ADAPTIVE CONTROL OF PDES

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Abstract: This paper presents several recently developed techniques for adaptive control of PDE systems. Three different design methods are employed—the Lyapunov design, the passivity-based design, and the swapping design. The basic ideas for each design are introduced through benchmark plants with constant unknown coefficients. It is then shown how to extend the designs to reaction-advection-diffusion PDEs in 2D. Finally, the PDEs with unknown spatially varying coefficients and with boundary sensing are considered, making the adaptive designs applicable to PDE systems with an infinite relative degree, infinitely many unknown parameters, and open loop unstable.

Keywords: backstepping, distributed parameter systems, adaptive control

1. INTRODUCTION

In systems with thermal, fluid, or chemically reacting dynamics, which are usually modelled by parabolic partial differential equations (PDEs), physical parameters are often unknown. Thus a need exists for developing adaptive controllers that are able to stabilize a potentially unstable, parametrically uncertain plant. While adaptive control of finite dimensional systems is a mature area that has produced adaptive control methods for most LTI systems of interest (Ioannou and Sun 1996), adaptive control techniques have been developed for only a few of the classes of PDE for which non-adaptive controllers exist. The existing results (Bentsman and Orlov 2001, Bohm *et al.* 1998, Hong and Bentsman 1994) focus on model reference (MRAC) type schemes and the control action distributed in the PDE domain (see (Krstic 2005) for a more detailed literature review). One of the major obstacles in developing adaptive schemes for PDEs is the absence of parametrized families of stabilizing controllers. In a recent paper (Smyshlyaev and Krstic 2004), the *explicit* formulae were introduced for *boundary* control of parabolic PDEs. Those formulae are not only explicit functions of the spatial coordinates, but also depend explicitly on the physical parameters of the plant. In this paper we overview three different design methods based on those explicit controllers—Lyapunov method, and certainty equivalence approaches with passive and swapping identifiers. For tutorial reasons, the presentation proceeds through a series of one-unknown-parameter benchmark examples with sketches of the proofs. The detailed proofs for the designs presented here are given in (Krstic 2005, Smyshlyaev and Krstic 2006*a*, Smyshlyaev and Krstic 2006*b*, Smyshlyaev and Krstic 2007*a*, Smyshlyaev and Krstic 2007*b*). We then extend the presented approaches to reaction-advection-diffusion plants in 2D and plants with spatially varying (functional) parametric uncertainties. We end the paper with the *output*-feedback adaptive design for reaction-advection-diffusion systems with only boundary sensing and actuation. These systems have an infinite relative degree, infinitely many unknown parameters and are open-loop unstable, representing the ultimate challenge in adaptive control for PDEs.

2. LYAPUNOV DESIGN

Consider the following plant

$$u_t(x,t) = u_{xx}(x,t) + \lambda u(x,t), \quad 0 < x < 1 \quad (1)$$
$$u(0,t) = 0\,, \quad (2)$$

where λ is an unknown constant parameter. We use a Neumann boundary controller designed in (Smyshlyaev and Krstic 2004) in the form [1]

$$u_x(1) = -\frac{\hat\lambda}{2}u(1) - \hat\lambda \int_0^1 \xi \frac{I_2\left(\sqrt{\hat\lambda(1-\xi^2)}\right)}{1-\xi^2} u(\xi)d\xi, \quad (3)$$

which employs the measurements of $u(x)$ for $x \in [0,1]$ and an estimate $\hat\lambda$ of λ. Consider an invertible change of variable

$$w(x) = u(x) - \int_0^x \hat k(x,\xi)u(\xi)\,d\xi\,, \quad (4)$$
$$\hat k(x,\xi) = -\hat\lambda\xi \frac{I_1\left(\sqrt{\hat\lambda(x^2-\xi^2)}\right)}{\sqrt{\hat\lambda(x^2-\xi^2)}}\,. \quad (5)$$

One can show that (4) maps (1)–(3) into

$$w_t = w_{xx} + \dot{\hat\lambda}\int_0^x \frac{\xi}{2} w(\xi)\,d\xi + \tilde\lambda w\,, \quad (6)$$
$$w(0) = w_x(1) = 0\,, \quad (7)$$

where $\tilde\lambda = \lambda - \hat\lambda$ is the parameter estimation error.

We will show that the update law

$$\dot{\hat\lambda} = \gamma\frac{\|w\|^2}{1+\|w\|^2}\,, \qquad 0 < \gamma < 1 \quad (8)$$

achieves regulation of $u(x,t)$ to zero for all $x \in [0,1]$, for arbitrarily large initial data $u(x,0)$ and for an arbitrarily poor initial estimate $\hat\lambda(0)$.

Theorem 1. Suppose that the system (1)–(3), (8) has a well defined classical solution for all $t \geq 0$. Then, for any initial condition $u_0 \in H^1(0,1)$ compatible with boundary conditions, and any $\hat\lambda(0) \in \mathbb{R}$, the solutions $u(x,t)$ and $\hat\lambda(t)$ are uniformly bounded and $\lim_{t\to\infty} u(x,t) = 0$ uniformly in $x \in [0,1]$.

Remark 1. It is important to note that the update law (8) contains normalization. Normalization is uncommon in Lyapunov designs and is the result of including the logarithm in the Lyapunov function (Praly 1992). Normalization is necessary because the control law (3) is of certainty equivalence type—unlike the Lyapunov adaptive controllers in (Krstic *et al.* 1995) which employ normalized adaptation and strengthened nonlinear controllers that compensate for time-varying effects of adaptation. An additional measure of preventing overly fast adaptation in (8) is the restriction on the adaptation gain ($\gamma < 1$).

[1] In the sequel, to reduce notational overload, the dependence on time will be suppressed whenever possible.

Proof of Theorem 1 (Sketch). Consider a Lyapunov function candidate

$$V = \frac{1}{2}\log\left(1+\|w\|^2\right) + \frac{1}{2\gamma}\tilde\lambda^2\,. \quad (9)$$

The time derivative along the solutions of (6)–(8) can be shown to be

$$\dot V = -\frac{\|w_x\|^2}{1+\|w\|^2} + \frac{\dot{\hat\lambda}}{2}\frac{\int_0^1 w(x)\left(\int_0^x \xi w(\xi)d\xi\right)dx}{1+\|w\|^2} \quad (10)$$

(the calculation involves one step of integration by parts). Using Cauchy and Poincare inequalities, one gets

$$\left|\int_0^1 w(x)\left(\int_0^x \xi w(\xi)d\xi\right)dx\right| \leq \frac{2}{\sqrt3}\|w_x\|^2\,. \quad (11)$$

Substituting (11) and (8) into (10) and using the fact that $|\dot{\hat\lambda}| < \gamma$ (see (8)), we get

$$\dot V \leq -\left(1-\frac{\gamma}{\sqrt3}\right)\frac{\|w_x\|^2}{1+\|w\|^2}\,. \quad (12)$$

This implies that $V(t)$ remains bounded for all time whenever $0 < \gamma \leq \sqrt3$. From the definition of V it follows that $\|w\|$ and $\hat\lambda$ remain bounded for all time. To show that $w(x,t)$ is bounded for all time and for all x, we estimate (using Agmon, Young, and Poincare inequalities):

$$\begin{aligned}\frac12\frac{d}{dt}\|w_x\|^2 &= -\|w_{xx}\|^2 + \tilde\lambda\|w_x\|^2 \\ &\quad + \frac{\dot{\hat\lambda}}{4}\left(w(1)^2 - \|w\|^2\right) \\ &\leq -(1-\gamma)\|w_{xx}\|^2 + \tilde\lambda\|w_x\|^2 \\ &\leq \tilde\lambda\|w_x\|^2\,. \end{aligned} \quad (13)$$

Integrating the last inequality, we obtain

$$\|w_x(t)\|^2 \leq \|w_x(0)\|^2 + 2\sup_{0\leq\tau\leq t}|\tilde\lambda(\tau)|\int_0^t \|w_x(\tau)\|^2 d\tau. \quad (14)$$

Using (12) and the fact that $\|w\|$ is bounded, we get

$$\int_0^t \|w_x(\tau)\|^2 d\tau \leq (1+C)\int_0^t \frac{\|w_x(\tau)\|^2}{1+\|w(\tau)\|^2}d\tau < \infty, \quad (15)$$

where C is the bound on $\|w\|^2$. From (14) and (15) we get that $\|w_x\|^2$ is bounded. Combining Agmon and Poincare inequalities, we get $\max_{x\in[0,1]}|w(x)|^2 \leq 4\|w_x\|^2$, thus $w(x,t)$ is bounded for all x and t.

Next, we prove regulation of $w(x,t)$ to zero. Using (6)–(7), it is easy to show that

$$\frac{1}{2}\left|\frac{d}{dt}\|w\|^2\right| \le \|w_x\|^2 + \left(|\tilde\lambda| + \frac{\gamma}{4\sqrt{3}}\right)\|w\|^2 . \quad (16)$$

Since $\|w\|$ and $\|w_x\|$ have been proven bounded, it follows that $\frac{d}{dt}\|w\|^2$ is bounded, and thus $\|w(t)\|$ is uniformly continuous. From (15) and Poincare inequality we get that $\|w\|^2$ is integrable in time over the infinite time interval. By Barbalat's lemma it follows that $\|w\| \to 0$ as $t \to \infty$. The regulation in the maximum norm follows from Agmon inequality.

Having proved the boundedness and regulation of w, we now set out to establish the same for u. We start by noting that the inverse transformation to (4) is (Smyshlyaev and Krstic 2004)

$$u(x) = w(x) + \int_0^x \hat l(x,\xi)w(\xi)d\xi \quad (17)$$

$$\hat l(x,\xi) = -\hat\lambda\xi \frac{J_1\left(\sqrt{\hat\lambda(x^2-\xi^2)}\right)}{\sqrt{\hat\lambda(x^2-\xi^2)}} . \quad (18)$$

Since $\hat\lambda$ is bounded, the function $\hat l(x,\xi)$ has bounds

$$L_1 = \max_{0\le\xi\le x\le 1} \hat l(x,\xi)^2, \; L_2 = \max_{0\le\xi\le x\le 1} \hat l_x(x,\xi)^2 . \quad (19)$$

It is straightforward to show that

$$\|u_x\|^2 \le 2\left(1 + \hat\lambda^2 + 4L_2\right)\|w_x\|^2 , \quad (20)$$

Noting that $u(x,t)^2 \le 4\|u_x\|^2$ for all $(x,t) \in [0,1]\times[0,\infty)$ and using the fact that $\|w_x\|$ is bounded, we get uniform boundedness of u. To prove regulation of u, we estimate from (17)

$$\|u\|^2 \le 2(1+L_1)\|w\|^2 . \quad (21)$$

Since $\|w\|$ is regulated to zero, so is $\|u\|$. By Agmon's inequality $u(x,t)^2 \le 2\|u\|\|u_x\|$, where $\|u_x\|$ is bounded. Therefore $u(x,t)$ is regulated to zero for all $x \in [0,1]$. The proof is completed.

The Lyapunov design incorporates all the states of the closed loop system into a single Lyapunov function and therefore Lyapunov adaptive controllers possess the best transient performance properties. However, this method is not applicable as broadly as the *certainty equivalence* approaches, which we consider next.

3. CERTAINTY EQUIVALENCE DESIGN WITH PASSIVE IDENTIFIER

Consider the plant

$$u_t = u_{xx} + \lambda u \quad (22)$$
$$u(0) = 0 , \quad (23)$$

where a constant parameter λ is *unknown*. We use a Dirichlet controller designed in (Smyshlyaev and Krstic 2004):

$$u(1) = -\hat\lambda\int_0^1 \xi \frac{I_1\left(\sqrt{\hat\lambda(1-\xi^2)}\right)}{\sqrt{\hat\lambda(1-\xi^2)}} u(\xi)d\xi , \quad (24)$$

Following the certainty equivalence principle, first we need to design an identifier which will provide the estimate $\hat\lambda$.

3.1 Identifier

Consider the following auxiliary system:

$$\hat u_t = \hat u_{xx} + \hat\lambda u + \gamma^2(u-\hat u)\int_0^1 u^2(x)\,dx \quad (25)$$
$$\hat u(0) = 0 \quad (26)$$
$$\hat u(1) = u(1) . \quad (27)$$

Such an auxiliary system is often called an "observer", even though it is not used here for state estimation (the entire state u is available for measurement in our problem). The purpose of this "observer" is to help identify the unknown parameter; we will refer to the system (25)–(27) as "passive identifier". This identifier employs a copy of the PDE plant and an additional nonlinear term. The term "passive identifier" comes from the fact that an operator from the parameter estimation error $\tilde\lambda = \lambda - \hat\lambda$ to the inner product of u with $u - \hat u$ is strictly passive. The additional term in (25) acts as nonlinear damping whose task is to slow down the adaptation.

Let us introduce the error signal $e = u - \hat u$. Using (22)–(23) and (25)–(27), we obtain the following PDE for $e(x,t)$:

$$e_t = e_{xx} + \tilde\lambda u - \gamma^2 e\|u\|^2 \quad (28)$$
$$e(0) = e(1) = 0 . \quad (29)$$

Consider a Lyapunov function

$$V = \frac{1}{2}\int_0^1 e^2(x)\,dx + \frac{\tilde\lambda^2}{2\gamma}. \quad (30)$$

The time derivative of V along the solutions of (28)–(29) is

$$\dot V = -\|e_x\|^2 - \gamma^2\|e\|^2\|u\|^2 + \tilde\lambda\int_0^1 e(x)u(x)\,dx - \frac{\tilde\lambda\dot{\hat\lambda}}{\gamma}. \quad (31)$$

Let us choose the update law

$$\dot{\hat\lambda} = \gamma\int_0^1 (u(x)-\hat u(x))u(x)\,dx . \quad (32)$$

Then the last two terms in (31) cancel out and we obtain

$$\dot V = -\|e_x\|^2 - \gamma^2\|e\|^2\|u\|^2 , \quad (33)$$

which implies $V(t) \le V(0)$. By the definition of V, this means that $\tilde\lambda$ and $\|e\|$ are bounded functions of time.

Integrating (33) with respect to time from zero to infinity we get that the spatial norms $\|e_x\|$ and $\|e\|\|u\|$ are square integrable over infinite time (belong to $\mathcal{L}_2$). From the update law (32) we get $|\dot{\hat{\lambda}}| \le \gamma\|e\|\|u\|$ which shows that $\dot{\hat{\lambda}}$ is also square integrable in time, i.e. the identifier indirectly slows down the adaptation.

Lemma 2. The identifier (25)–(27) with update law (32) guarantees the following properties:

$$\|e_x\|,\ \|e\|\|u\|,\ \|e\|,\ \dot{\hat{\lambda}} \in \mathcal{L}_2, \quad \|e\|,\ \tilde{\lambda} \in \mathcal{L}_\infty. \quad (34)$$

3.2 Main result

Theorem 3. Suppose that a closed loop system that consists of (22)-(24), identifier (25)–(27), and update law (32), has a classical solution ($\hat{\lambda}$, u, $\hat{u}$). Then for any $\hat{\lambda}(0)$ and any initial conditions $u_0, \hat{u}_0 \in H^1(0,1)$, the signals $\hat{\lambda}$, u, $\hat{u}$ are bounded and u is regulated to zero for all $x \in [0,1]$:

$$\lim_{t\to\infty} \max_{x\in[0,1]} |u(x,t)| = 0. \quad (35)$$

Proof. Consider the transformation

$$\hat{w}(x) = \hat{u}(x) - \int_0^x \hat{k}(x,y)\hat{u}(y)\,dy \quad (36)$$

with $\hat{k}$ given by (5). One can show that the above transformation maps (25)–(27) into the following target system

$$\hat{w}_t = \hat{w}_{xx} + \dot{\hat{\lambda}} \int_0^x \frac{\xi}{2}\hat{w}(\xi)\,d\xi + (\hat{\lambda} + \gamma^2\|u\|^2)e_1 \quad (37)$$

$$\hat{w}(0) = \hat{w}(1) = 0\,, \quad (38)$$

where e_1 is the "transformed" estimation error

$$e_1(x) = e(x) - \int_0^x \hat{k}(x,y)e(y)\,dy\,. \quad (39)$$

We observe that in comparison to non-adaptive target system (plain heat equation) two additional terms appeared in (37), one is proportional to $\dot{\hat{\lambda}}$ and the other is proportional to the estimation error e. The identifier guarantees that both of these terms are square integrable in time, which means that they decay to zero barring some occasional "spikes."

Since $\hat{\lambda}$ is bounded, and the functions $\hat{k}(x,y)$ and $\hat{l}(x,y)$ are twice continuously differentiable with respect to x and y, there exist constants M_1, M_2, M_3 such that

$$\|e_1\| \le M_1\|e\| \quad (40)$$

$$\|u\| \le \|\hat{u}\| + \|e\| \le M_2\|\hat{w}\| + \|e\| \quad (41)$$

$$\|u_x\| \le \|\hat{u}_x\| + \|e_x\| \le M_3\|\hat{w}_x\| + \|e_x\|. \quad (42)$$

Before we proceed, we need the following lemma.

Lemma 4. ((Krstic *et al.* 1995)). Let v, l_1, and l_2 be real-valued functions of time defined on $[0,\infty)$, and let c be a positive constant. If l_1 and l_2 are nonnegative and integrable on $[0,\infty)$ and satisfy the differential inequality

$$\dot{v} \le -cv + l_1(t)v + l_2(t), \quad v(0) \ge 0 \quad (43)$$

then v is bounded and integrable on $[0,\infty)$.

Using Young, Cauchy-Schwartz, and Poincare inequalities along with the identifier properties (34) and (40)–(42) one can obtain the following estimate

$$\frac{1}{2}\frac{d}{dt}\|\hat{w}\|^2 \le -\frac{1}{16}\|\hat{w}\|^2 + l_1\|\hat{w}\|^2 + l_2, \quad (44)$$

where l_1, l_2 are some integrable functions of time on $[0,\infty)$. Using Lemma 4 we get the boundedness and square integrability of $\|\hat{w}\|$. From (41) and (34) we get boundedness and square integrability of $\|u\|$ and $\|\hat{u}\|$, and (32) then gives boundedness of $\dot{\hat{\lambda}}$.

In order to get pointwise in x boundedness, one estimates

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \hat{w}_x^2\,dx \le -\frac{1}{8}\|\hat{w}_x\|^2 + \frac{|\dot{\hat{\lambda}}|^2\|\hat{w}\|^2}{4} + (\lambda_0 + \gamma^2\|u\|^2)^2 M_1\|e\|^2 \quad (45)$$

$$\frac{1}{2}\frac{d}{dt}\int_0^1 e_x^2\,dx \le -\frac{1}{8}\|e_x\|^2 + \frac{1}{2}|\tilde{\lambda}|^2\|u\|^2\,. \quad (46)$$

Since the right hand sides of (45) and (46) are integrable, using Lemma 4 we get boundedness and square integrability of $\|\hat{w}_x\|$ and $\|e_x\|$. Using the inverse transformation

$$\hat{u}(x) = \hat{w}(x) + \int_0^x \hat{l}(x,y)\hat{w}(y)\,dy \quad (47)$$

with $\hat{l}$ given by (18), we get boundedness and square integrability of $\|\hat{u}_x\|$ and (42) then gives the same properties for $\|u_x\|$. By Agmon inequality, we get the boundedness of $\hat{u}$ and u for all $x \in [0,1]$.

To show the regulation of u to zero, note that

$$\frac{1}{2}\frac{d}{dt}\|e\|^2 \le -\|e_x\|^2 + |\tilde{\lambda}|\|e\|\|u\| < \infty \quad (48)$$

The boundedness of $(d/dt)\|w\|^2$ follows from (44). By Barbalat's lemma, we get $\|\hat{w}\| \to 0$, $\|e\| \to 0$ as $t \to \infty$. It follows from (47) that $\|\hat{u}\| \to 0$ and therefore (41) gives $\|u\| \to 0$. Using Agmon inequality and the fact that $\|u_x\|$ is bounded, we get the regulation of $u(x,t)$ to zero for all $x \in [0,1]$. The proof is completed.

4. CERTAINTY EQUIVALENCE DESIGN WITH SWAPPING IDENTIFIER

The certainty equivalence design with *swapping* identifer is perhaps the most common method of parameter estimation in adaptive control. Filters of the "regressor" and of the measured part of the plant are implemented to convert a dynamic parameterization of the problem (given by the plant's dynamic model) into a static parametrization where standard gradient and least squares estimation techniques can be used.

Consider the plant

$$u_t = u_{xx} + \lambda u, \quad 0 < x < 1 \tag{49}$$
$$u(0) = 0\,, \tag{50}$$

with unknown constant parameter λ. We start by employing two filters: the state filter

$$v_t = v_{xx} + u \tag{51}$$
$$v(0) = v(1) = 0 \tag{52}$$

and the input filter

$$\eta_t = \eta_{xx} \tag{53}$$
$$\eta(0) = 0 \tag{54}$$
$$\eta(1) = u(1)\,. \tag{55}$$

The "estimation" error

$$e = u - \lambda v - \eta \tag{56}$$

is then exponentially stable:

$$e_t = e_{xx} \tag{57}$$
$$e(0) = e(1) = 0\,. \tag{58}$$

Using the static relationship (56) as a parametric model, we implement a "prediction error" as

$$\hat{e} = u - \hat{\lambda} v - \eta, \qquad \hat{e} = e + \tilde{\lambda} v\,. \tag{59}$$

We choose the gradient update law with normalization

$$\dot{\hat{\lambda}} = \gamma \frac{\int_0^1 \hat{e}(x) v(x)\,dx}{1 + \|v\|^2}\,. \tag{60}$$

With a Lyapunov function

$$V = \frac{1}{2}\int_0^1 e^2\,dx + \frac{1}{8\gamma}\tilde{\lambda}^2 \tag{61}$$

we get

$$\begin{aligned}\dot{V} &\le -\int_0^1 e_x^2\,dx - \frac{\int_0^1 \hat{e}^2(x)\,dx}{4(1+\|v\|^2)} + \frac{\int_0^1 \hat{e}(x) e(x)\,dx}{4(1+\|v\|^2)} \\ &\le -\|e_x\|^2 - \frac{\|\hat{e}\|^2}{4(1+\|v\|^2)} + \frac{\|e_x\|\|\hat{e}\|}{2\sqrt{1+\|v\|^2}} \\ &\le -\frac{1}{2}\|e_x\|^2 - \frac{1}{8}\frac{\|\hat{e}\|^2}{1+\|v\|^2}\,. \end{aligned} \tag{62}$$

This gives the following properties

$$\frac{\|\hat{e}\|}{\sqrt{1+\|v\|^2}} \in \mathcal{L}_2 \cap \mathcal{L}_\infty, \tag{63}$$

$$\tilde{\lambda} \in \mathcal{L}_\infty, \quad \dot{\hat{\lambda}} \in \mathcal{L}_2 \cap \mathcal{L}_\infty. \tag{64}$$

In contrast with the passive identifier, the normalization in the swapping identifier is employed in the update law. This makes $\dot{\hat{\lambda}}$ not only square integrable but also bounded.

We use the controller (24) with the state u replaced by its estimate $\hat{\lambda} v + \eta$:

$$u(1) = -\hat{\lambda}\int_0^1 \xi \frac{I_1\left(\sqrt{\hat{\lambda}(1-\xi^2)}\right)}{\sqrt{\hat{\lambda}(1-\xi^2)}}(\hat{\lambda} v(\xi) + \eta(\xi))d\xi. \tag{65}$$

Theorem 5. Suppose that a closed loop system that consists of the plant (49)–(50), the controller (65), the filters (51)–(55), and the update law (60), has a classical solution ($\hat{\lambda}$, u, v, η). Then for any $\hat{\lambda}(0)$ and any initial conditions $u_0, v_0, \eta_0 \in H^1(0,1)$, the signals $\hat{\lambda}$, u, v, η are bounded and u is regulated to zero for all $x \in [0,1]$:

$$\lim_{t\to\infty} \max_{x\in[0,1]} |u(x,t)| = 0. \tag{66}$$

Proof (Sketch). Consider the transformation

$$\begin{aligned}\hat{w}(x) &= \hat{\lambda} v(x) + \eta(x) \\ &\quad - \int_0^x \hat{k}(x,\xi)(\hat{\lambda} v(\xi) + \eta(\xi))\,d\xi \end{aligned} \tag{67}$$

with the same $\hat{k}(x,\xi)$ as in (36). Using (51)–(55) and the inverse transformation

$$\hat{\lambda} v(x) + \eta(x) = \hat{w}(x) + \int_0^x \hat{l}(x,\xi)\hat{w}(\xi)\,d\xi \tag{68}$$

$$\hat{l}(x,\xi) = -\hat{\lambda}\xi \frac{J_1\left(\sqrt{\hat{\lambda}(x^2-\xi^2)}\right)}{\sqrt{\hat{\lambda}(x^2-\xi^2)}} \tag{69}$$

one can get the following PDE for $\hat{w}$:

$$\hat{w}_t = \hat{w}_{xx} + \hat{\lambda}\left(\hat{e}(x) - \int_0^x \hat{k}(x,\xi)\hat{e}(\xi)\,d\xi\right) + \dot{\hat{\lambda}}v(x) + \dot{\hat{\lambda}}\int_0^x\left(\frac{\xi}{2}\hat{w}(\xi) - \hat{k}(x,\xi)v(\xi)\right)d\xi \quad (70)$$

$$\hat{w}(0) = \hat{w}(1) = 0\,. \quad (71)$$

In order to prove boundedness of all signals we rewrite the filter (51)–(52) as follows

$$v_t = v_{xx} + \hat{e} + \hat{w} + \int_0^x \hat{l}(x,\xi)\hat{w}(\xi)\,d\xi \quad (72)$$

$$v(0) = v(1) = 0\,. \quad (73)$$

We have now two interconnected systems for v and $\hat{w}$, (70)–(73), which are driven by the signals $\dot{\hat{\lambda}}$, $\hat{\lambda}$, and $\hat{e}$ with properties (64). Note that the situation here is more complicated than in the passive design where we had to analyze only the $\hat{w}$-system (37)–(38). While the signal v feeds into $\hat{w}$-system (70)–(71) through a "convergent-to-zero" signal $\dot{\hat{\lambda}}$, the signal $\hat{w}$ feeds into the v-system (72)–(73) through a bounded but possibly large gain $\hat{l}$. Therefore to prove the boundedness of $\|\hat{w}\|$ and $\|v\|$ we use a weighted Lyapunov function

$$W = A\|\hat{w}\|^2 + \|v\|^2\,, \quad (74)$$

where A is a large enough constant. One can then show that

$$\dot{W} \le -\frac{1}{4A}W + l_1 W\,, \quad (75)$$

and with the help of Lemma 4 we get the boundedness of $\|\hat{w}\|$ and $\|v\|$. Using this result it can be shown that

$$\frac{d}{dt}\left(\|\hat{w}_x\|^2 + \|v_x\|^2\right) \le -\|\hat{w}_{xx}\|^2 - \|v_{xx}\|^2 + l_1,$$

which proves that $\|\hat{w}_x\|$ and $\|v_x\|$ are bounded. From Agmon's inequality we get that $\hat{w}$ and v are bounded pointwise in x. By Barbalat's lemma we get $\|\hat{w}\| \to 0$, $\|v\| \to 0$ as $t \to \infty$. From (68) and (56) we get the pointwise boundedness of η and u and $\|u\| \to 0$. Finally, the pointwise regulation of u to zero follows from Agmon's inequality. The proof is completed.

The swapping method uses the highest order of dynamics of all identifier approaches. Lyapunov is the lowest in this respect as it only incorporates the dynamics of the parameter update, and passivity-based is better than swapping because it uses only one filter, as opposed to 'one-filter-per-unknown-parameter' in the case of the swapping approach. Despite its high dynamic order, the swapping approach is popular because it is the most transparent (its stability proof is the simplest due to the static parametrization) and it is the only method that allows least-squares estimation.

5. EXTENSION TO REACTION-ADVECTION-DIFFUSION SYSTEMS IN HIGHER DIMENSIONS

All the approaches presented in Sections 2–4 can be readily extended to reaction-advection-diffusion plants and higher dimensions (2D and 3D). As an illustration, consider a 2D plant with four unknown parameters ε, b_1, b_2, and λ:

$$u_t = \varepsilon(u_{xx} + u_{yy}) + b_1 u_x + b_2 u_y + \lambda u \quad (76)$$

on the rectangle $0 \le x \le 1$, $0 \le y \le L$ with actuation applied on the side with $x = 1$ and Dirichlet boundary conditions on the other three sides.

We choose to design the scheme with passive identifier. We introduce the following "observer"

$$\hat{u}_t = \hat{\varepsilon}(\hat{u}_{xx} + \hat{u}_{yy}) + \hat{b}_1\hat{u}_1 + \hat{b}_2\hat{u}_2 + \hat{\lambda}u + \gamma^2(u - \hat{u})\|\nabla u\|^2 \quad (77)$$

$$\hat{u} = 0\,, (x,y) \in \{[0,1]\times[0,1]\}\backslash\{x = 1\} \quad (78)$$

$$\hat{u} = u\,, \quad x = 1,\ 0 \le y \le 1\,. \quad (79)$$

There are two main differences compared to 1D case with one parameter considered in Section 3. First, since the diffusion coefficient ε is unknown we must use projection to ensure $\hat{\varepsilon} > \underline{\varepsilon} > 0$. We define the projection operator as

$$\mathrm{Proj}_{\underline{\varepsilon}}\{\tau\} = \begin{cases} 0\ , & \hat{\varepsilon} = \underline{\varepsilon} \text{ and } \tau < 0 \\ \tau\ , & \text{else}\ . \end{cases} \quad (80)$$

Although this operator is discontinuous it is possible to introduce a small boundary layer instead of a hard switch which will avoid dealing with Filippov solutions and noise due to frequent switching of the update law (see (Krstic 2005) for more details). However, we use (80) here for notational clarity. Note that $\hat{\varepsilon}$ does not require the projection from above and all other parameters do not require projection at all.

Second, we can see in (77) that while the diffusion and advection coefficients multiply the operators of $\hat{u}$, the reaction coefficient multiplies u in the observer. This is necessary in order to eliminate any λ-dependence in the error system so that it is stable.

The update laws are

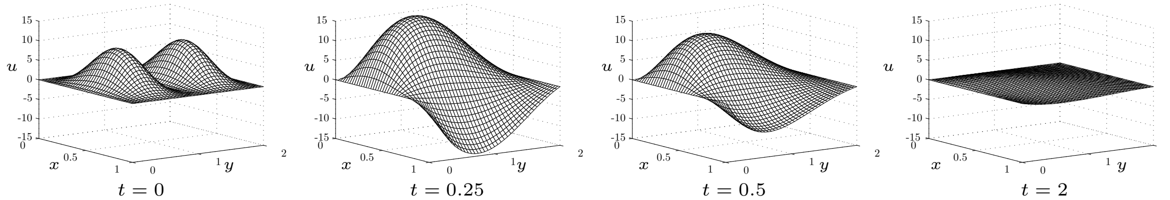

Fig. 1. The closed loop state for the plant (76) at different times.

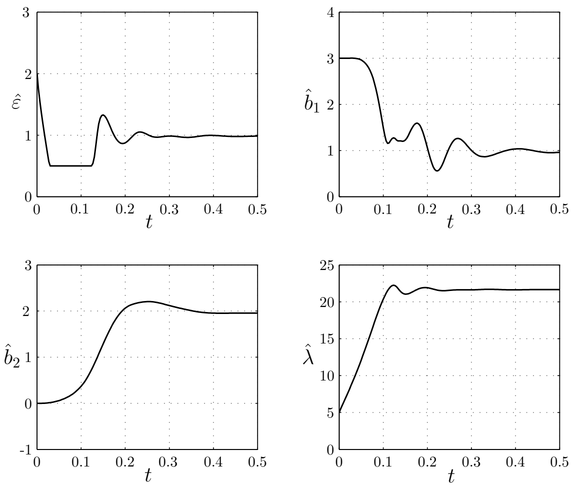

Fig. 2. The parameter estimates for the plant (76).

$$\dot{\hat{\varepsilon}} = -\gamma_0 \mathrm{Proj}_{\underline{\varepsilon}} \left\{ \int_0^1 \int_0^1 u_x(u_x - \hat{u}_x) + u_y(u_y - \hat{u}_y)\, dx\, dy \right\} \quad (81)$$

$$\dot{\hat{b}}_1 = \gamma_1 \int_0^1 \int_0^1 (u - \hat{u}) u_x \, dx\, dy \quad (82)$$

$$\dot{\hat{b}}_2 = \gamma_1 \int_0^1 \int_0^1 (u - \hat{u}) u_y \, dx\, dy \quad (83)$$

$$\dot{\hat{\lambda}} = \gamma_2 \int_0^1 \int_0^1 (u - \hat{u}) u \, dx\, dy\,, \quad (84)$$

and the controller is

$$u(1,y) = -\int_0^1 \frac{\hat{\lambda}}{\hat{\varepsilon}} \xi e^{-\frac{\hat{b}_1(1-\xi)}{2\hat{\varepsilon}}} \times \frac{I_1\left(\sqrt{\frac{\hat{\lambda}}{\hat{\varepsilon}}(1-\xi^2)}\right)}{\sqrt{\frac{\hat{\lambda}}{\hat{\varepsilon}}(1-\xi^2)}} \hat{u}(\xi, y)\, d\xi\,. \quad (85)$$

The results of the simulation of the above scheme are presented in Fig. 1 and Fig. 2. The true parameters are set to $\varepsilon = 1$, $b_1 = 1$, $b_2 = 2$, $\lambda = 22$, $L = 2$. With this choice the open-loop plant has two unstable eigenvalues at 8.4 and 1. All estimates come close to the true values at approximately $t = 0.5$ and after that the controller stabilizes the system.

6. PLANTS WITH SPATIALLY-VARYING UNCERTAINTIES

The designs presented in Sections 2–4 can be extended to the plants with spatially-varying unknown parameters. For example, for the plant

$$u_t = u_{xx} + \lambda(x) u \quad (86)$$

$$u_x(0) = 0 \quad (87)$$

the Lyapunov adaptive controller would be

$$u(1) = \hat{k}(1,1)u(1) + \int_0^1 \hat{k}_x(1,\xi)u(\xi)d\xi \quad (88)$$

with

$$\hat{\lambda}_t(t,x) = \gamma \frac{u(t,x)\left(w(t,x) - \int_x^1 \hat{k}(\xi,x)w(t,\xi)d\xi\right)}{1 + \|w(t)\|^2}$$

where $\hat{\lambda}(t,x)$ is the online functional estimate of $\lambda(x)$, $w(x) = u(x) - \int_0^x \hat{k}(x,\xi)u(\xi)d\xi$, and the kernel $\hat{k}(x,\xi) = \hat{k}_n(x,\xi)$ is obtained recursively from

$$\hat{k}_0 = -\frac{1}{2} \int_{\frac{x-\xi}{2}}^{\frac{x+\xi}{2}} \hat{\lambda}(\zeta)\, d\zeta \quad (89)$$

$$\hat{k}_{i+1} = \hat{k}_i + \int_{\frac{x-\xi}{2}}^{\frac{x+\xi}{2}} \int_0^{\frac{x-\xi}{2}} \hat{\lambda}(\zeta - \sigma)\, \hat{k}_i(\zeta + \sigma, \zeta - \sigma) \times d\sigma d\zeta, \quad i = 0, 1, \ldots, n$$

for each new update of $\hat{\lambda}(t,x)$. Stability is guaranteed for sufficiently small γ and sufficiently high n. The recursion (89) was proved convergent in (Smyshlyaev and Krstic 2004). The certainty equivalence designs with passive and swapping identifiers can also be extended to the case of functional unknown parameters using the same recursive procedure. For further details, the reader is referred to (Smyshlyaev and Krstic 2006*a*).

7. OUTPUT-FEEDBACK DESIGN

Consider the plant

$$u_t = u_{xx} + \lambda(x) u, \qquad 0 < x < 1 \quad (90)$$

$$u_x(0,t) = 0, \quad (91)$$

$$u(1,t) = U(t)\,. \quad (92)$$

where $\lambda(x)$ is an unknown continuous function and only the boundary value $u(0,t)$ is measured.

The key step in our design is the transformation of the original plant (90)–(92) into a system in which unknown parameters multiply the measured output.

7.1 Transformation to Observer Canonical Form

Consider the transformation

$$v(x) = u(x) - \int_0^x p(x,y)u(y)\,dy \tag{93}$$

where $p(x,y)$ is a solution of the PDE

$$p_{xx}(x,y) - p_{yy}(x,y) = \lambda(y)p(x,y) \tag{94}$$

$$p(1,y) = 0 \tag{95}$$

$$p(x,x) = \frac{1}{2}\int_x^1 \lambda(s)\,ds\,. \tag{96}$$

One can show that this transformation maps the system (90)–(92) into

$$v_t = v_{xx} + \theta(x)v(0) \tag{97}$$

$$v_x(0) = \theta_1 v(0) \tag{98}$$

$$v(1) = u(1) \tag{99}$$

where

$$\theta(x) = -p_y(x,0), \qquad \theta_1 = -p(0,0) \tag{100}$$

are the new unknown parameters.

The system (97)–(99) is the PDE analog of observer canonical form. Note from (93) that $v(0) = u(0)$ and therefore $v(0)$ is measured. The transformation (93) is invertible so that stability of v implies stability of u. Therefore it is enough to design the stabilizing controller for v–system and then use the condition $u(1) = v(1)$ (which follows from (95)) to obtain the controller for the original system. We are going to directly estimate the new uknown parameters $\theta(x)$ and θ_1 instead of estimating $\lambda(x)$. Thus, we do not need to solve the PDE (94)–(96) for the control scheme implementation.

7.2 Estimator

The unknown parameters θ and $\theta(x)$ enter the boundary condition and the domain of the v–system. Therefore we will need the following output filters:

$$\phi_t = \phi_{xx} \tag{101}$$

$$\phi_x(0) = u(0) \tag{102}$$

$$\phi(1) = 0 \tag{103}$$

and

$$\Phi_t = \Phi_{xx} + \delta(x-\xi)u(0) \tag{104}$$

$$\Phi_x(0) = \Phi(1) = 0\,. \tag{105}$$

Here the filter $\Phi = \Phi(x,\xi)$ is parametrized by $\xi \in [0,1]$ and $\delta(x-\xi)$ is a delta function. The reason for this parametrization is the presence of the functional parameter $\theta(x)$ in the domain. Therefore, loosely speaking we need an infinite "array" of filters, one for each $x \in [0,1]$ (since the swapping design normally requires one filter per unknown parameter). We also introduce the input filter

$$\psi_t = \psi_{xx} \tag{106}$$

$$\psi_x(0) = 0 \tag{107}$$

$$\psi(1) = u(1)\,. \tag{108}$$

It is straightforward to show now that the error

$$\bar{e}(x) = v(x) - \psi(x) - \theta_1\phi(x) - \int_0^1 \theta(\xi)\Phi(x,\xi)\,d\xi \tag{109}$$

satisfies the exponentially stable PDE

$$\bar{e}_t = \bar{e}_{xx} \tag{110}$$

$$\bar{e}_x(0) = \bar{e}(1) = 0\,. \tag{111}$$

Typically the swapping method requires one filter per unknown parameter and since we have functional parameters, infinitely many filters are needed. However, we reduce their number down to only two by representing the state $\Phi(x,\xi)$ algebraically through $\phi(x)$ at each moment in time.

Lemma 6. The signal

$$e(x) = v(x) - \psi(x) - \theta_1\phi(x) - \int_0^1 \theta(\xi)F(x,\xi)\,d\xi \tag{112}$$

where $F(x,\xi)$ is given by

$$F_{xx}(x,\xi) = F_{\xi\xi}(x,\xi) \tag{113}$$

$$F(0,\xi) = -\phi(\xi) \tag{114}$$

$$F_x(0,\xi) = F_\xi(x,0) = F(x,1) = 0 \tag{115}$$

is governed by the exponentially stable heat equation:

$$e_t = e_{xx} \tag{116}$$

$$e_x(0) = e(1) = 0 \tag{117}$$

Proof. The initial conditions for the filters ϕ and Φ are the design choice so let us assume that they are continuous functions in x and ξ. We now write down the explicit solutions to the filters:

$$\phi(x,t) = 2\sum_{n=0}^{\infty}\cos(\sigma_n x)\, e^{-\sigma_n^2 t}\int_0^1 \phi_0(s)\cos(\sigma_n s)\, ds - 2\sum_{n=0}^{\infty}\cos(\sigma_n x)\int_0^t u(0,\tau)e^{-\sigma_n^2(t-\tau)}\, d\tau, \quad (118)$$

and

$$\Phi(x,\xi,t) = 2\sum_{n=0}^{\infty}\cos(\sigma_n x)\cos(\sigma_n \xi)\int_0^t u(0,\tau)e^{-\sigma_n^2(t-\tau)}\, d\tau + 2\sum_{n=0}^{\infty}\cos(\sigma_n x)\, e^{-\sigma_n^2 t}\int_0^1 \Phi_0(s,\xi)\cos(\sigma_n s)\, ds \quad (119)$$

where $\sigma_n = \pi(n+1/2)$. Multiplying (118) with $\cos(\sigma_m x)$ and using the orthogonality of these functions on [0,1] we can rewrite the Φ–filter in the form

$$\Phi(x,\xi,t) = -2\sum_{n=0}^{\infty}\cos(\sigma_n x)\cos(\sigma_n \xi) \times \int_0^1 \cos(\sigma_n s)\,\phi(s,t) ds + 2\sum_{n=0}^{\infty}\cos(\sigma_n x)\, e^{-\sigma_n^2 t}\int_0^1 \cos(\sigma_n s) \times (\phi_0(s)\cos(\sigma_n\xi) + \Phi_0(s,\xi))\, ds \quad (120)$$

Here the first term represents the explicit solution of the system (113)–(115) and the second term is the effect of filters' initial conditions. Therefore we can represent Φ as

$$\Phi(x,\xi,t) = F(x,\xi,t) + \Delta F(x,\xi,t)\,, \quad (121)$$

where ΔF satisfies

$$\Delta F_t = \Delta F_{xx} \quad (122)$$

$$\Delta F_x(0,\xi,t) = \Delta F(1,\xi,t) = 0\,, \quad (123)$$

and using (110)–(111) we get (116)–(117).

Lemma 6 allows us to avoid solving an infinite "array" of parabolic equations (104)–(105) by computing the solution of the standard wave equation (113)–(115) at each time step. Therefore we only have two dynamic equations to solve.

7.3 Update laws

We take the following equation as a parametric model

$$e(0) = v(0) - \psi(0) - \theta_1\phi(0) + \int_0^1 \theta(\xi)\phi(\xi)\, d\xi\,. \quad (124)$$

The estimation error is

$$\hat e(0) = v(0) - \psi(0) - \hat\theta_1\phi(0) + \int_0^1 \hat\theta(\xi)\phi(\xi)\, d\xi \quad (125)$$

We employ the gradient update laws with normalization

$$\hat\theta_t(x,t) = -\gamma(x)\frac{\hat e(0)\phi(x)}{1+\|\phi\|^2+\phi^2(0)} \quad (126)$$

$$\dot{\hat\theta}_1 = \gamma_1\frac{\hat e(0)\phi(0)}{1+\|\phi\|^2+\phi^2(0)}\,, \quad (127)$$

where $\gamma(x)$ and γ_1 are positive adaptation gains.

Lemma 7. The adaptive laws (126)–(127) guarantee the following properties:

$$\frac{\hat e(0)}{\sqrt{1+\|\phi\|^2+\phi^2(0)}} \in \mathcal{L}_2\cap\mathcal{L}_\infty \quad (128)$$

$$\|\tilde\theta\|,\ \tilde\theta_1 \in \mathcal{L}_\infty, \quad \|\hat\theta_t\|,\ \dot{\hat\theta}_1 \in \mathcal{L}_2\cap\mathcal{L}_\infty\,. \quad (129)$$

Proof. Using a Lyapunov function

$$V = \frac{1}{2}\|e\|^2 + \frac{1}{2\gamma_1}\tilde\theta_1^2 + \int_0^1 \frac{\tilde\theta^2(x)}{2\gamma(x)}\, dx \quad (130)$$

we get

$$\begin{aligned}\dot V &= -\int_0^1 e_x^2\, dx + \frac{\int_0^1 \tilde\theta(x)\phi(x)\, dx - \tilde\theta_1\phi(0)}{1+\|\phi\|^2+\phi^2(0)}\hat e(0)\\ &\le -\|e_x\|^2 + \frac{e(0)\hat e(0) - \hat e^2(0)}{1+\|\phi\|^2+\phi^2(0)}\\ &\le -\|e_x\|^2 + \frac{\|e_x\||\hat e(0)|}{\sqrt{1+\|\phi\|^2+\phi^2(0)}} - \frac{\hat e^2(0)}{1+\|\phi\|^2+\phi^2(0)}\\ &\le -\frac{1}{2}\|e_x\|^2 - \frac{1}{2}\frac{\hat e^2(0)}{1+\|\phi\|^2+\phi^2(0)}\,. \end{aligned} \quad (131)$$

This gives

$$\frac{\hat e(0)}{\sqrt{1+\|\phi\|^2+\phi^2(0)}} \in \mathcal{L}_2, \quad \|\tilde\theta\|,\ \tilde\theta_1 \in \mathcal{L}_\infty \quad (132)$$

The rest of the properties (128)–(129) follows from the relation $\hat e(0) = e(0) + \tilde\theta_1\phi(0) - \int_0^1 \tilde\theta(x)\phi(x)dx$ and the update laws.

7.4 Main result

Theorem 8. Consider the system (90)–(92) with the controller

$$u(1) = \int_0^1 \left(\psi(y) + \hat\theta_1\phi(y) + \int_0^1 F(y,\xi)\hat\theta(\xi)\, d\xi\right) \times \hat k(1,y)\, dy \quad (133)$$

where $\hat{k}(x,y) = \hat{\kappa}(x-y)$ with $\hat{\kappa}(x)$ determined from the equation

$$\hat{\kappa}'(x) = -\hat{\theta}_1\hat{\kappa}(x) - \hat{\theta}(x) + \int_0^x \hat{\kappa}(x-y)\hat{\theta}(y)\,dy \quad (134)$$

$$\hat{\kappa}(0) = \hat{\theta}_1\,, \quad (135)$$

the filters ϕ and ψ are given by (101)–(103), (106)–(108) and the update laws for $\hat{\theta}(x)$ and $\hat{\theta}_1$ are given by (126)–(127). If the closed loop system has a solution $(u,\,\phi,\,\psi,\,\hat{\theta},\,\hat{\theta}_1)$ with $u,\phi,\psi \in H_1(0,1)$ then for any $\hat{\theta}(x,0)$, $\hat{\theta}_1(0)$ and any initial conditions $u_0,\phi_0,\psi_0 \in H_1(0,1)$ the signals $\|\hat{\theta}\|$, $\hat{\theta}_1$, $\|u\|$, $\|\phi\|$, $\|\psi\|$ are bounded and $\|u\|$ is regulated to zero:

$$\lim_{t\to\infty} \|u\| = 0\,. \quad (136)$$

Proof (Sketch).

Denote

$$h(x) = \psi(x) + \hat{\theta}_1\phi(x) + \int_0^1 F(x,\xi)\hat{\theta}(\xi)\,d\xi \quad (137)$$

and use the following backstepping transformation

$$w(x) = h(x) - \int_0^x \hat{k}(x,y)h(y)\,dy := T[h](x)\,. \quad (138)$$

One can show that the inverse transformation to (138) is

$$h(x) = w(x) + \int_0^x \hat{l}(x,y)w(y)\,dy \quad (139)$$

where

$$\hat{l}(x,y) = \hat{\theta}_1 - \int_0^{x-y} \hat{\theta}(\xi)\,d\xi\,. \quad (140)$$

Using Lemma 6, the equations for the plant, filters ϕ and ψ, and the Volterra relationship between $\hat{l}$ and $\hat{k}$

$$\hat{l}(x,y) = \hat{k}(x,y) + \int_y^x \hat{l}(x,\xi)\hat{k}(\xi,y)\,d\xi\,, \quad (141)$$

one can derive the following target system

$$\begin{aligned} w_t &= w_{xx} + \hat{e}(0)\hat{k}_y(x,0) \\ &- \int_0^x w(y)\left(\hat{l}_t(x,y) - \int_y^x \hat{k}(x,\xi)\hat{l}_t(\xi,y)\,d\xi\right)dy \\ &+ \dot{\hat{\theta}}_1 T[\phi] + T\left[\int_0^1 F(x,\xi)\hat{\theta}_t(\xi)\,d\xi\right] \quad (142) \\ w_x(0) &= \hat{\theta}_1\hat{e}(0) \quad (143) \\ w(1) &= 0\,. \quad (144) \end{aligned}$$

Let us rewrite ϕ filter as

$$\phi_t = \phi_{xx} \quad (145)$$

$$\phi_x(0) = w(0) + \hat{e}(0) \quad (146)$$

$$\phi(1) = 0\,. \quad (147)$$

We now have interconnection of two systems ϕ and w with forcing terms that have properties (128)–(129).

Let us establish bounds on the gains $\hat{k}(x,y)$ and $\hat{l}(x,y)$. The boundedness of the parameter estimates $\hat{\theta}_1$ and $\|\hat{\theta}\|$ has been shown in Lemma 7. From (140) we get

$$|\hat{l}(x,y)| \le \bar{\theta}_1 + \bar{\theta}\,, \quad (148)$$

where we denote $\bar{\theta}_1 = \max_{t\ge 0}|\hat{\theta}_1|$ and $\bar{\theta} = \max_{t\ge 0}\|\hat{\theta}\|$.

Using (141) and Gronwall inequality it is easy to get the following bound

$$|\hat{k}(x,y)| \le (\bar{\theta}_1 + \bar{\theta})\mathrm{e}^{\bar{\theta}_1+\bar{\theta}} := K_1 \quad (149)$$

If we look at the right hand side of the w-system, we can see that the estimates for $\hat{k}_y(x,0)$ and $\hat{l}_t(x,y)$ are also needed. They are readily obtained from (134) and (140):

$$|\hat{k}_y(x,0)| \le (\bar{\theta}_1 + \bar{\theta})K_1 + \bar{\theta} := K_2 \quad (150)$$

$$|\hat{l}_t(x,y)| \le |\dot{\hat{\theta}}_1| + \|\hat{\theta}_t\|\,. \quad (151)$$

We are now ready to start with stability analysis of (142)–(147). Consider a Lyapunov function

$$V_1 = \frac{1}{2}\int_0^1 \phi^2\,dx\,. \quad (152)$$

Computing its derivative along the solutions of ϕ–system and using Young, Poincare, and Agmon inequalities, we get

$$\begin{aligned} \dot{V}_1 =& -\phi(0)w(0) - \phi(0)\hat{e}(0) - \int_0^1 \phi_x^2\,dx \\ \le& \frac{1}{2}w^2(0) + \frac{1}{2}\phi^2(0) - \|\phi_x\|^2 \\ &+ \frac{|\phi(0)\hat{e}(0)|}{1+\|\phi\|^2+\phi^2(0)}(1 + \|\phi\|^2 + 2\|\phi\|\|\phi_x\|) \\ \le& -\frac{1}{2}\|\phi_x\|^2 + \frac{1}{2}\|w_x\|^2 + c_1\|\phi_x\|^2 \\ &+ \frac{1}{4c_1}\frac{\hat{e}^2(0)}{1+\|\phi\|^2+\phi^2(0)} + \frac{c_1}{4}\|\phi\|^2 + c_1\|\phi_x\|^2 \\ &+ \frac{3}{c_1}\left(\frac{|\phi(0)\hat{e}(0)|}{1+\|\phi\|^2+\phi^2(0)}\right)^2\|\phi\|^2 \\ \le& -\left(\frac{1}{2} - 3c_1\right)\|\phi_x\|^2 + \frac{1}{2}\|w_x\|^2 + l_1\|\phi\|^2 + l_1\,. \end{aligned}$$

Here c_1 is a positive constant that will be chosen later and l_1 denotes a generic bounded and square integrable function of time.

With a Lyapunov function

$$V_2 = \frac{1}{2}\int_0^1 w^2\,dx \tag{153}$$

we get

$$\begin{aligned}\dot V_2 = &-\int_0^1 w_x^2\,dx + \hat e(0)\int_0^1 \hat k_y(x,0)w(x)\,dx\\ &+\int_0^1 w(x)T\left[\int_0^1 F(x,\xi)\hat\theta_t(\xi)\,d\xi\right]dx\\ &+\hat\theta_1 w(0)\hat e(0) + \dot{\hat\theta}_1\int_0^1 w(x)T[\phi](x)\,dx\\ &+\int_0^1 w(x)\int_0^x w(y)\\ &\times\left(\hat l_t(x,y) - \int_y^x \hat k(x,\xi)\hat l_t(\xi,y)\,d\xi\right)dy\,dx\end{aligned}$$

Separately estimating each term in the last equality, one can show

$$\begin{aligned}\dot V_2 \le &(c_4)\,\|\phi_x\|^2 + l_1\|w\|^2 + l_1\|\phi\|^2 + l_1\\ &-(1-c_2-c_3-4c_5-4c_7-8c_9)\,\|w_x\|^2\end{aligned} \tag{154}$$

For $V = V_1 + V_2$ we get

$$\begin{aligned}\dot V \le &-\left(\frac{1}{2}-c_2\right)\|w_x\|^2 - \left(\frac{1}{2}-3c_1-c_3\right)\|\phi_x\|^2\\ &+l_1\|w\|^2 + l_1\|\phi\|^2 + l_1\,.\end{aligned} \tag{155}$$

Choosing $c_2 = 1/5$, $3c_1 = 1/16$, $c_3 = 3/16$, we get

$$\dot V \le -\frac{1}{8}V + l_1 V + l_1 \tag{156}$$

and by Lemma 4 we get $\|w\|$, $\|\phi\| \in \mathcal{L}_2\cap\mathcal{L}_\infty$. From the transformation (139) we get $\|h\| \in \mathcal{L}_2\cap\mathcal{L}_\infty$ and therefore $\|\psi\| \in \mathcal{L}_2\cap\mathcal{L}_\infty$ follows from (137). From (112) and (93) we get $\|v\|$, $\|u\| \in \mathcal{L}_2\cap\mathcal{L}_\infty$.

It is easy to see from (156) that V is bounded from above. By using an alternative to Barbalat's lemma (Liu and Krstic 2001, Lemma 3.1) we get $V \to 0$, that is $\|\hat w\| \to 0$, $\|\phi\| \to 0$. From the transformation (139) we get $\|h\| \to 0$ and from (137) $\|\psi\| \to 0$ follows. From (112) and (93) we get $\|v\| \to 0$ and $\|u\| \to 0$. The proof of Theorem 8 is completed.

7.5 Reaction-Advection-Diffusion Systems

The approach presented in the paper can also be applied to general reaction–advection–diffusion system

$$\begin{aligned}u_t = &\varepsilon(x)u_{xx} + b(x)u_x + \lambda(x)u\\ &+g(x)u(0) + \int_0^x f(x,y)u(y)\,dy\end{aligned} \tag{157}$$

$$u_x(0) = -qu(0)\,, \tag{158}$$

where $\varepsilon(x)$, $b(x)$, $\lambda(x)$, $g(x)$, $f(x,y)$, q are unknown parameters.

The parameters $g(x)$, $f(x,y)$, and q can be easily handled because the observer canonical form (97)–(99) is not changed in this case, only the PDE (94)–(96) and expressions (100) for the new unknown parameters are modified. Since we are not concerned with identification, the adaptive scheme stays exactly the same.

With unknown parameters $b(x)$ and $\varepsilon(x)$, however, additional difficulties arise. The transformed plant is changed to

$$v_t = \theta_0 v_{xx} + \theta(x)v(0) \tag{159}$$

$$v_x(0) = \theta_1 v(0) \tag{160}$$

$$v(1) = \theta_2 u(1) \tag{161}$$

where the new constant parameters θ_0 and θ_2 appear due to $\varepsilon(x)$ and $b(x)$ respectively. We can see that one of the issues is the need of projection to keep the estimates of θ_0 and θ_2 positive since the filters should be stable and the controller is given as $u(1) = \hat\theta_2^{-1}v(1)$. This issue, although making the closed loop stability proof more challenging, does not pose a conceptual problem. The real difficulty comes from the fact that the parameter θ_0, which comes from the unknown $\varepsilon(x)$, multiplies the second derivative of the state which is not measured. Therefore, while an unknown $b(x)$ is allowed, $\varepsilon(x)$ should be known.

7.6 Simulations

We now present the results of numerical simulations of the designed adaptive scheme. The parameters of the plant (157)–(158) are taken to be $b(x) = 3 - 2x^2$ and $\lambda(x) = 16 + 3\sin(2\pi x)$, $\varepsilon \equiv 1$, $g(x) = q = 0$, so that the plant is unstable. The evolution of the closed loop state is shown in Fig. 3. We can see that the regulation is achieved. The parameter estimates, shown in Figs. 4–5, converge to some stabilizing values.

8. CONCLUSION

We presented several approaches to adaptive control of parabolic PDEs. In future work, the extension to hyperbolic PDEs (strings, beams, plates) will be considered. Another interesting problem is the identification of spatially-varying (functional) unknown parameters using only boundary sensing and actuation.

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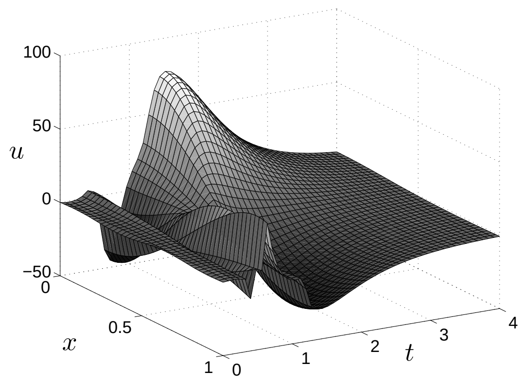

Fig. 3. The closed loop state $u(x, t)$.

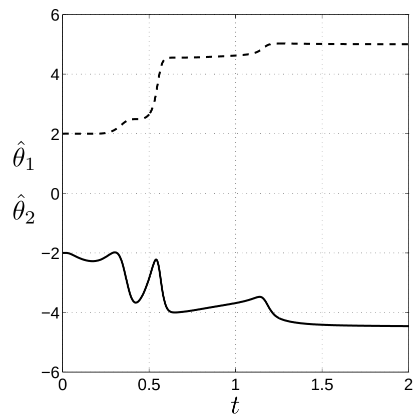

Fig. 4. The parameter estimates $\hat{\theta}_1(t)$ (solid) and $\hat{\theta}_2(t)$ (dashed).

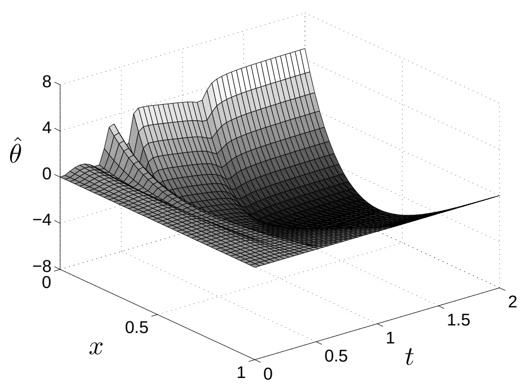

Fig. 5. The parameter estimate $\hat{\theta}(x, t)$.

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