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APPROXIMATION AND RELAXATION OF MECHANICAL SYSTEMS WITH DISCONTINUOUS VELOCITIES

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Abstract
We study mechanical systems controlled by "shock impacts", i.e., external signals of, possibly, negligible duration and very high intensity. From a physical point of view, the signals are due to fast vibrations of segments of a rigid body, and impactive blocking of a part of its degrees of freedom. A mathematical idealization of such phenomena leads to systems with discontinuous velocities described by distributional (measure differential) equations with square and affine impulses, at that blocking of degrees of freedom formally results in a "complementarity" constraint relating states with the affine impulsive control. We raise two closely connected issues: first, we provide a correct approximation of the prototypical impulsive system by ordinary control processes; second, seeing that the trajectory tube of the system occurs to lose the property of compactness, we design its constructive relaxation (a compactification). The final goal is to discover the limit behavior of the system driven by the two types of impulsive controls.

Key words
Impulsive control, impact mechanics, blockable degrees of freedom, vibrating controls, trajectory relaxation, approximation.

1 Introduction: Mechanical Systems Controlled by Shock Impacts and Fast Vibrations

The object of our study originates in the framework of *acceleration controlled* mechanical systems and contact dynamics [Aldo Bressan and Motta, 1993, Bressan, 2008, Bressan and Rampazzo, 1993, Bressan and Rampazzo, 2010, Kozlov and Treshchev, 1991, Marle, 1991, May, 1987, Miller and Bentsman, 2006] and relies on the mathematical apparatus of impulsive control theory [Arutyunov, Karamzin, and Lobo Pereira, 2011, Dykhta, 1990, Dykhta and Samsonyuk, 2000, Filippova, 2005, Gurman, 1972, Gurman, 1977, Karamzin et al., 2014, Krotov, 1996, Miller, 1996, Miller and Rubinovich, 2003, Lobo Pereira, Silva and Oliveira, 2008, Rishel, 1965, Zavalishchin and Sesekin, 1997, Warga, 1987]. Our model involves control external forces: the complementary forces of frictional or stress type produced by collisions, and forces produced by frictionless holonomic constraints.

To illustrate the model's features, we start with two simple but representative cases.

1.1 Inspiring Examples

Example 1.1. *A double pendulum with a vibrating pinned link and an impactively blockable joint.* Consider a control double pendulum with the links of unit length and unit mass moving in the vertical plane, in the gravity field. The system configuration is represented by the angular coordinates $\varphi = (\varphi_1, \varphi_2)$ of the links. The system has two degrees of freedom associated with $\varphi_{1,2}$. In the Lagrangian formalism, the time evolution of the system is described by a standard Lagrange equation with the Lagrangian $L = K - P$, the total kinetic energy K, the total potential energy P, and the generalized vector of external forces F:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\varphi}\right) - \frac{\partial L}{\partial \varphi} = F. \tag{1}$$

In local coordinates of the inertial reference system $(\varphi_1, \varphi_2, \dot\varphi_1, \dot\varphi_2)$, being the angular coordinates and the angular velocities $\dot\varphi_{1,2} \doteq \frac{d}{dt}\varphi_{1,2}$, the terms K and P are given by $K = \dot\varphi_1^2 + 1/2\,\dot\varphi_2^2 + \varphi_1\varphi_2\cos(\varphi_1 - \varphi_2)$, and $P = -\mathrm{g}\,(2\cos\varphi_1 + \cos\varphi_2)$, where g is the acceleration due to gravity.

Consider the behavior of the angular position φ_2, and assume that it can be artificially affected by signals of two types, namely: by choosing the angle φ_1 as a desired continuous function of time, and by instantaneous blocking/releasing the angle $\varphi_1 - \varphi_2$ between the links. The dynamics of the *control* mechanical system then takes the form:

$$\begin{aligned}(1+\sin^2(\varphi_1-\varphi_2))\,\ddot\varphi_2 - 2\sin(\varphi_1-\varphi_2)\,\dot\varphi_1^2\\ -1/2\sin 2(\varphi_1-\varphi_2)\,\dot\varphi_2^2\\ +\,\mathrm{g}\,(\sin\varphi_2 - \sin\varphi_1\cos(\varphi_1-\varphi_2)) = F,\end{aligned} \qquad (2)$$

and the impactive blocking/releasing principle is performed by the "mixed" constraint

$$\dot\varphi_2(t) = 0 \quad |F|\text{-a.e.} \qquad (3)$$

relating the state $\dot\varphi_2$, and the shock control F. The derivative $\dot F$ is here regarded in a generalized sense, namely, as a distribution or a Lebesgue-Stieltjes measure induced by the function F of bounded variation, and "-a.e." means "almost everywhere with respect to a measure". Recall that the mapping $t \mapsto \varphi_1(t)$ is the input signal, but not a trajectory of the system, and note that the derivative $\dot\varphi$ participates in the equation of motion as a quadratic term.

Example 1.2. *A telescopic arm with a blockable degree of freedom and control vibrations.* Consider a telescopic manipulator wherein a unit-mass point slides without friction along a massless pinned link. The length h of the link is controlled by setting the angle position φ of the link as a desired function of time, and by instantaneous blocking/releasing the linear motion of the point mass. This results in the control system

$$\ddot h - h\,\dot\varphi^2 = -F, \qquad (4)$$
$$\dot h(t) = 0 \quad F\text{-a.e.} \qquad (5)$$

Here, F is an impulsive frictional force.

1.2 Impact Dynamics and Distributional (Measure Differential) Equations

Consider Lagrangian dynamical system (1). Following [Brogliato, 2000], a generalized version of this model – admitting non-smooth velocities – takes the form

$$\mathbf{M}(q)\,\ddot q - \mathbf{h}(q,\dot q) = \mathbf{f} + \mathbf{B}(q)\,u. \qquad (6)$$

Here, the force F is decomposed into a uncontrolled *external* force $\mathbf{f}$ and a control force u. The (symmetric, positive definite) matrix $\mathbf{M} = \mathbf{M}(q)$, called a generalized mass matrix, is related with (1) by $\mathbf{M} = \partial^2_{\dot q\dot q}K$. The function $\mathbf{h} = \mathbf{h}(q,\dot q)$ includes all finite smooth forces (spring, damper, centripetal, gyroscopic, coriolis etc.) and is defined by $\mathbf{h} = \partial^2_{\dot q q}K - \partial_q L$. The linear map $\mathbf{B} = \mathbf{B}(q)$ represents directions of control forces.

For the case of discontinuous velocities, which is typical in impact mechanics, equation (6) has to be further generalized to the form [Moreau, 1966]

$$\mathbf{M}(q)\,d(\dot q) - \mathbf{h}(q,\dot q)\,dt = \pi(dt) + \mathbf{B}(q)\,\mu(dt), \quad (7)$$

where π and μ are distributions (measures), representing the generalized uncontrolled and control external forces, respectively.

2 Impulsive Control Systems with Square and Mixed-Constrained Affine Impulses

Now we focus on the mathematical formalism behind the mechanical systems with blockable degrees of freedom and vibrating controls discussed in Section 1. For ease of presentation, we will consider the case of scalar-valued input signals.

2.1 Complementarity Problem for Impulsive Control Systems

Given $T, M_1, M_2 > 0$, $x_0 \in \mathbb{R}^n$, and functions $f_i, g : \mathbb{R}^n \to \mathbb{R}^n$, $i = 0,1,2$, defined on a finite time interval $\mathcal{T} \doteq [0,T]$, consider the following control distributional equation:

$$\begin{aligned}\dot x = f_0(x) + f_1(x)w + f_2(x)w^2 + g(x)\dot V,\\ x(0^-) = x_0.\end{aligned} \qquad (8)$$

Here, states are $x(t) \in \mathbb{R}^n$, $x(t^-)$ denotes the left one-sided limit of a function x at a point t, and inputs are of different types: (i) $w : \mathcal{T} \to \mathbb{R}$ are Borel measurable L_2-functions, and (ii) $V \in BV \doteq BV^+(\mathcal{T},\mathbb{R})$, i.e., right continuous functions $\mathcal{T} \to \mathbb{R}$ with bounded variation. The inputs are subject to the standard "energetic" constraints:

$$|\dot V|(\mathcal{T}) \preceq \nu(\mathcal{T}) \leqslant M_1, \quad \|w\|^2_{L_2} \leqslant M_2. \qquad (9)$$

In what follows, $\dot V$ is regarded as a measure, i.e., $\dot V \doteq \mu \in C^*$, where C^* is the dual of the space C of continuous functions. In (9), $|\dot V| \doteq |\mu|$ denotes the total variation of the measure $|\mu|$, ν is yet another nonnegative Lebesgue-Stieltjes measure to be chosen by the controller, i.e., ν is an extra control, $\preceq$ is a usual order on the set of measures, and $\|\cdot\|_{L_2}$ denotes the norm in L_2.

Introduce a generalized form of constraints (5), (3):

$$W_-\big(x(t^-)\big) + W_+\big(x(t)\big) = 0 \quad \nu\text{-a.e. on } \mathcal{T}, \quad (10)$$

where $W_\pm : \mathbb{R}^n \to \mathbb{R}$ are certain *nonnegative continuous* functions. Note that condition (10) is a generalization of orthogonality in L_2, and it establishes a certain complementarity of states with respect to the control measure ν.

2.2 Understanding of the Affine Impulsive Control and the Distributional Equation

We make standard assumptions (H) about the Lipschitz continuity and sublinear growth of the functions f_i, $i = 0, 1, 2$, and g.

Following [Arutyunov, Karamzin, and Lobo Pereira, 2011, Arutyunov, Karamzin, and Lobo Pereira, 2010, Karamzin, 2006], by an impulsive control we mean a collection $\vartheta \doteq (\mu, \nu, \{v_\tau\}_{\tau\in D_\nu})$, wherein the measures $\mu = |\dot{V}|$ and ν meet (9), and $\{v_\tau\}_{\tau\in D_\nu}$, $D_\nu \doteq \{\tau \in \mathcal{T} : \ |\mu|(\mathcal{T}) \neq 0\}$, is a family of Borel measurable functions $v_\tau : \mathcal{T}_\tau \doteq [0, T_\tau] \to \mathbb{R}^m$, parameterized by atoms of ν, such that

$$|v_\tau(\theta)| = 1 \quad \lambda\text{-a.e. on } \mathcal{T}_\tau, \tag{11}$$

$$\int_{\mathcal{T}_\tau} v_\tau(\theta)\, d\theta = \mu(\{\tau\}) \tag{12}$$

for all $\tau \in D_\nu$. Here, $T_\tau \doteq \nu(\{\tau\})$. Functions v_τ are "controls of jumps" [Arutyunov, Karamzin, and Lobo Pereira, 2011].

Let Θ be the set of impulsive controls satisfying (9), (11), (12), and $\mathcal{W}$ the set of Borel measurable L_2-functions $\mathcal{T} \to \mathbb{R}$ satisfying (9), and $\mathcal{P} = \Theta \times \mathcal{W}$.

Denote $f(x, w) \doteq f_0(x) + f_1(x)w + f_2(x)w^2$. Given an input $\varrho = (\vartheta \doteq (\mu, \nu, \{v_\tau\}), w) \in \mathcal{P}$, by a solution to the Cauchy problem (8) under the control input ϱ, we mean a function $x \in BV^+(\mathcal{T}, \mathbb{R}^n)$ satisfying, for each $t \in \mathcal{T}$, the following integral relation:

$$x(t) = x_0 + \int_0^t f(x, w)\, d\theta + \int_0^t g(x)\, \mu_c(d\theta) + \sum_{\tau\in D_\nu} \left[x(\tau) - x(\tau^-)\right]. \tag{13}$$

Here, $\mu_c \doteq \mu_{ac} + \mu_{sc}$ is the continuous component of the Lebesgue decomposition of measure μ being the sum of its absolutely continuous and singular continuous parts μ_{ac} and μ_{sc}, and the integral with respect to μ_c is understood in the Lebesgue-Stieltjes sense; jump exit points $x(\tau)$ of a function x at points $\tau \in D_\nu$ are defined as $x(\tau) = \varkappa_\tau(T_\tau)$, where $\varkappa_\tau$ is a Carathéodory solution of the *limit control system*:

$$\frac{d}{d\varsigma}\varkappa(\varsigma) = g\big(\varkappa(\varsigma)\big)\, v_\tau(\varsigma), \quad \varkappa(0) = x(\tau^-). \tag{14}$$

Under assumptions (H), the existence and uniqueness of a solution $x[\varrho]$ to (13), (14) for any input $\varrho \in \mathcal{P}$ is guaranteed by the general theorem stated in [Miller and Rubinovich, 2003].

A collection $\sigma = (x, \varrho)$ with $x = x[\varrho]$, $\varrho = (\vartheta = (\mu, \nu, \{v_\tau\}), w) \in \mathcal{P}$ is called an impulsive control process. We have to assume that the set $\{\sigma = (x, \varrho) : \ x = x[\varrho],\ \varrho \in \mathcal{P}\}$ is nonempty.

2.3 Quadratic Impulses. Relaxation of the Dynamics

In the model, "affine impulses", performed by the distribution $\dot{V}$, are *postulated* (due to the mentioned mechanical origin, see [Moreau, 1966]). At the same time, the values of the term w^2 with $\|w\|^2_{L_2} \le M$ are not uniformly bounded. One can consider sequences $\{w_n\}$ of controls such that the squares w_n^2 tend, in the sense of distributions, to a Dirac "point-mass" measure (a "δ"-type distribution). Such signals are said to be quadratic impulsive controls. They produce extra discontinuities of state solutions, in addition to the discontinuities caused by affine impulses. As we will see below, the interplay between the two impulsive controls of different natures results in a rather sophisticated behavior of the system state in the phase of jump.

The trajectory tube of measure-driven system (13), (14) is not compact in the strong topology of BV, and therefore it requires a relaxation in a certain weak topology, and the relaxation should correlate with constraint (10). In the following section we design such a relaxation in the weak* topology of BV, and give its constructive representation by means of a discontinuous time reparameterization of solutions to a terminally constrained ordinary differential inclusion. For this, we first need to define a proper relaxation of solutions to system (13), (14), (10).

3 Approximation and Relaxation of the Complementarity Property

The following definition establishes ε-complementarity of a process with respect to its small weak* perturbation.

Definition 3.1. *Given $\varepsilon > 0$, an impulsive control process $\sigma = (x, \varrho)$ is said to be an ε-approximate solution of the complementary system (13), (14), (10), iff there exists another process $\sigma = (\widetilde{x}, \widetilde{\varrho})$, $x = x[\widetilde{\varrho}]$, $\widetilde{\varrho} = (\widetilde{\vartheta} = (\widetilde{\mu}, \widetilde{\nu}, \{\widetilde{v}_\tau\}), w) \in \mathcal{P}$, such that the following relations hold:*

1. *$(\widetilde{x}, F_{\widetilde{\nu}})$ belongs to an ε-neighborhood of (x, F_ν) in the weak* topology of BV, i.e.,*

$$\left\|(x, F_\nu)(t) - (\widetilde{x}, F_{\widetilde{\nu}})(t)\right\| \le \varepsilon \quad \text{for all } t \in \big([0, T) \setminus D_\nu\big) \cup \{T\}. \tag{15}$$

2. *The ε-"ordering" condition*

$$\int_{\mathcal{T}} Q(F_\nu, F_{\widetilde{\nu}})\, d\nu_c + \int_{\mathcal{T}} Q(F_\nu, F_{\widetilde{\nu}})\, d\widetilde{\nu}_c + \sum_{\tau \in D_\nu} Q\big(F_\nu(\tau), F_{\widetilde{\nu}}(\tau)\big)\, \nu(\{\tau\}) + \sum_{\tau \in D_{\widetilde{\nu}}} Q\big(F_\nu(\tau), F_{\widetilde{\nu}}(\tau)\big)\, \widetilde{\nu}(\{\tau\}) \le \varepsilon \quad (16)$$

is met. Here, F_ν and $F_{\widetilde{\nu}}$ are the distribution functions of the measures ν and $\widetilde{\nu}$, and $Q = Q(\eta_+, \eta_-)$ is an arbitrary fixed continuous non-negative function $\mathbb{R}^2_+ \to \mathbb{R}$ vanishing only on the set $\{(\eta_+, \eta_-) \in \mathbb{R}^2_+ : \eta_- \le \eta_+\}$.

3. *The ε-complementarity condition*

$$\int_{\mathcal{T}} W_-(\widetilde{x})\, d\nu + \int_{\mathcal{T}} W_+(x)\, d\widetilde{\nu} \le \varepsilon \quad (17)$$

is satisfied.

4. *The relation with the quadratic control*

$$\int_{\mathcal{T}} \big\|(x, F_\nu) - (\widetilde{x}, F_{\widetilde{\nu}})\big\|\, w^2\, dt \leqslant \varepsilon \quad (18)$$

holds true.

The role of ε-solutions can be played by ordinary control processes $(x, v, w) = (x, \varrho = (\vartheta, w))$ with $\vartheta = (\mu, \nu)$, $\mu(dt) = v\, dt$, $v \in L_\infty$, $\nu = |\mu|$. At the same time, simple examples show that the complementarity condition (10) generically does not hold for ordinary control processes, even approximately.

Theorem 3.1. *Any solution to (13), (14), (10) can be approximated in the weak* topology of BV by ordinary control processes. In other words, for any $\varrho \in \mathcal{P}$, there exists a sequence $(x, v, w)_\varepsilon$ such that $x_\varepsilon \rightharpoonup x[\varrho]$ as $\varepsilon \to 0$. Here and further, $\rightharpoonup$ indicates the convergence in the weak* topology of BV.*

The proof is similar to [Goncharova and Staritsyn, 2017].

Let $\overline{\mathbb{X}}$ denote the set of functions $x \in BV^+(\mathcal{T}, \mathbb{R}^n)$ such that there exists a sequence $\{\sigma_\varepsilon\}_{\varepsilon > 0}$ of impulsive control processes $\sigma_\varepsilon = (x, \varrho)_\varepsilon$ with the following properties: For any $\varepsilon > 0$, σ_ε is an approximate ε-solution of (13), (14), (10) in the sense of Definition 3.1, and $x_\varepsilon \rightharpoonup x$ as $\varepsilon \to 0$.

3.1 Space-Time Transformation

Consider a new, "extended" time interval $\mathcal{S} \doteq [0, S]$, $S \geqslant T$, and introduce on $\mathcal{S}$ the following ordinary control system (called the *reduced* system):

$$\frac{d}{ds}\,\xi = \alpha^2,\ \frac{d}{ds}\, y_\pm = \alpha^2 f_0(y_\pm) + \alpha\beta f_1(y_\pm) + \beta^2 f_2(y_\pm) + \omega_\pm\, g(y_\pm), \quad (19)$$

$$\frac{d}{ds}\,\eta_\pm = |\omega_\pm|,\ \frac{d}{ds}\,\zeta = \beta^2, \quad (20)$$

$$\frac{d}{ds}\,\iota = (\alpha^2 + \beta^2)\big[\Delta^\pm \eta + |\Delta^\pm y|\big] + |\omega_+|\, W_-(y_-) + |\omega_-|\, W_+(y_+) + Q(\eta), \quad (21)$$

$$y_\pm(0) = x_0, \quad (\xi, \eta, \zeta, \iota)(0) = 0 \in \mathbb{R}^5, \quad (22)$$

$$\xi(S) = T,\ \Delta^\pm(y, \eta)(S) = 0 \in \mathbb{R}^{n+1},\ \iota(S) = 0, \quad (23)$$

$$\eta_+(S) \leqslant M_1,\ \zeta(S) \leqslant M_2, \quad (24)$$

$$\mathbf{u} \doteq (\alpha, \beta, \omega) \in \mathbf{U}. \quad (25)$$

Here, s is new time variable; $\mathbf{U} \doteq \mathbf{U}(S)$ is the set of controls $\mathbf{u} = (\alpha, \beta, \omega)$ with $\omega \doteq (\omega_+, \omega_-)$, $\alpha, \beta, \omega_\pm : \mathcal{S} \to \mathbb{R}$, being Borel functions such that $\alpha(s) \geqslant 0$, $\alpha^2(s) + \beta^2(s) + |\omega_+(s)| + |\omega_-(s)| = 1$ λ-a.e. on $\mathcal{S}$. The new states are $\mathbf{x} \doteq (\xi, y, \eta, \zeta, \iota) \in \mathbb{R}^{2n+5}$, where $y \doteq (y_+, y_-)$ and $\eta \doteq (\eta_+, \eta_-)$, $\xi(s), \eta_\pm(s), \zeta(s), \iota(s) \in \mathbb{R}_+$, $y_\pm(s) \in \mathbb{R}^n$. The operation $\Delta^\pm$ applied to a vector $c \doteq (c_+, c_-) \in \mathbb{R}^{2r}$ defines the vector $c_+ - c_- \in \mathbb{R}^r$, and $\Delta^\mp = -\Delta^\pm$.

By $\mathbf{x}[\mathbf{u}]$ we denote the Carathéodory solution of system (20)–(22) on $\mathcal{S}$, under control $\mathbf{u} \in \mathbf{U}$.

Given $\varrho \in \mathcal{P}$, denote $\hat{\mu} \doteq \lambda + \|w\|^2_{L_2} + 2\nu$ and introduce a strictly increasing function $\Upsilon : \mathcal{T} \to [0, \hat{\mu}(\mathcal{T})]$ as follows: $\Upsilon(t) = F_{\hat{\mu}}(t)$, $t \in \mathcal{T}$. Let υ denote the inverse of Υ.

The following result provides an embedding of the set of solutions to the complementarity system (13), (14), (10) into the trajectory tube of system (19)–(25).

Theorem 3.2. *Let $\varrho \in \mathcal{P}$ be such that the solution $x = x[\varrho]$ of (13), (14) meets condition (10). Then, there exist a real $S \ge T$ and a control $\mathbf{u} \in \mathbf{U}(S)$ such that the respective solution $\mathbf{x} = (\xi, y = (y_+, y_-), \eta = (\eta_+, \eta_-), \zeta, \iota)[\mathbf{u}]$ of control system (20)–(22) satisfies the right-point constraints (23), (24), and*

$$y_- \circ \Upsilon = y_+ \circ \Upsilon = x, \quad \upsilon = \xi. \quad (26)$$

Proofs of all the assertions in this section are rather technical and will appear in our forthcoming paper. They employ a combination of arguments similar to [Goncharova and Staritsyn, 2012] and [Goncharova and Staritsyn, 2017].

The inverse transform is performed as stated in the following

Theorem 3.3. *Assume that $S \ge T$ and $\mathbf{u} \in \mathbf{U}(S)$ are such that the solution $\mathbf{x} \doteq (\xi, y = (y_+, y_-), \eta =$*

$(\eta_+, \eta_-), \zeta, \iota)[\mathbf{u}]$ *of system (19)–(22) satisfies constraints (23), (24). Let* $(x, V, \pi) \in BV^+(\mathcal{T}, \mathbb{R}^{n+2})$ *be defined by the composition*

$$(x, V, \pi) = (y_+, \eta_+, \zeta) \circ \Xi \ \text{ on } \mathcal{T}, \qquad (27)$$

where $\Xi : \mathcal{T} \to \mathcal{S}$ *is given by* $\Xi(t) = \inf\{s \in \mathcal{S} : \xi(s) > t\}$, $t \in [0, T)$, $\Xi(T) = S$. *Set* $\nu \doteq dV$, $\varpi \doteq d\pi$ *(by definitions the measures are nonnegative, and* $\nu(\mathcal{T}) \leqslant M_1$, $\varpi(\mathcal{T}) \leqslant M_2$*). Then, there exist a scalar Borel measure* μ *with* $|\mu| \preceq \nu$ *and* $|\mu|_c = \nu_c$*; a Borel measurable function* $l : \mathcal{T} \to \mathbb{R}$ *with* $\int_0^t l^2\, d\theta = \varpi_{ac}([0, t])$*, and a family* $\{(v, w)_\tau\}$ *of Borel functions* $v_\tau : \check{\mathcal{T}}_\tau \to \mathbb{R}$, $w_\tau : \check{\mathcal{T}}_\tau \to \mathbb{R}_+$*, parameterized by atoms of the measure* $\check{\mu} \doteq \nu + \mu_2$ *(here* $\check{\mathcal{T}}_\tau \doteq [0, \check{T}_\tau]$, $\check{T}_\tau \doteq \check{\mu}(\{\tau\})$*) and satisfying the relations*

$$\int_{\check{\mathcal{T}}_\tau} v_\tau(\theta) d\theta = \mu(\{\tau\}), \ \int_{\check{\mathcal{T}}_\tau} w_\tau(\theta) d\theta = \varpi(\{\tau\}),$$
$$u_\tau + |v_\tau| = 1 \ \lambda\text{-a.e. over } \check{\mathcal{T}}_\tau,$$

such that, for all $t \in \mathcal{T}$, x *meets condition (10) with the measure* ν *and satisfies the following measure differential equation:*

$$x(t) = x_0 + \int_0^t \big(f_0(x) + f_1(x)\, l\big)\, d\theta + \int_0^t f_2(x)\, \varpi_c(d\theta) + \int_0^t g(x)\, \mu_c(d\theta) + \sum_{\tau \in D_{\check{\mu}}} \big[x(\tau) - x(\tau^-)\big].$$

Here, $x(\tau) = \varkappa_\tau(\check{T}_\tau)$, $\tau \in D_{\check{\mu}}$*, where* $\varkappa_\tau$ *solves the Cauchy problem*

$$\tfrac{d}{d\theta}\varkappa = f_2(\varkappa)\, v_\tau + g(\varkappa)\, w_\tau, \ \ \varkappa(0) = x(\tau^-). \quad (28)$$

Now we establish a relation between the trajectory relaxation $\overline{\mathbb{X}}$ and the reduced system (19)–(25). Let $\mathcal{F} = \mathcal{F}(\mathbf{x}, \mathbf{u})$ denote the right-hand side of system (19)–(22). Consider the relaxed dynamics

$$\frac{\mathrm{d}\mathbf{x}(s)}{\mathrm{d}s} \in \overline{\mathrm{co}}\, \{\mathcal{F}(\mathbf{x}, \mathbf{u}) |\, \mathbf{u} \in \mathbf{U}\}. \qquad (29)$$

Here, $\overline{\mathrm{co}}\, A$ denotes the closed convex hull of a set A. As is stated below, $\overline{\mathbb{X}}$ coincides with the trajectory tube of (29), (22)–(24), up to a discontinuous time change.

Theorem 3.4. *1) For any* $x \in \overline{\mathbb{X}}$*, there exist* $S \geq T$ *and a solution* $\mathbf{x} = (\xi, y = (y_+, y_-), \eta = (\eta_+, \eta_-), \zeta, \iota)$ *of terminally constrained differential inclusion (29), (22)–(24), such that relations (26) hold.*

2) Let $\mathbf{x} = (\xi, y = (y_+, y_-), \eta = (\eta_+, \eta_-), \zeta, \iota)$ *be a solution to the Cauchy problem for differential inclusion (29) on a time interval* $\mathcal{S} = [0, S]$, $S \geq T$*, such that conditions (23), (24) hold. Define* x *by formula (27). Then,* $x \in \overline{\mathbb{X}}$.

The following assertion generalizes Theorem 3.1.

Theorem 3.5. *Any* $x \in \overline{\mathbb{X}}$ *can be approximated in the weak* topology of BV by ordinary control processes.*

4 Limit Behavior of Extended System

On condition that the affine and quadratic impulses occur at different instants, constraint (10) remains in force. In general case, there may arise *multiple jumps at the same time, produced by different measures*. Indeed, suppose, for example, that $W_-\big(x(\tau-)\big) > 0$, but τ is an atom of ϖ such that the respective trajectory $\varkappa_\tau$ of limit system (28) under $v_\tau = 0$ reaches the set $\mathcal{Z}_- \doteq \{x \in \mathbb{R}^n : W_-(x) = 0\}$ at a certain moment θ within the "fast time" interval $\mathcal{T}_\tau$, i.e., $W\big(\varkappa_\tau(\theta)\big) = 0$. Assumed further that the set $\mathcal{Z}_+ \doteq \{x \in \mathbb{R}^n : W_+(x) = 0\}$ is reachable from the position $\varkappa_\tau(\theta)$ along a trajectory of (28) with a nontrivial control v_τ, one can find another control $(v, w)_\tau$ with $v_\tau = 0$ that brings us back to $\mathcal{Z}_-$, and so on. We observe that jumps of a state are decomposed into a series of subjumps caused by different impulsive actions, and the subjumps corresponding to the affine impulses enjoy the property given by (10), whereas the total jump does not satisfy this condition anymore.

The limit form of (10) is exhibited by the following

Theorem 4.1. *Given* $x \in \overline{\mathbb{X}}$*, let* $\{\sigma_\varepsilon \doteq (x_\varepsilon, \varrho_\varepsilon)\}_{\varepsilon>0}$, $\varrho_\varepsilon \doteq (\vartheta_\varepsilon, w_\varepsilon)$, $\vartheta_\varepsilon = (\mu_\varepsilon, \nu_\varepsilon, \{v_{\tau_\varepsilon}\})$*, be its approximation in the sense of Definition 3.1, given by Theorem 3.5, and* $\mathbf{x} \doteq (\xi, y = (y_+, y_-), \eta = (\eta_+, \eta_-), \zeta, \iota)$ *be a solution to the constrained differential inclusion, presented by Theorem 3.4. Denote by* (ν, ϖ) *a weak* limit of the sequence* $\{(\nu_\varepsilon, \|w_\varepsilon\|^2_{L_2})\}_{\varepsilon>0}$ *and define* $v_\tau \doteq \frac{2\frac{d}{ds}\zeta}{\frac{d}{ds}(2\zeta + \eta_+ + \eta_-)} \circ s_\tau$, $w_\tau \doteq \frac{\frac{d}{ds}(\eta_+ + \eta_-)}{\frac{d}{ds}(2\zeta + \eta_+ + \eta_-)} \circ s_\tau$*, on* $\check{\mathcal{T}}_\tau$*, where* $s_\tau \doteq \theta_\tau^{-1}$ *and* $\theta_\tau(s) \doteq [\eta_+(s) + \eta_-(s)]/2 + \zeta(s) - \check{\mu}([0, \tau))$, $s \in \Xi_\tau \doteq [\Xi(\tau^-), \Xi(\tau)]$.

There exists a subset $\mathcal{R}_\tau \subseteq \operatorname{supp} v_\tau \setminus \operatorname{supp} w_\tau$ *("*supp*" stands for the support of a function) being the union* $\cup_{i \in \mathbb{I}_\tau} \Omega^i_\tau$ *of disjoint open intervals* $\Omega^i_\tau \doteq (\underline{\theta}_\tau, \overline{\theta}_\tau)^i$ *(the index set* $\mathbb{I}_\tau$ *is at most countable), such that, for any* $\tau \in D_\nu$*, it holds:*

$$W_-\big(x(t)\big) = W_+\big(x(t)\big) = 0 \quad \nu_c\text{-a.e. on } \mathcal{T},$$
$$W_-\big(\varkappa_\tau(\theta)\big) = W_+\big(\varkappa_\tau(\theta)\big) = 0 \ \lambda\text{-a.e. } \operatorname{supp} v_\tau \setminus \mathcal{R}_\tau,$$
$$W_-\big(\varkappa_\tau(\underline{\theta}^i_\tau)\big) = W_+\big(\varkappa_\tau(\overline{\theta}^i_\tau)\big) = 0 \ \forall i \in \mathbb{I}_\tau.$$

In other words, the complementarity condition (10) does hold during the phase of jumps, for “fast motions” produced by affine impulses.

5 Example 1.2: Revisited

To illustrate the designed trajectory extension, we present a relaxed form of the control mechanical system figuring in Example 1.2. Conditions (2), (3) have to be rewritten as

$$\dot{h} = \mathrm{v}, \quad d\mathrm{v} = h\varpi(dt) - \nu(dt),$$
$$\mathrm{v}(t) = 0 \quad \nu\text{-a.e.},$$

where ν and ϖ are the linear and quadratic control measures, respectively, and the angle φ satisfies the relation $\dot{\varphi} = \left(\dot{F}_{\varpi_{ac}}\right)^{1/2}$.

The interaction between the two controls is as follows: by changing the angle φ, we produce a centrifugal force, which shifts the point mass along the link, and this motion can be instantaneously stopped by the “frictional” force ν.

Let us comment on the part played by the quadratic impulse control. Given an initial state $(h_0, \mathrm{v}_0, \varphi_0)$, assume that ϖ is nontrivial, but $\varpi_{ac} = 0$. Then, $\varphi(t) \equiv \varphi_0$, i.e., the angular coordinate stays in rest. At the same time, the states (h, v) do evolve. For instance, if $\varpi = \varpi_{sc}$, we get that $\mathrm{v} = \mathrm{v}_0 + F_{\varpi^{sc}}$, and $h = h_0 + \int \mathrm{v}$ are *increasing* functions. This example reveals a singular continuous behavior. Physically, this is a manifestation of fast vibrations of invisibly small magnitude.

6 Conclusion

The developed mathematical tools can be used for analytical and numerical investigation of optimal control problems for the addressed class of mechanical systems. The obtained results can be easily generalized to the case of vector-valued control inputs.

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