevered rod subject to a support vertical motion in the form $z(t)\boldsymbol{e}_1$. The boundary conditions are

$$\begin{aligned} &\boldsymbol{p}(0,t) = z(t)\boldsymbol{e}_1,\ \theta(0,t) = 0, \\ &\boldsymbol{n}(l,t) = \boldsymbol{o},\ M(l,t) = 0 \end{aligned} \qquad (35)$$

This problem is interesting in that, depending on the basis onto which we choose to project the equations of motion, the base motion can appear as a parametric forcing or a direct external forcing. Were we to choose the fixed basis $(\boldsymbol{e}_1, \boldsymbol{e}_2)$, where $\boldsymbol{e}_1$ is taken collinear with the rod axis,

$$\begin{aligned} &(N_s - \theta_s H)\cos\theta - (H_s + \theta_s N)\sin\theta = \rho A(u_{tt} + z_{tt}) \\ &(N_s - \theta_s H)\sin\theta + (H_s + \theta_s N)\cos\theta = \rho A v_{tt} \end{aligned} \qquad (36)$$

The balance of angular momentum is still given by (14).

On the other hand, projecting the equations of motion into the rod section-fixed basis, $(\boldsymbol{b}_1, \boldsymbol{b}_2)$, yields

$$\begin{aligned} N_s - \theta_s H &= \rho A(u_{tt} + z_{tt})\cos\theta + \rho A v_{tt}\sin\theta \\ H_s + \theta_s N &= -\rho A(u_{tt} + z_{tt})\sin\theta + \rho A v_{tt}\cos\theta \end{aligned} \qquad (37)$$

The boundary conditions are $u(0,t) = 0,\ v(0,t) = 0,\ \theta(0,t) = 0,\ \boldsymbol{n}(l,t) = \boldsymbol{o},\ M(l,t) = 0$.

A constrained version is typically used in the literature enforcing the inextensibility and unshearbility. The equation of motion of the constrained rod would again be an integro-partial-differential equation of motion with the support motion appearing as a time-varying coefficient in agreement with previous observations.

5 Spherical and cylindrical shells subject to pulsating pressures on the inner and outer surfaces

The equations governing radial motions of visco-elastic shells suffering in-surface stretching and transverse stretching were obtained in [Antman and Lacarbonara, 2008] within the three-dimensional elasticity theory as well as within the geometrically exact Cosserat theory.

We identify material points of the cylindrical shell by their cylindrical coordinates r, ϕ, z (local basis $\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3 \equiv \boldsymbol{k}$) and we identify material points of a spherical shell by their spherical coordinates r, θ, ϕ (local basis $\boldsymbol{b}_1, \boldsymbol{b}_2, \boldsymbol{b}_3$).

Plane-strain radial motions of cylindrical shells are taken in the form

$$\boldsymbol{p}_{\mathrm{C}}(r, \phi, z, t) = f(r,t)\boldsymbol{a}_1(\phi) + z\boldsymbol{k}, \qquad (38)$$

and radial motions of spherical shells are of the form

$$\boldsymbol{p}^{\mathrm{S}}(r, \theta, \phi, t) = f(r,t)\boldsymbol{b}_1(\theta, \phi). \qquad (39)$$

where the indices C and S, written either as subscripts or superscripts, here and henceforth, will identify distinctive quantities for cylindrical and spherical shells.

The ensuing right Cauchy-Green deformation tensors

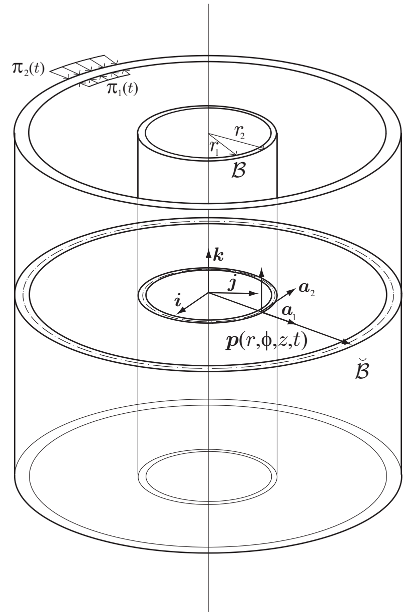

Figure 4. Stress-free configuration $\mathcal{B}$ of the cylindrical shell and actual configuration $\breve{\mathcal{B}}$ under the pressures $\pi_1(t)$ and $\pi_2(t)$.

are

$$\boldsymbol{C}^{\mathrm{c}} = {f_r}^2 \boldsymbol{a}_1\boldsymbol{a}_1 + \frac{f^2}{r^2}\boldsymbol{a}_2\boldsymbol{a}_2 + \boldsymbol{k}\boldsymbol{k}, \tag{40}$$

$$\boldsymbol{C}^{\mathrm{s}} = {f_r}^2 \boldsymbol{b}_1\boldsymbol{b}_1 + \frac{f^2}{r^2}[\boldsymbol{b}_2\boldsymbol{b}_2 + \boldsymbol{b}_3\boldsymbol{b}_3] \tag{41}$$

The second Piola-Kirchhoff stress tensors $\boldsymbol{S}^{\mathrm{v}}$ have enough isotropy to ensure that their components with respect to the bases $\{\boldsymbol{a}_k\}$ and $\{\boldsymbol{b}_k\}$, respectively, are diagonal when $\boldsymbol{C}$ has the diagonal forms. Thus the first Piola-Kirchhoff stress tensor $\boldsymbol{T}$ is diagonal with respect to these bases. Consequently, the only nonzero components of $\boldsymbol{T}$ for radially symmetric motions a cylindrical shell are T_{11} and T_{22}, and the only nonzero components of $\boldsymbol{T}$ for radially symmetric motions of a spherical shell are T_{11} and $T_{22} = T_{33}$. We denote the constitutive equations for these stresses by $T_{11}(r,t) = T_{11}^{\mathrm{v}}(f_r(r,t), f(r,t)/r, f_{rt}(r,t), f_t(r,t)/r, r)$, etc. Thus the only nontrivial equations of motion are the scalar equations

$$\partial_r(rT_{11}^{\mathrm{vc}}) - T_{22}^{\mathrm{vc}} = \rho r f_{tt} \tag{42}$$

$$\partial_r(r^2 T_{11}^{\mathrm{vs}}) - 2rT_{22}^{\mathrm{vs}} = \rho r^2 f_{tt} \tag{43}$$

We assume that the inner surfaces $r = r_1$ of the shells are subjected to a hydrostatic pressure $\pi_1(t)$ taken positive when it is acting radially outward, and that the outer surfaces $r = r_2$ of the shells are subjected to a hydrostatic pressure $\pi_2(t)$ taken positive when it is acting radially inward. For the cylindrical shell, the dimensions of these pressures are those of force per actual area of the cylindrical segments of unit length and radii $f(r_1,t)$ and $f(r_2,t)$. For the cylinder, the mechanical boundary conditions are

$$r_j T_{11}^{\mathrm{c}}(r_j,t) = -\pi_j(t) f(r_j,t), \quad j = 1,2 \tag{44}$$

We assume that the cylinder is doubly infinite, or equivalently that the ends $z = z_1, z_2$ lie on fixed planes that are lubricated, so that they offer no resistance to radial motions. For the spherical shell, the dimensions of the pressures are those of force per actual area of the spheres of radii $f(r_1,t)$ and $f(r_2,t)$. Thus, its mechanical boundary conditions are

$${r_j}^2 T_{11}^{\mathrm{s}}(r_j,t) = -\pi_j(t) f(r_j,t)^2, \quad j = 1,2 \tag{45}$$

Clearly, the equations of motion for both cylindrical and spherical shells do not exhibit time-varying coefficients. The time-dependent pressures appear as direct excitation terms in the boundary conditions.

It is within the context of the Cosserat shell theory that the governing equations of motion exhibit parametric-type terms. We constrain f to have the form $f(r,t) = g(t) + \zeta(r)h(t), \quad g(t) := f(r_0,t), \quad \zeta(r) := r - r_0, \quad r_1 \le r_0 \le r_2$, with g and h satisfying $h > 0, \; g > (r_0 - r_1)h$, for the preservation of orientation of the reference configuration. The functions g and h denote the current radius of the reference circle of undeformed radius r_0 and the ratio of cross-sectional thickness to that in the reference configuration, respectively. Thus, $\boldsymbol{p}^{\mathrm{c}}$ and $\boldsymbol{p}^{\mathrm{s}}$ are constrained to have the forms

$$\boldsymbol{p}^{\mathrm{c}}(r,\phi,z,t) = [g(t) + \zeta(r)h(t)]\boldsymbol{a}_1(\phi) + z\boldsymbol{k} \tag{46}$$

$$\boldsymbol{p}^{\mathrm{s}}(r,\theta,\phi,t) = [g(t) + \zeta(r)h(t)]\boldsymbol{b}_1(\theta,\phi) \tag{47}$$

By applying a procedure that mimics the Bubnov-Galerkin method, we obtain the governing equations of motion. For cylindrical shells [Antman and Lacarbonara, 2008],

$$\begin{aligned} \rho A^c g_{tt} + \rho I^c h_{tt} + G^c(g,h,g_t,h_t) &= \alpha(t)g + \beta(t)h, \\ \rho I^c g_{tt} + \rho J^c h_{tt} + H^c(g,h,g_t,h_t) &= \beta(t)g + \gamma(t)h \end{aligned} \tag{48}$$

For spherical shells [Antman and Lacarbonara, 2008],

$$\begin{gathered} \rho A^s g_{tt} + \rho I^s h_{tt} + G^s(g,h,g_t,h_t) = \\ \alpha(t)g^2 + 2\beta(t)gh + \gamma h^2 \\ \rho I^s g_{tt} + \rho J^s h_{tt} + H^s(g,h,g_t,h_t) = \\ \beta(t)g^2 + 2\gamma(t)gh + \delta(t)h^2 \end{gathered} \tag{49}$$

In Eqs. (48)-(49), G^c, G^s and H^c, H^s denote the generalized resultant contact forces as defined in [Antman and Lacarbonara, 2008] and the pressure terms are expressed as

$$\begin{aligned}\alpha(t) &:= \pi_1(t) - \pi_2(t),\\ \beta(t) &:= (r_1 - r_0)\pi_1(t) - (r_2 - r_0)\pi_2(t),\\ \gamma(t) &:= (r_1 - r_0)^2\pi_1(t) - (r_2 - r_0)^2\pi_2(t),\\ \delta(t) &:= (r_1 - r_0)^3\pi_1(t) - (r_2 - r_0)^3\pi_2(t)\end{aligned} \tag{50}$$

For shells that are constrained to be transversally inextensible, the kinematic constraint $h = 1$ leads to the following two nonlinear versions of Hill's equations:

$$\begin{aligned}\rho A^c g_{tt} + G^c(g, g_t) &= \alpha(t)g + \beta(t)\\ \rho A^s g_{tt} + G^s(g, g_t) &= \alpha(t)g^2 + 2\beta(t)g + \gamma(t)\end{aligned} \tag{51}$$

By considering a pressure applied onto the outer surface only ($\pi_1 \equiv 0$) and the reference radius coinciding with the outer radius ($r_0 = r_2$), the pressure terms are simplified into $\alpha(t) = -\pi_2(t)$ and $\beta(t) = \gamma(t) = 0$. Moreover, letting $p(t) := \pi_2(t)$, the equations of motion are further transformed into a simpler version

$$\rho A g_{tt} + G(g, g_t) + p(t)g^\nu = 0 \tag{52}$$

where $\nu = 1$ for cylindrical shells and $\nu = 2$ for spherical shells, respectively. The obtained ordinary-differential equation is a remarkable nonlinear version of Hill's equation. However, mention must be made of the fact that, strictly speaking, the governing equation for spherical shells is a modified version of Hill's equation not only for the presence of the nonlinear term $G(g, g_t)$ but also for the nonlinear parametric term $p(t)g^2$.

6 Concluding remarks

The fundamental problem of how to recognize parametric excitation terms has been discussed within the context of a geometrically exact treatment of structural problems [Antman, 2005]. In particular, the problem is intriguing for those structural problems with unconstrained motions whose excitation terms, either pulsating boundary forces or displacements, appear as inhomogeneities in the boundary conditions. Yet, these excitations, under pertinent conditions, may cause parametric Hill-type instabilities. We have treated several paradigmatic problems, as cables subject to horizontal motions of the supports or simply supported rods subject to an end pulsating thrust or cantilvered rods subject to vertical base excitations and shells, both cylindrical and spherical, subject to pulsating pressures.

A general circumstance is that the parametrically excited motions are orthogonal to the excitations. These motions are excited by part of the excitation that couples with the structural motions in a way that can shift the eigenfrequency and can create resonance forces.

It is shown that when suitable coordinate transformations are introduced (such that the inhomogeneous boundary conditions are rendered homogeneous), the boundary forcing terms may become time-varying coefficients of the governing equations of motion. At the same time, when a constrained version of the motions is sought (e.g., via a Bubnov-Galerkin approach), the parametric nature of the excitation is similarly revealed explicitly and unambiguously.

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References

Faraday, M. (1831) On a peculiar class of acoustical figures; and on certain forms assumed by a group of particles upon vibrating elastic surfaces, *Philos. Tr. R. Soc. S-A* **121**, pp. 299–.

Melde, W. (1859) Über Erregung stehender Wellen eines fadenförmigen Körpers, *Ann. Phys. Chem.* **109**, pp. 193–.

Strutt, J. W. (1883) On maintained vibrations, *Phil. Mag.* **15**, pp. 229 .

Stephenson, A. (1908) On a new type of dynamic stability, *Mem. Proc. Manch. Lit. Phil. Soc.* **52**, pp. 1907–.

Belayev, N. M. (1924) Stability of prismatic rods, subject to variable longitudinal forces, Collection of Papers: *Eng. Construct. Struct. Mech.*, Put'. Leningrad, pp. 149–.

Einaudi, R. (1936) Sulle configurazioni di equilibrio instabile di una piastra sollecitata da sforzi tangenziali pulsanti, *Atti Accad. Gioenia Catania* **1** (serie 6), mem. XX.

Chelomei, V. N. (1939) *The dynamic stability of elements of aircraft structures*. Aeroflot, Moscow.

Bolotin, V. V. (1964) *The dynamic stability of elastic systems.* Holden-Day, San Francisco.

Yakubovich, V. A. and Starzhinskii, V. M. (1975) *Linear differential equations with periodic coefficients*, Vol. 2 Wiley & Sons, New York.

Nayfeh, A. H. and Mook, D. T. (1979) *Nonlinear Oscillations*. Wiley, New York.

Cartmell M. (1990) *Introduction to linear, parametric and nonlinear vibrations*, Chapman and Hall, London.

Seyranian, A. P. (2001) Regions of resonance for Hill's equation with damping, *Dokl. Ross. Akad. Nauk.* **376**, pp. 44–.

Yabuno, H., Okhuma, M., Lacarbonara, W. (2003) An experimental investigation of the parametric resonance in a buckled beam, Paper. No. VIB-48615, in: *Proc. of the 19th ASME Biennial Conf. on Mech. Vib. and Noise*, Chicago, Illinois, Sept. 2-6.

Zavodney, L. D. and Nayfeh, A. H. (1989) The nonlinear response of a slender beam carrying a lumped mass to a principal parametric excitation: theory and experiment, *Int. J. Non-Linear Mech.* **24**, pp. 105.

Lacarbonara, W. and Yabuno, H. (2007) Nonlinear cancellation of the parametric resonance in elastic beams: theory and experiment, *Int. J. Solids Struct.* **44**, pp. 2209–.

Irvine, H. M. and Caughey, T. K. (1974) The linear theory of free vibrations of a suspended cable, *Proc. of the Royal Soc. London, Series A* **341**, pp. 299-315.

Mettler, E. (1962) *Dynamic buckling*. In *Handbook of Engineering Mechanics* (ed. Flugge). McGraw-Hill, New York.

Antman, S. S. and Lacarbonara, W. (2008) Forced radial motions of nonlinearly viscoelastic cylindrical and spherical shells, to be submitted.

Antman, S. .S. (2005) *Nonlinear problems of elasticity*, 2nd ed. Springer-Verlag, New York.