GRADIENT-LIKE PROPERTIES OF DISTRIBUTED AND DISCRETE PHASE SYSTEMS.

Aleksey A. Perkin
Dept. of Mathematics
St. Petersburg State University of
Architecture and Civil Engineering
Russia
ofercinn@gmail.com

Vera B. Smirnova
Dept. of Mathematics
St. Petersburg State University of
Architecture and Civil Engineering
Russia
root@al2189.spb.edu

Alexander I. Shepeljavyi
Dept. of Mathematics and Mechanics
St.Petersburg State University
Russia
as@as1020.spb.edu

Abstract
Global asymptotic behavior of control systems with periodic vector nonlinearities and denumerable sets of equilibria is investigated. Distributed systems described by integrodifferential Volterra equations and discrete systems described by difference equations are examined. New kinds of Popov-type functionals and Lyapunov-type sequences are offered. New frequency-domain criteria for gradient-like behavior of the systems are obtained.

Key words
Distributed phase system, discrete phase system, gradient-like behavior, Lyapunov-type sequence, Popov-type functional, frequency-domain criterion.

1 Introduction

Nonlinear systems with non-unique equilibria are widespread among control systems, mechanical systems, electrical and radio-engineering systems. The qualitative analysis of various systems with non-unique equilibria generated a number of new stability problems and new Lyapunov-type theorems.

This paper is devoted to systems with denumerable equilibria set and periodic nonlinear functions. They are often called phase systems.

The stability of multidimensional phase systems was for the first time investigated in [Gelig, Leonov and Yakubovich, 1978], where two types of stability characteristic of phase systems are considered. They are Lagrange stability and gradient-like behavior, which means that every solution of the system tends to a certain equilibrium state as the argument-time goes to infinity. In [Gelig, Leonov and Yakubovich, 1978] new classes of Lyapunov functions specially constructed for phase systems were introduced. They gave the opportunity to establish a number of sufficient conditions for Lagrange stability and gradient-like behavior of the systems. These conditions have often the form of frequency-domain inequalities with varying parameters.

By means of special Lyapunov-type sequences all the stability theorems for autonomous systems were extended to discrete phase systems [Leonov, Smirnova, 2000]. With the help of the method of a priori integral estimates and the Popov functionals they were extended to infinite dimensional phase systems [Leonov, Ponomarenko and Smirnova, 1996].

Lyapunov-type sequences and Popov-type functionals destined for discrete and distributed phase systems are generated by the same technique as Lyapunov functions for lumped systems. In particular, for the purpose the technique of periodic Lyapunov functions [Bakaev, Guzh, 1965], [Gelig, Leonov and Yakubovich, 1978], [Brockett, 1982], [Perkin, Smirnova and Shepeljavyi, 2009], [Duan, Wang and Huang, 2007] is used.

In this paper the idea of generalized periodic Lyapunov functions from the paper [Perkin, Smirnova and Shepeljavyi, 2009] is used to construct Popov-type functionals for phase systems described by Volterra integrodifferntial equations. For this type of systems a multiparametric frequency-domain criterion for gradient-like behavior from [Perkin, Smirnova and Shepeljavyi, 2009] is extended. As a result a generalization of frequency-domain criterion of [Leonov, Ponomarenko and Smirnova, 1996] is obtained. For distributed systems a certain modification of generalized criterion is also proved.

This modification is spread to discrete phase systems. The modification includes less number of variable parameters than the basic criterion. It is easier to apply it to concrete systems.

New frequency-domain criterion is applied in the paper to radio-engineering and mechanical systems. The stability regions obtained by new criterion are compared with results of other investigations.

2 Asymptotic behavior of distributed systems with phase control

Let us consider a control system which is described by a system of Volterra integrodifferential equations

$$\dot{\sigma}(t) = a(t) + Rf(\sigma(t-h)) - \int_0^t \gamma(t-\tau) f(\sigma(\tau)) d\tau. \tag{1}$$

Here $t \geq 0, h \geq 0; \sigma(t) = \|\sigma_j(t)\|_{j=1,\dots,l}$, $a(t) = \|a_j(t)\|_{j=1,\dots,l}$, $f(\sigma) = \|\varphi_j(\sigma_j)\|_{j=1,\dots,l}$ are vector-functions, R is a matrix and $\|\gamma_{kj}(t)\|_{k,j=1,\dots,l}$ is a matrix function. For the system (1) the initial condition

$$\sigma(t)|_{t\in[-h,0]} = \sigma^0(t). \tag{2}$$

is given.

We suppose that the following requirements are satisfied:

1. $a_j(t) \in C[0,+\infty) \bigcap L_1[0,+\infty)$, $a_j(t) \to 0$ as $t \to +\infty$ $(j = 1,\dots,l)$;
2. functions γ_{jk} are measurable and $e^{ct}\gamma_{jk}(t) \in L_2[0,+\infty)$ $\quad (k,j = 1,2,\dots,l)$ for a certain $c > 0$;
3. $\sigma^0(t) \in C^1[-h,0]$;
4. function $\varphi_j(\sigma_j)$ is a continually differentiable, Δ_j-periodic function and

$$\int_0^{\Delta_j} \varphi_j(\sigma) d\sigma < 0 \qquad (j = 1,2,\dots,l); \tag{3}$$

5. function $\varphi_j(\sigma_j)$ has two zeros $\sigma_{1j} < \sigma_{2j}$ on interval $[0,\Delta_j)$ and

$$\varphi_j^2(\sigma_{kj}) + (\varphi_j'(\sigma_{kj}))^2 \neq 0 \qquad (k = 1,2; j = 1,2,\dots,l); \tag{4}$$

6.

$$\int_0^\infty \gamma(t) d\sigma \neq R. \tag{5}$$

System (1) is a phase system. It has a denumerable set of equilibria. The basic characteristic of the linear part of system (1) is the transfer matrix

$$K(p) = -Re^{-ph} + \int_0^\infty \gamma(t) e^{-pt} dt \qquad (p \in \mathbf{C}). \tag{6}$$

Let us introduce diagonal matrices $A_1 = diag\{\alpha_{11},\dots,\alpha_{1l}\}$ and $A_2 = diag\{\alpha_{21},\dots,\alpha_{2l}\}$ where numbers α_{1j}, α_{2j} satisfy the inequalities

$$\alpha_{1j} \leq \frac{d\varphi_j(\sigma)}{d\sigma} \leq \alpha_{2j} \qquad (j = 1,2,\dots,l). \tag{7}$$

Note that $\alpha_{1j}\alpha_{2j} < 0$. Let us also introduce numbers

$$\nu_j = \frac{\int_0^{\Delta_j} \varphi_j(\sigma) d\sigma}{\int_0^{\Delta_j} |\varphi_j(\sigma)| d\sigma} \qquad (j = 1,\dots,l), \tag{8}$$

$$\nu_{0j} = \frac{\int_0^{\Delta_j} \varphi_j(\sigma) d\sigma}{\int_0^{\Delta_j} |\varphi_j(\sigma)| \sqrt{\left(1-\alpha_{1j}^{-1}\varphi_j'(\sigma)\right)\left(1-\alpha_{2j}^{-1}\varphi_j'(\sigma)\right)} d\sigma} \tag{9}$$

$$(j = 1,\dots,l).$$

We shall designate by symbol $(*)$ the Hermite conjugation. We shall also use for $n \times n$ - matrix H the denotation $\Re e H = 1/2(H + H^*)$.

Theorem 1. Suppose there exist such positive definite diagonal matrices $æ = diag\{æ_1,\dots,æ_l\}$, $\delta = diag\{\delta_1,\dots,\delta_l\}$, $\varepsilon = diag\{\varepsilon_1,\dots,\varepsilon_l\}$ and such numbers $a_k \in [0,1]$ $(k = 1,\dots,l)$, that the following conditions are satisfied:

1) for all $\omega \in \mathbf{R}$ the inequality

$$\Re e \left\{ æK(i\omega) - K^*(i\omega)\varepsilon K(i\omega) - \left(K(i\omega) + A_1^{-1} i\omega\right)^* \tau \cdot \left(K(i\omega) + A_2^{-1} i\omega\right) \right\} - \delta > 0 \qquad (i^2 = -1) \tag{10}$$

is true;

2) matrices

$$\left\| \begin{array}{ccc} \varepsilon_k & \frac{æ_k a_k \nu_k}{2} & 0 \\ \frac{æ_k a_k \nu_k}{2} & \delta_k & \frac{æ_k a_{0k} \nu_{0k}}{2} \\ 0 & \frac{æ_k a_{0k} \nu_k}{2} & \tau_k \end{array} \right\|$$

where $a_{0k} = 1 - a_k$, are positive definite.

Then

$$\dot{\sigma}_k \to 0, \quad \sigma_k \to c_k \text{ as } t \to +\infty, \tag{11}$$

where $\varphi_k(c_k) = 0 \quad (k = 1,\dots,l)$.

Proof. Let $\sigma(t)$ be an arbitrary solution of (1) and T be a positive number. Let us introduce the following functions

$$\mu(t)=\left\{\begin{array}{l} 0 \text{ for } t<0 \\ t \text{ for } t\in[0,1] \\ 1 \text{ for } t>1 \end{array}\right\}, \tag{12}$$

$$\eta(t)=f(\sigma(t)), \tag{13}$$

$$\xi_T(t)=\left\{\begin{array}{ll} \eta(t) & t\le T \\ \eta(T)e^{\lambda(T-t)} & t>T>1\,(\lambda>0) \end{array}\right\}, \tag{14}$$

$$\eta_T(t)=\mu(t)\xi_T(t), \tag{15}$$

$$\sigma_T(t)=R\eta_T(t-h)-\int\limits_0^t\gamma(t-\tau)\eta_T(\tau)d\tau, \tag{16}$$

$$\begin{array}{l}\sigma_0(t)=a(t)+(1-\mu(t-h))R\xi_T(t-h) \\ -\int\limits_0^t(1-\mu(\tau))\gamma(t-\tau)\xi_T(\tau)d\tau.\end{array} \tag{17}$$

For $t\in[0,T]$ we have

$$\dot\sigma(t)=\sigma_0(t)+\sigma_T(t) \tag{18}$$

Let $\eta_T(t) = \|\eta_{Tj}\|_{j=1,\ldots,l}$, $\eta(t) = \|\eta_j\|_{j=1,\ldots,l}$ $\sigma_T(t)=\|\sigma_{Tj}\|_{j=1,\ldots,l}$. It follows from the properties of $\eta_T(t)$, $\gamma_{ij}(t)$ that

$$\sigma_{Tj},\eta_{Tj},\dot\eta_{Tj}\in L_2[0,+\infty)\quad(j=1,\ldots,l) \tag{19}$$

for each $T>0$.

Let us consider a one-parameter set of functionals

$$\begin{array}{l}\rho_T=\int\limits_0^\infty\{\sigma_T^*(t)æ\eta_T(t)+\eta_T^*(t)\delta\eta_T(t)+\sigma_T^*(t)\varepsilon\sigma_T(t)+ \\ +\left(\sigma_T(t)-A_1^{-1}\dot\eta_T(t)\right)^*\tau\left(\sigma_T(t)-A_2^{-1}\dot\eta_T(t)\right)\Big\}\,dt.\end{array} \tag{20}$$

By Parseval equation we have

$$\begin{array}{l}\rho_T=\dfrac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\{\tilde\sigma_T^*(i\omega)æ\tilde\eta_T(i\omega)+\tilde\eta_T^*(i\omega)\delta\tilde\eta_T(i\omega)+ \\ \tilde\sigma_T^*(i\omega)\varepsilon\tilde\sigma_T(i\omega)+\left(\tilde\sigma_T(i\omega)-A_1^{-1}\tilde{\dot\eta}_T(i\omega)\right)^* \\ \tau\left(\tilde\sigma_T(i\omega)-A_2^{-1}\tilde{\dot\eta}_T(i\omega)\right)\}\,d\omega,\end{array} \tag{21}$$

where by $\tilde\sigma_T(i\omega),\tilde\eta_T(i\omega),\tilde{\dot\eta}_T(i\omega)$ the Fourier transforms of $\sigma_T(t),\eta_T(t),\dot\eta_T(t)$ respectively are denoted. By means of equalities

$$\begin{array}{l}\tilde\sigma_T(i\omega)=-K(i\omega)\tilde\eta_T(i\omega), \\ \tilde{\dot\eta}_T(i\omega)=i\omega\tilde\eta_T(i\omega)\end{array} \tag{22}$$

we obtain that

$$\begin{array}{l}\rho_T=-\dfrac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\tilde\eta^*(i\omega)\Re e\,\{æK(i\omega)-\delta-K^*(i\omega)\cdot \\ \cdot\varepsilon K(i\omega)-(K(i\omega)+i\omega A_1^{-1})^*\tau(K(i\omega)+i\omega A_2^{-1})\}\cdot \\ \cdot|\tilde\eta(i\omega)|^2d\omega\end{array} \tag{23}$$

From the condition 1) of the theorem it follows that

$$\rho_T<0. \tag{24}$$

Let us represent the functional ρ_T as follows:

$$\rho_T=I_T+\rho_1+\rho_{2T}+\rho_{3T}+\rho_{4T}, \tag{25}$$

where

$$\begin{array}{l}I_T=\int\limits_0^T\{\dot\sigma^*æ\eta+\eta^*\delta\eta+\dot\sigma^*\varepsilon\dot\sigma^*+ \\ \quad+\left(\dot\sigma-A_1^{-1}\dot\eta\right)^*\tau\left(\dot\sigma-A_2^{-1}\dot\eta\right)\Big\}\,dt,\end{array} \tag{26}$$

$$\begin{array}{l}\rho_1=\int\limits_0^1\left\{(1-t)\dot\sigma^*æ\eta+(1-t^2)\eta^*\delta\eta+\right. \\ \quad+\dot\eta^*A_1^{-1}\tau A_2^{-1}\dot\eta-(\dot{\hat{t\eta}})^*A_1^{-1}\tau A_2^{-1}\dot{\hat{t\eta}} \\ \quad+\dot\sigma^*\left(A_1^{-1}+A_2^{-1}\right)\tau(\dot\eta_T-\dot{\hat{t\eta}})\Big\}\,dt,\end{array} \tag{27}$$

$$\begin{array}{l}\rho_{2T}=\int\limits_0^T(-\sigma_0^*æ\eta_T-2\dot\sigma^*(\varepsilon+\tau)\sigma_0+ \\ \quad\sigma_0^*(\varepsilon+\tau)\sigma_0-\sigma_0^*\left(A_1^{-1}+A_2^{-1}\right)\tau\dot\eta_T)\,dt,\end{array} \tag{28}$$

$$\rho_{3T}=\int\limits_T^\infty\sigma_T^*(t)(\varepsilon+\tau)\sigma_T(t)dt, \tag{29}$$

$$\begin{array}{l}\rho_{4T}=\int\limits_T^\infty\left(-\dot\eta_T^*A_1^{-1}\tau\sigma_T+\sigma_T^*æ\eta_T+\eta_T^*\delta\eta_T-\right. \\ \quad\left.-\sigma_T^*\tau A_2^{-1}\dot\eta_T+\dot\eta_T^*A_1^{-1}\tau A_2^{-1}\dot\eta_T\right)dt.\end{array} \tag{30}$$

It follows from the properties of $a_j(t)$, $\gamma_{kj}(t)$, $\eta_{Tj}(t)$ $(k,j=1,\ldots,l)$ that

$$\rho_{kT}<C_k\quad(k=2,4), \tag{31}$$

where C_k does not depend on T. From (24), (25), (31) with regard to the positiveness of ρ_{3T} it arises that

$$I_T < C_5, \tag{32}$$

where C_5 does not depend on T.

Let us introduce functions

$$F_i(\sigma) = \varphi_i(\sigma) - \nu_i \left|\varphi_i(\sigma)\right|, \tag{33}$$

$$\Phi_i(\sigma) = \sqrt{\left(1 - \alpha_{1i}^{-1}\varphi_i'(\sigma)\right)\left(1 - \alpha_{2i}^{-1}\varphi_i'(\sigma)\right)}, \tag{34}$$

$$\Psi_i(\sigma) = \varphi_i(\sigma) - \nu_{0i}\Phi_i(\sigma)\left|\varphi_i(\sigma)\right| \quad (i = 1, \dots, l). \tag{35}$$

It is clear that

$$\int_0^{\Delta_i} F_i(\sigma)d\sigma = 0, \ \int_0^{\Delta_i} \Psi_i(\sigma)d\sigma = 0 \quad (i = 1, \dots, l). \tag{36}$$

The functional I_T can be represented in the following way

$$I_T = \sum_{j=1}^{l}\int_0^T \left\{æ_j\varphi_j(\sigma_j(t))\dot\sigma_j(t) + \delta_j\varphi_j^2(\sigma_j(t)) + \right. \\ \left. + \varepsilon\dot\sigma_j^2(t) + \tau_j\Phi_j^2(\sigma_j(t))\dot\sigma_j^2(t)\right\} dt. \tag{37}$$

From the definition of F_i and Ψ_i it follows that

$$I_T = \sum_{j=1}^{l}\int_0^T \{æ_j a_j \nu_j|\varphi_j(\sigma)|\dot\sigma_j(t) + æ_j a_{0j}\nu_{0j}|\varphi_j(\sigma)|\cdot \\ \cdot\Phi_j(\sigma_j)\dot\sigma_j(t) + \varepsilon\dot\sigma_j^2(t) + \delta_j\varphi_j^2(\sigma_j(t)) + \tau_j\Phi_j^2(\sigma_j(t))\cdot \\ \left.\cdot\dot\sigma_j^2(t)\right\} dt + \sum_{j=1}^{l}\left[\int_0^T æ_j a_j F_j(\sigma_j(t))\dot\sigma_j(t)dt + \right. \\ \left.\int_0^T æ_j a_{0j}\nu_{0j}\Psi_{0j}(\sigma_j(t))\dot\sigma_j(t)dt\right]. \tag{38}$$

It follows from (36) that all integrals $\int_0^T F_j(\sigma_j(t))\dot\sigma_j(t)dt$ and $\int_0^T \Psi_j(\sigma_j(t))\dot\sigma_j(t)dt$ are bounded by constants which do not depend on T.

This assumption together with (32) implies that

$$\sum_{j=1}^{l}\int_0^T \{æ_j a_j\nu_j|\varphi_j(\sigma)|\dot\sigma_j(t) + æ_j a_{0j}\nu_{0j}|\varphi_j(\sigma_j)|\cdot \\ \cdot\Phi_j(\sigma_j)\dot\sigma_j(t) + \varepsilon\dot\sigma_j^2(t) + \delta_j\varphi_j^2(\sigma_j(t)) + \\ \left. + \tau_j\Phi_j^2(\sigma_j(t))\dot\sigma_j^2(t)\right\} dt < C_6, \tag{39}$$

where C_6 does not depend on T. By virtue of the condition 2 of the theorem every sum which stands under the integral sign in the left part of (39) is a positive definite quadratic form of $\dot\sigma_j$, $|\varphi_j(\sigma_j)|$, $\Phi_j(\sigma_j)\dot\sigma_j$. Then it follows from (39) that

$$\int_0^{+\infty} \varphi_j^2(\sigma_j(t))dt < +\infty, \tag{40}$$

$$\int_0^{+\infty} \dot\sigma_j^2(t)dt < +\infty. \tag{41}$$

It is proved in [Leonov, Ponomarenko and Smirnova, 1996] that (40) and (41) imply the validity of (11).

Theorem 2. Suppose there exist such positive diagonal matrices $æ = diag\{æ_1, \dots, æ_l\}$, $\delta = diag\{\delta_1, \dots, \delta_l\}$, $\varepsilon = diag\{\varepsilon_1, \dots, \varepsilon_l\}$, $\tau = diag\{\tau_1, \dots, \tau_l\}$, that for all $\omega \geq 0$ the frequency-domain inequality (10) is fulfilled. Suppose also that for varying parameters ε_j, δ_j, $æ_j$ the inequalities

$$2\sqrt{\varepsilon_j\delta_j} > |\nu_{2j}|æ_j \quad (j = 1, \dots, l), \tag{42}$$

where

$$\nu_{2j} = \frac{\int_0^{\Delta}\varphi_j(\sigma)d\sigma}{\int_0^{\Delta}|\varphi_j(\sigma)|\sqrt{1 + \frac{\tau_j}{\varepsilon_j}(1 - \alpha_{1j}^{-1}\varphi_j'(\sigma))(1 - \alpha_{2j}^{-1}\varphi_j'(\sigma))}d\sigma} \tag{43}$$

are valid. Then the conclusion of Theorem 1 is true.

Proof. Let us repeat the first part of the proof of theorem 1 and prove that the inequality (32), where C_5 does not depend on T, is true.

Let us now use the functions Φ_j, introduced in text of the proof of theorem 1 and introduce the functions

$$P_j(\sigma) = \sqrt{1 + \frac{\tau_j}{\varepsilon_j}\Phi_j^2(\sigma)}; \tag{44}$$

$$Y_j(\sigma) = \varphi_j(\sigma) - \nu_{2j}|\varphi_j(\sigma)|P_j(\sigma). \tag{45}$$

Note that the parameters ν_{2j} can be rewritten in the form

$$\nu_{2j} = \frac{\int_0^{\Delta}\varphi_j(\sigma)d\sigma}{\int_0^{\Delta_j}|\varphi_j(\sigma)|P_j(\sigma)d\sigma} \tag{46}$$

Note also that

$$\int_0^{\Delta_j} Y_j(\sigma)d\sigma = 0. \qquad (47)$$

Let us consider the function which stands under the integral sign in the functional I_T and transform it. Note that

$$\begin{aligned}&æ_j\dot\sigma_j\eta_j + \delta_j\eta_j^2 + \tau_j(\dot\sigma_j - \alpha_{1j}^{-1}\dot\eta_j)(\dot\sigma_j - \alpha_{2j}^{-1}\dot\eta_j) + \varepsilon_j\dot\sigma_j^2 = \\ &= æ_j\dot\sigma_j(Y_j(\sigma_j) + \nu_{2j}|\eta_j|P_j(\sigma_j)) + \\ &+\delta_j\eta_j^2 + \tau_j\Phi_j^2(\sigma_j)\dot\sigma_j^2 + \varepsilon_j\dot\sigma_j^2 = æ_j\dot\sigma_jY_j(\sigma_j) + \\ &+ \left(\delta_j\eta_j^2 + æ_j\nu_{2j}|\eta_j|\dot\sigma_jP_j(\sigma_j) + \varepsilon_j\dot\sigma_j^2P_j^2(\sigma_j)\right).\end{aligned} \qquad (48)$$

So

$$\begin{aligned}&I_T = \sum_{j=1}^{l}(æ_j\int_0^T \dot\sigma_j(t)Y_j(\sigma_j(t))dt + \\ &+\int_0^T\left(\delta_j\eta_j^2(t) + \nu_{2j}æ_j|\eta_j(t)|\dot\sigma_j(t)P_j(\sigma_j(t)) + \right.\\ &\left.+\varepsilon_j\dot\sigma_j^2P_j^2(\sigma(t))\right)dt).\end{aligned} \qquad (49)$$

By virtue of (47) we affirm that

$$\int_0^T \dot\sigma_j(t)Y_j(\sigma_j(t))dt = \int_{\sigma_j(0)}^{\sigma_j(T)} Y_j(\sigma_j)d\sigma_j \le C_{10}, \qquad (50)$$

where C_{10} does not depend on T. On the other hand by virtue of (42) the quadratic forms

$$\delta_j\eta_j^2(t) + \nu_{2j}æ_j|\eta_j(t)|\dot\sigma_j(t)P_j(\sigma_j(t)) + \varepsilon_j\dot\sigma_j^2P_j^2(\sigma_j(t))) \qquad (51)$$

are positive definite.

So it follows from (32) and (50) that

$$\int_0^T \eta_j^2(t)dt \le C_{10}, \quad \int_0^T \dot\sigma_j^2(t)dt \le C_{11}, \qquad (52)$$

where C_{10} and C_{11} do not depend on T. Now we can use the concluding part of the proof of theorem 1.

3 Gradient-like behavior of radio-engineering and mechanical systems

1) Theorem 1 was applied to stability investigation of a second order phase-locked loop with proportional-integrating filter and time delay in the loop. In this case $m = l = 1$ and the transfer function has the form

$$K(p) = T\frac{1+\beta Tp}{1+Tp}e^{-phT} \qquad (T > 0, h > 0, \beta \in (0,1)) \qquad (53)$$

For $\varphi(\sigma) = \sin\sigma - \gamma$ ($\gamma \in (0,1)$); $\beta = 0,2$; $h = 0,01; 0,1; 1$ the estimates for the boundaries of lock-in ranges on the plane $\{T^2, \gamma\}$ were obtained. These estimates were compared with the lock-in ranges obtained in [Belyustina, Kinyapina and Fishman, 1990] by qualitative-numerical methods. It turned out that the ranges received by means of theorem 1 have the same structure as those in [Belyustina, Kinyapina and Fishman, 1990]. For $T^2 < h^{-1}$ the ranges obtained by theorem 1 are 15-25% smaller than the ranges received in [Belyustina, Kinyapina and Fishman, 1990].

2) Theorem 1 was also applied to the problem of self-synchronization of two rotors on a vibrator with one degree of freedom. The equations describing the change of the slowly variable components $\Theta_s(t)$ $(s = 1,2)$ of the phase of the rotor motion are

$$\left\{\begin{array}{l} I_1\ddot\Theta_1 + K_1\dot\Theta_1 + A\sin(\Theta_1 - \Theta_2) - \beta = 0, \\ I_2\ddot\Theta_2 + K_2\dot\Theta_2 - A\sin(\Theta_1 - \Theta_2) + \beta = 0 \end{array}\right\} \qquad (54)$$

where I_1, I_2, K_1, K_2, β are positive parameters. The self-synchronization of the rotors means that the difference $\sigma = \Theta_1 - \Theta_2$ tends to a zero of $\varphi(\sigma) = \sin\sigma - \beta/A$ as $t \to +\infty$. The system (54) can be reduced to (1) with $m = l = 1, R = 0$ and the transfer function

$$K(p) = A\left(\frac{1}{I_1p + K_1} + \frac{1}{I_2p + K_2}\right). \qquad (55)$$

In monograph [Leonov, Smirnova, 2000] various requirements on the coefficients of (54) are given which guarantee that the relations (11) are true. These requirements are such that the conditions of theorem 1 are satisfied in case $a_1 = 1 (a_{01} = 0)$. Varying the parameter a_1 in theorem 1 we can weaken these requirements. Let us introduce the parameter

$$y = \frac{K_1K_2(K_1I_2 + K_2I_1)}{AI_1I_2(K_1 + K_2)}.$$

Suppose that

$$A > \sqrt{\frac{K_1K_2\left(K_1^2I_2^2 + K_2^2I_1^2\right)}{2\left(K_1+K_2\right)\left(K_1^2I_2^2 + K_2^2I_1^2\right)}}\cdot\frac{I_1K_2 + I_2K_1}{I_1\cdot I_2}. \qquad (56)$$

In this case for $A = 2\beta$ theorem 1 guarantees (11) if $y > 0,97$ and theorems of [Leonov, Smirnova, 2000] give $y > 1.13$

4 Discrete systems

Consider a discrete phase system

$$\begin{aligned}&z(n+1) = Az(n) + Bf(\sigma(n)), \\ &\sigma(n+1) = \sigma(n) + C^*z(n) + Rf(\sigma(n)) \\ &(n = 0,1,2,\ldots),\end{aligned} \qquad (57)$$

where A, B, C, R are described in section 2. We suppose that the pair (A, B) is controllable, the pair (A, C) is observable and all eigenvalues of matrix A are situated inside the open unit circle. All the properties of $f(\sigma)$ are just the same as in section 2. The transfer matrix $K(p)$ for the linear part of system (57) has the form

$$K(p) = -R + C^*(A - pE_m)^{-1}B \quad (p \in \mathbf{C}), \quad (58)$$

where E_m is a unite $m \times m$ - matrix.

We shall present in this section certain analogues of theorems 2 and 1. We shall need numbers $k_{1j} = 2\alpha_{1j} - \alpha_{2j}$ and $k_{2j} = 2\alpha_{2j} - \alpha_{1j}$ and diagonal matrices $K_1 = diag(k_{11}, \ldots, k_{1l})$ and $K_2 = diag(k_{21}, \ldots, k_{2l})$.

Theorem 3. Suppose there exist such positive definite diagonal matrices $\varepsilon = diag\{\varepsilon_1, \ldots, \varepsilon_l\}$, $\tau = diag\{\tau_1, \ldots, \tau_l\}$, $\delta = diag\{\delta_1, \ldots, \delta_l\}$, a diagonal matrix æ $= diag\{æ_1, \ldots, æ_l\}$ that the following requirements are fulfilled:

1) for all $p \in \mathbf{C}$, $|p| = 1$ the inequality

$$\begin{aligned}&\Re e\left\{ æK(p) - \left(K(p) + (p-1)K_1^{-1}\right)^* \right.\\ &\left.\tau\left(K(p) + (p-1)K_2^{-1}\right)\right\} - K^*(p)\varepsilon K(p) - \delta \geq 0\end{aligned} \quad (59)$$

is valid;

2) the inequalities

$$(1-\frac{\alpha_{1k}\alpha_{2k}}{k_{1k}k_{2k}})\varepsilon_k > \frac{æ_k\alpha_{0k}}{2}(1-\nu_{2k}\sqrt{1+\frac{\tau_k}{\varepsilon_k}\frac{(\alpha_{2k}-\alpha_{1k})^2}{|\alpha_{1k}|\alpha_{2k}}}), \quad (60)$$

where $\alpha_{0k} = \alpha_{2k}$ if $æ_k > 0$, and $\alpha_{0k} = \alpha_{1k}$ if $æ_k < 0$, and

$$2\sqrt{\varepsilon_k\delta_k\frac{\alpha_{2k}\alpha_{1k}}{k_{1k}k_{2k}}} > |\nu_{2k}æ_k| (k = 1, 2, \ldots, l) \quad (61)$$

are true.

Then

$$\begin{aligned}&\lim_{n\to\infty} f(\sigma(n)) = 0,\\ &\lim_{n\to\infty} z(n) = 0,\\ &\lim_{n\to\infty} (\sigma(n+1) - \sigma(n)) = 0,\\ &\lim_{n\to\infty} (\sigma(n)) = c,\end{aligned}$$

where $f(c) = 0$.

Proof. The proof is based on the proof of theorem 5.4.1 from [Leonov, Smirnova, 2000]. Its first step is the extension of the state space of the system. For the purpose we introduce the notations

$$y = \left\| \begin{matrix} z \\ f(\sigma) \end{matrix} \right\|, \qquad P = \left\| \begin{matrix} A & B \\ O & E_l \end{matrix} \right\|,$$

$$L = \left\| \begin{matrix} O \\ E_l \end{matrix} \right\|, \qquad D^* = \| C^*, R \|,$$

$$\xi_1(n) = f(\sigma(n+1)) - f(\sigma(n)).$$

Then system (57) can be represented as

$$\begin{aligned}&y(n+1) = Py(n) + L\xi_1(n),\\ &\sigma(n+1) = \sigma(n) + D^*y(n),\\ &(n = 0, 1, 2, \ldots).\end{aligned} \quad (62)$$

The second step is to determine the quadratic form of $y \in \mathbf{R^{m+l}}$ and $\xi_1 \in \mathbf{R^l}$

$$\begin{aligned}&M(y, \xi_1) = (Py + L\xi_1)^*H(Py + L\xi_1) -\\ &y^*Hy + y^*LæDy + y^*D\varepsilon D^*y + y^*L\delta L^*y -\\ &(D^*y - K_1^{-1}\xi_1)^*\tau(K_2^{-1}\xi_1 - D^*y),\end{aligned} \quad (63)$$

where H is a symmetric $(m+l) \times (m+l)$ matrix and ε, æ, δ, τ are diagonal matrices from the text of theorem 3.

It follows from [Leonov, Smirnova, 2000] that if the condition 1) of theorem 3 is true then there exists matrix $H = H^*$ such that for all $y \in R^{m+l}$ and $\xi_1 \in R^l$

$$M(y, \xi_1) \leq 0. \quad (64)$$

Since all the eigenvalues of matrix A are situated inside the unit circle and function $f(\sigma)$ is bounded we can affirm that sequence $W(n) = y^*(n)Hy(n)$, where $y(n)$ satisfies (62) is bounded as well.

Let us use functions $Y_j(\sigma)$ which were introduced in the proof of theorem 2 and define a Lyapunov-type sequence

$$V(n) = W(n) + \sum_{k=1}^{l} æ_k \int\limits_{\sigma_k(0)}^{\sigma_k(n)} Y_k(\sigma)d\sigma. \quad (65)$$

Let us consider the difference

$$\begin{aligned}&V(n+1) - V(n) = W(n+1) - W(n) +\\ &\sum_{k=1}^{l} æ_k \int\limits_{\sigma_k(n)}^{\sigma_k(n+1)} Y_k(\sigma)d\sigma.\end{aligned} \quad (66)$$

It follows from (64) that

$$\begin{array}{l} W(n+1)-W(n)\leq \\ \sum\limits_{k=1}^{l}\{-æ_k\varphi_k(\sigma_k(n))(\sigma_k(n+1)-\sigma_k(n))- \\ \varepsilon_k(\sigma_k(n+1)-\sigma_k(n))^2-\delta_k\varphi_k^2(\sigma_k(n))- \\ \tau_k[k_{1k}^{-1}(\varphi_k(\sigma_k(n+1))-\varphi_k(\sigma_k(n)))- \\ (\sigma_k(n+1)-\sigma_k(n))][k_{2k}^{-1}(\varphi_k(\sigma_k(n+1))- \\ \varphi_k(\sigma_k(n)))-(\sigma_k(n+1)-\sigma_k(n))]. \end{array} \quad (67)$$

On the other hand we can establish the estimate [Leonov, Smirnova, 2000]

$$\begin{array}{l} æ_k\int\limits_{\sigma_k(n)}^{\sigma_k(n+1)}Y_k(\sigma)d\sigma\leq æ_k(\varphi_k(\sigma_k(n))+ \\ \Theta_k|\varphi_k(\sigma_k(n))|)(\sigma_k(n+1)- \\ \sigma_k(n))+æ_k\dfrac{\alpha_{0k}}{2}(1+\Theta_k)(\sigma_k(n+1)-\sigma_k(n))^2 \end{array} \quad (68)$$

where

$$\Theta_k=|\nu_{2k}P_k(\sigma'_{kn})| \quad (69)$$

and

$$\sigma_k(n)\lessgtr\sigma'_{kn}\lessgtr\sigma_k(n+1). \quad (70)$$

Note that

$$\Phi_k(\sigma)<\frac{\alpha_{2k}-\alpha_{1k}}{\sqrt{|\alpha_{1k}|\alpha_{2k}}}. \quad (71)$$

Hence

$$P_k(\sigma'_{kn})<\sqrt{1+\frac{\tau_j(\alpha_{2k}-\alpha_{1k})^2}{\varepsilon_j\alpha_{2k}|\alpha_{1k}|}} \quad (72)$$

It is established in [Smirnova, Shepeljavyi, 2007] that

$$\begin{array}{l} \left[k_{2k}^{-1}(\varphi_k(\sigma_k(n+1))-\varphi_k(\sigma_k(n)))-\right. \\ \left.-(\sigma_k(n+1)-\sigma_k(n))\right] \\ \left[k_{1k}^{-1}(\varphi_k(\sigma_k(n+1))-\varphi_k(\sigma_k(n)))-\right. \\ \left.-(\sigma_k(n+1)-\sigma_k(n))\right] \\ \geq\dfrac{\alpha_{2k}\alpha_{1k}}{k_{1k}k_{2k}}\Phi_k^2(\sigma'_{kn})(\sigma_k(n+1)-\sigma_k(n))^2= \\ =\frac{\alpha_{2k}\alpha_{1k}}{k_{1k}k_{2k}}(P^2(\sigma'_{kn})-1)\frac{\varepsilon_k}{\tau_k}(\sigma_k(n+1)-\sigma_k(n))^2. \end{array} \quad (73)$$

Formulae (66)-(73) imply that

$$V(n+1)-V(n)\leq\sum_{k=1}^{l}Z_k(n), \quad (74)$$

where

$$\begin{array}{l} Z_k(n)=-\left(\varepsilon_k-\dfrac{æ_k\alpha_{0k}}{2}\left(1+\right.\right. \\ \left.|\nu_{2k}|\sqrt{1+\dfrac{(\alpha_{2k}-\alpha_{1k})^2\tau_k}{|\alpha_{1k}|\alpha_{2k}\varepsilon_k}}\right)- \\ \left.\dfrac{\varepsilon_k\alpha_{1k}\alpha_{2k}}{k_{1k}k_{2k}}\right)(\sigma_k(n+1)-\sigma_k(n))^2- \\ \delta_k\varphi_k^2(\sigma_k(n))-\varepsilon_k\dfrac{\alpha_{1k}\alpha_{2k}}{k_{1k}k_{2k}}P_k^2(\sigma'_{kn})(\sigma_k(n+1)-\sigma_k(n))^2 \\ +æ_k|\nu_{2k}\varphi_k(\sigma_k(n))P_k(\sigma'_{kn})|(\sigma_k(n+1)-\sigma_k(n)). \end{array} \quad (75)$$

By virtue of condition 2) of the theorem we have that

$$V(n+1)-V(n)\leq-\delta_0\,|f(\sigma(n))|^2 \quad (\delta_0>0), \quad (76)$$

where by $|f|$ the Euclidian norm of vector f is designated. Since sequence $W(n)$ $(n=0,1,2,\dots)$ is bounded and functions $Y_k(\sigma)$ $(k=1,2,\dots,l)$ satisfy (47) we can affirm that sequence $V(n)$ $(n=0,1,2,\dots)$ is bounded as well. Then is follows from (76) that the series

$$\sum_{n=1}^{+\infty}|f(\sigma(n))|^2 \quad (77)$$

converges. Hence

$$\lim_{n\to+\infty}|f(\sigma(n))|=0 \quad (78)$$

and consequently as soon as all eigenvalues of A are situated inside the unit circle we can affirm that

$$\lim_{n\to+\infty}z(n)=0. \quad (79)$$

Then from (62) it follows that

$$\sigma(n+1)-\sigma(n)\to0 \quad as \quad n\to+\infty. \quad (80)$$

From (78) and (80) it follows that

$$\sigma(n)\to\hat\sigma \quad as \quad n\to+\infty, \quad (81)$$

with $f(\hat\sigma)=0$. Theorem 3 is proved.

Theorem 4.
Suppose there exist such positive definite diagonal matrices $\varepsilon=diag\,\{\varepsilon_1,\dots,\varepsilon_l\}$, $\tau=diag\,\{\tau_1,\dots,\tau_l\}$, $\delta=diag\,\{\delta_1,\dots,\delta_l\}$, a diagonal matrix $æ=diag\,\{æ_1,\dots,æ_l\}$ and numbers $a_k\in[0,1]$ $(k=$

$1,\dots,l$) that the requirement 1) from theorem 3 is fulfilled and matrices

$$\left\| \begin{matrix} \varepsilon_k - \frac{æ_k \alpha_{0k}}{2}\,(a_k(1+|\nu_k|)+ \\ a_{0k}\left(1-\frac{\alpha_{2k}-\alpha_{1k}}{\sqrt{|\alpha_{1k}|\alpha_{2k}}}\right)\Big) \end{matrix}, & \frac{æ_k\nu_k a_k}{2}, & 0 \\ \frac{æ_k\nu_k a_k}{2}, & \delta_k, & \frac{æ_k a_{0k}\nu_{0k}}{2} \\ 0, & \frac{æ_k a_{0k}\nu_{0k}}{2}, & \tau_k\frac{\alpha_{1k}\alpha_{2k}}{k_{1k}k_{2k}} \right\|, \tag{82}$$

where $a_{0k} = 1 - a_k$ and α_{0k} are defined in the text of theorem 3, are positive definite. Then the conclusion of theorem 3 is true.

The proof of theorem 4 is alike the proof of theorem 3. It is based on the Lyapunov-type sequence

$$V(n) = W(n) + \sum_{k=1}^{l} æ_k \left(a_k \int_{\sigma_k(0)}^{\sigma_k(n)} F_k(\sigma)d\sigma + a_{0k} \int_{\sigma_k(0)}^{\sigma_k(n)} \Psi_k(\sigma)d\sigma \right), \tag{83}$$

where the sequence $W(n)$ is defined in the text of theorem 3 and functions F_k and Ψ_k $(k = 1,\dots,l)$ are borrowed from the text of theorem 1. The estimates for $a_k \int_{\sigma_k(0)}^{\sigma_k(n)} F_k(\sigma)d\sigma$ and $a_{0k} \int_{\sigma_k(0)}^{\sigma_k(n)} \Psi_k(\sigma)d\sigma$ are taken from [Leonov, Smirnova, 2000] and [Smirnova, Shepeljavyi, 2007] respectively. The full text of this proof can be found in [Perkin, Smirnova and Shepeljavyi, 2009].

5 Conclusion

The paper is devoted to the problem of gradient-like behavior for distributed and discrete phase systems. The problem is investigated by two methods traditionally used in absolute stability. They are the method of a priori integral estimates for distributed systems and Lyapunov direct method for discrete systems. In the paper certain generalization of periodic Popov-type functionals as well as of periodic Lyapunov-type sequence is exploited. As a result new frequency-domain criteria are obtained. They are applied to concrete radio-engineering and mechanical systems. The new criteria give the opportunity to improve the estimates for the regions of gradient-like behavior in the space of parameters of the systems.

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