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OPTIMAL BOUNDARY CONTROL OF STRING VIBRATIONS WITH INTERMEDIATE VALUES OF THE DEFLECTION FUNCTION UNDER MINIMIZATION OF THE BOUNDARY ENERGY INTEGRAL

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Abstract

The paper studies optimal boundary control problems with given intermediate values of the deflection function for the string vibration equation with given initial and final conditions. The boundary energy integral serves as the quality criterion. The controls are either a displacement at one end with the other end fixed or a displacement at both ends of the string. The quality criterion is set over the entire time interval. We rely on the methods of separation of variables and optimal control theory with multipoint intermediate conditions to propose a constructive approach for deriving an optimal boundary control of string vibrations. We also perform a computational experiment and analyze its results.

Key words

Boundary control, optimal control of vibrations, intermediate conditions, boundary energy integral, separation of variables.

1 Introduction

Mathematical models of many physical and technological processes with important theoretical and practical implications are described by the one-dimensional wave equation. In particular, the wave equation describes vibrations of strings, manipulator links, crane booms, airplane wings, and a number of other processes. Therefore, there is a need to study control and optimal control problems of oscillatory processes described by the wave equation which includes different problems statements and both lumped (boundary) and distributed controls. Such problems were discussed in [Butkovsky, 1965; Butkovsky, 1975; Sirazetdinov, 1977; Znamenskaya, 2004; Krabs, 1995; Yang et al., 2004; Li, 2008; Dreglea et al., 2018; Sadybekov et al., 2016; Ilyin et al., 2005; Moiseev et al., 2013; Abdukarimov, 2013; Abdukarimov, 2014; Gibkina et al., 2016; Arguchintsev et al., 2023; Arguchintsev et al., 2024]. Research monographs [Butkovsky, 1965; Butkovsky, 1975; Sirazetdinov, 1977; Znamenskaya, 2004; Krabs, 1995], to name just a few studies, provided an overview of the theory of control of distributed vibrations. Studies [Ilyin et al., 2005; Moiseev et al., 2013] (and other contributions by the same authors) dealt with the problem of boundary control (optimal control) of wave processes as part of the class of generalized solutions. Studies [Abdukarimov, 2013; Abdukarimov, 2014] considered the problems of optimal boundary control of displacements at the string ends based on minimization of the boundary energy integral, with such controls transferring the string vibration process from an arbitrarily given initial state to a given final state within an arbitrary and sufficiently large time interval. The above studies have contributed various methods for solving control and optimal control problems, such as the Fourier method, method of harmonics, and method of moments.

Many dynamic control processes pose multipoint boundary value control and optimal control problems where, along with classical (initial and final) boundary value conditions, multipoint intermediate (both separated and non-separated) conditions are also given [Aschepkov, 1981; Barseghyan, 2016; Korzyuk, 2010; Korzyuk, 2011; Barseghyan, 2021a; Barseghyan, 2019; Barseghyan, 2021b; Barseghyan, 2012; Barseghyan, 2022; Barseghyan, 2021c]. Studies [Korzyuk, 2010; Korzyuk, 2011] considered a boundary-value problem for the equation of string vibration with a given velocity at some point in time and constructed a solution to the problem. There is an extensive body of research, in particular studies [Barseghyan, 2021a; Barseghyan, 2019; Barseghyan, 2021b; Barseghyan, 2012; Barseghyan, 2022; Barseghyan, 2021c], on control and optimal control problems of oscillatory processes with multipoint intermediate (both separated and non-separated) conditions and both distributed and boundary controls under different types of boundary conditions. However, there is a gap in research of the problems of optimal boundary control of oscillatory processes with multipoint intermediate conditions with the functional being the integral of the squares of derivatives of boundary displacements, i.e., the boundary energy integral.

This paper seeks to design a constructive approach to deriving an optimal boundary control function for controlling string vibrations with given initial and final conditions as well as intermediate values of deflection of string points, where the quality criterion is the boundary energy integral specified over the entire time interval. We consider problems where the control is realized both by displacement of the left end when the right end is fixed and by displacement at the two ends of the string. The problems are reduced to distributed action control problems with zero boundary conditions. We construct the optimal boundary control by relying on the method of separation of variables and methods of the theory of optimal control of finite-dimensional systems with multipoint intermediate conditions for arbitrary numbers of first harmonics. We perform a computational experiment and present the resulting plots with their comparative analysis.

2 Problem Statement

Let the state of a distributed oscillating system (small transverse vibrations of a taut string), i.e., the deviation from the equilibrium state, be described by the function $Q(x,t)$, $0 \le x \le l$, $0 \le t \le T$, which conforms to the wave equation

$$\frac{\partial^2 Q}{\partial t^2} = a^2 \frac{\partial^2 Q}{\partial x^2}, \quad 0 < x < l, \quad t > 0, \tag{1}$$

with the following initial and final conditions

$$Q(x,0) = \varphi_0(x),\ \left.\frac{\partial Q}{\partial t}\right|_{t=0} = \psi_0(x),\ 0 \le x \le l, \tag{2}$$

$$\begin{gathered} Q(x,T) = \varphi_T(x) = \varphi_{m+1}(x), \\ \left.\frac{\partial Q}{\partial t}\right|_{t=T} = \psi_T(x),\ 0 \le x \le l, \end{gathered} \tag{3}$$

and boundary conditions:
with displacement of the left end while the right end is fixed

$$Q(0,t) = \mu(t),\ Q(l,t) = 0,\ 0 \le t \le T, \tag{4}$$

with displacement of both ends

$$Q(0,t) = \mu(t),\ Q(l,t) = \nu(t),\ 0 \le t \le T. \tag{5}$$

Here the functions $\mu(t)$ and $\nu(t)$ are the boundary controls, $a^2 = \frac{T_0}{\rho}$, where T_0 is the tension and ρ is the density of the string.

Let at intermediate time instants t_k $(k = 1,\ldots,m)$, $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T$,

$$Q(x,t_i) = \varphi_i(x),\ 0 \le x \le l,\ i = 1,\ldots,m. \tag{6}$$

The boundary energy integrals that are to be minimized have the following form:
with displacement of the left end while the right end is fixed

$$\int_0^T [\dot{\mu}(t)]^2\, dt, \tag{7}$$

with displacement of both ends

$$\int_0^T \left\{ [\dot{\mu}(t)]^2 + [\dot{\nu}(t)]^2 \right\} dt. \tag{8}$$

We assume that the function $Q(x,t) \in C^2(\Omega_T)$, where the set $\Omega_T = \{(x,t) : x \in [0,l],\ t \in [0,T]\}$, functions $\varphi_i(x) \in C^2[0,l]$ $(i = 0,1,\ldots,m+1)$, and functions $\psi_0(x)$ and $\psi_T(x)$ belong to the space $C^1[0,l]$. We also assume that all functions are such that the following consistency conditions are satisfied.

For the problems with displacement of the left end while the right end is fixed:

$$\begin{gathered} \varphi_0(0) = \mu(0),\ \psi_0(0) = \dot{\mu}(0), \\ \varphi_0(l) = \psi_0(l) = 0, \\ \mu(t_i) = \varphi_i(0),\ i = 1,\ldots,m, \\ \varphi_T(0) = \mu(T),\ \psi_T(0) = \dot{\mu}(T), \\ \varphi_T(l) = \psi_T(l) = 0. \end{gathered} \tag{9}$$

For the problems with displacement of both ends:

$$\begin{gathered} \varphi_0(0) = \mu(0),\ \psi_0(0) = \dot{\mu}(0), \\ \nu(0) = \varphi_0(l),\ \dot{\nu}(0) = \psi_0(l), \\ \mu(t_i) = \varphi_i(0),\ \varphi_i(l) = \nu(t_i),\ i = 1,\ldots,m, \\ \varphi_T(0) = \mu(T),\ \psi_T(0) = \dot{\mu}(T), \\ \varphi_T(l) = \nu(T),\ \psi_T(l) = \dot{\nu}(T). \end{gathered} \tag{10}$$

Let us state the following problem of optimal boundary control of string vibrations.

Problem 1 (displacement of the left end while the right end is fixed). It is required to find an optimal boundary control $\mu^0(t)$, $0 \le t \le T$ that transfers the oscillatory motion of the system (1) from a given initial state (2) to a final state (3), while satisfying condition (6) and minimizing functional (7).

Problem 2 (displacement of both ends). It is required to find optimal boundary controls and $\mu^0(t)$ and $\nu^0(t)$, $0 \le t \le T$ that transfer the oscillatory motion of the system (1) from a given initial state (2) to a final state (3), while satisfying condition (6) and minimizing functional (8).

The paper contributes a constructive approach for solving the investigated optimal control problems that properly consider intermediate conditions while minimizing the boundary energy integrals.

3 Reduction of Original Problems to Problems with Zero Boundary Conditions

The stated problems with non-homogeneous (non-zero) boundary conditions ((4) or (5)) with homogeneous equation (1) are reduced to optimal control problems with distributed controls (non-homogeneous equation) with zero boundary conditions [Tikhonov, 2011]. Details of the above approach with all the proofs are omitted from the paper. However, a proper presentation of further derivations of the solutions to the problems warrant the inclusion of some of the formulas.

3.1 Reduction of Non-homogeneous Boundary Conditions to Zero Boundary Conditions

The solution to equation (1) is to be found in the form

$$Q(x,t) = V(x,t) + W(x,t), \tag{11}$$

where $V(x,t)$ is an unknown function with boundary conditions

$$V(0,t) = V(l,t) = 0. \tag{12}$$

Given the boundary conditions $Q(0,t) = \mu(t)$, $Q(l,t) = 0$:

$$W(0,t) = \mu(t),\ W(l,t) = 0. \tag{13}$$

Given the boundary conditions $Q(0,t) = \mu(t)$, $Q(l,t) = \nu(t)$:

$$W(0,t) = \mu(t),\ W(l,t) = \nu(t). \tag{14}$$

The function $W(x,t)$ for boundary conditions (13) and (14), respectively, is represented as

$$W(x,t) = \left(1 - \frac{x}{l}\right)\mu(t), \tag{15}$$

$$W(x,t) = (\nu(t) - \mu(t))\frac{x}{l} + \mu(t). \tag{16}$$

To determine the function $V(x,t)$ we obtain the equation

$$\frac{\partial^2 V}{\partial t^2} = a^2 \frac{\partial^2 V}{\partial x^2} + F(x,t), \tag{17}$$

where for Problem 1 with the function $W(x,t)$ of type (15)

$$F(x,t) = \left(\frac{x}{l} - 1\right)\ddot{\mu}(t), \tag{18}$$

and for Problem 2 with the function $W(x,t)$ of type (16)

$$F(x,t) = (\ddot{\mu}(t) - \ddot{\nu}(t))\frac{x}{l} - \ddot{\mu}(t). \tag{19}$$

3.2 The Reduction of Initial, Intermediate, and Final Conditions to the Corresponding Conditions for the Non-homogeneous Equation

Given the expressions for the function $W(x,t)$ (15), (16) and the consistency conditions, we obtain the corresponding conditions for the function $V(x,t)$ from the initial (2), intermediate (6), and final conditions (3).

In the case of the problem of boundary control of string vibrations by displacement of the left end while the right end is fixed, i.e. for the function $V(x,t)$, we obtain the following initial conditions:

$$\begin{aligned} V(x,0) &= \varphi_0(x) + \left(\frac{x}{l} - 1\right)\varphi_0(0), \\ \left.\frac{\partial V}{\partial t}\right|_{t=0} &= \psi_0(x) + \left(\frac{x}{l} - 1\right)\psi_0(0), \end{aligned} \tag{20}$$

intermediate conditions:

$$\begin{gathered} V(x,t_i) = \varphi_i(x) + \left(\frac{x}{l} - 1\right)\varphi_i(0), \\ i = 1,\dots,m, \end{gathered} \tag{21}$$

final conditions:

$$\begin{aligned} V(x,T) &= \varphi_T(x) + \left(\frac{x}{l} - 1\right)\varphi_T(0), \\ \left.\frac{\partial V}{\partial t}\right|_{t=T} &= \psi_T(x) + \left(\frac{x}{l} - 1\right)\psi_T(0). \end{aligned} \tag{22}$$

For the problems of boundary control of string vibrations by displacement of two ends, i.e., for the function $V(x,t)$, we obtain the following initial conditions:

$$\begin{aligned} V(x,0) &= \varphi_0(x) - (\varphi_0(l) - \varphi_0(0))\frac{x}{l} - \varphi_0(0), \\ \left.\frac{\partial V}{\partial t}\right|_{t=0} &= \psi_0(x) - (\psi_0(l) - \psi_0(0))\frac{x}{l} - \psi_0(0), \end{aligned} \tag{23}$$

intermediate conditions:

$$\begin{gathered} V(x,t_i) = \varphi_i(x) - (\varphi_i(l) - \varphi_i(0))\frac{x}{l} - \varphi_i(0), \\ i = 1,\dots,m, \end{gathered} \tag{24}$$

final conditions

$$\begin{aligned} V(x,T) &= \varphi_T(x) - (\varphi_T(l) - \varphi_T(0))\frac{x}{l} - \varphi_T(0), \\ \left.\frac{\partial V}{\partial t}\right|_{t=T} &= \psi_T(x) - (\psi_T(l) - \psi_T(0))\frac{x}{l} - \psi_T(0). \end{aligned} \tag{25}$$

Thus, we obtain the following optimal control problems with zero boundary conditions.

Problem 1_0 (displacement of the left end while the right end is fixed). It is required to find the optimal boundary control $\mu^0(t)$, $0 \le t \le T$, that transfers the oscillatory motion described by equation (17), (18) with boundary conditions (12), from a given initial state (20) to a final state (22), while satisfying intermediate conditions (21) and minimizing functional (7).

Problem 2_0 (displacement of both ends). It is required to find optimal boundary controls $\mu^0(t)$ and $\nu^0(t)$, $0 \le t \le T$ that transfer the oscillatory motion described by equation (17), (19) with boundary conditions (12), from a given initial state (23) to a final state (25), while satisfying intermediate conditions (24) and minimizing functional (8).

4 Application of the Method of Separation of Variables and Reduction of the Solution to a Moment Problem

The solution to the equation (17) is to be found in the form

$$V(x,t) = \sum_{k=1}^{\infty} V_k(t) \sin \frac{\pi k}{l} x. \tag{26}$$

Let us represent the functions $F(x,t)$, $\varphi_i(x)$ ($i = 0, 1, \dots, m+1$), $\psi_0(x)$, and $\psi_T(x)$ as Fourier series; by substituting their values together with $V(x,t)$ in equations (17), (18), (19) and in conditions (23)–(25) we obtain

$$\begin{gathered} \ddot{V}_k^{(s)}(t) + \lambda_k^2 V_k^{(s)}(t) = F_k^{(s)}(t), \\ \lambda_k^2 = \left(\frac{a\pi k}{l}\right)^2, \ s = 1,2, \ k = 1,2,\dots, \end{gathered} \tag{27}$$

$$F_k^{(1)}(t) = -\frac{2a}{\lambda_k l}\ddot{\mu}(t), \tag{28}$$

$$F_k^{(2)}(t) = \frac{2a}{\lambda_k l}\left[(-1)^k \ddot{\nu}(t) - \ddot{\mu}(t)\right]. \tag{29}$$

Here and in what follows the letter "s", in the superscript means "'Problem 1"' if $s = 1$ and "'Problem 2"' if $s = 2$.

In the case of problems with the displacement of the left end and the fixed right end, the initial, intermediate, and final conditions are represented as

$$\begin{gathered} V_k^{(1)}(0) = \varphi_k^{(0)} - \frac{2a}{\lambda_k l}\varphi_0(0), \\ \dot{V}_k^{(1)}(0) = \psi_k^{(0)} - \frac{2a}{\lambda_k l}\psi_0(0), \end{gathered} \tag{30}$$

$$V_k^{(1)}(t_i) = \varphi_k^{(i)} - \frac{2a}{\lambda_k l}\varphi_i(0), \ i = 1,\dots,m, \tag{31}$$

$$\begin{gathered} V_k^{(1)}(T) = \varphi_k^{(T)} - \frac{2a}{\lambda_k l}\varphi_T(0), \\ \dot{V}_k^{(1)}(T) = \psi_k^{(T)} - \frac{2a}{\lambda_k l}\psi_T(0). \end{gathered} \tag{32}$$

In the case of problems with two-end displacement, the initial, intermediate, and final conditions are represented as

$$\begin{gathered} V_k^{(2)}(0) = \varphi_k^{(0)} - \frac{2a}{\lambda_k l}\left[\varphi_0(0) - (-1)^k \varphi_0(l)\right], \\ \dot{V}_k^{(2)}(0) = \psi_k^{(0)} - \frac{2a}{\lambda_k l}\left[\psi_0(0) - (-1)^k \psi_0(l)\right], \end{gathered} \tag{33}$$

$$\begin{gathered} V_k^{(2)}(t_i) = \varphi_k^{(i)} - \frac{2a}{\lambda_k l}\left[\varphi_i(0) - (-1)^k \varphi_i(l)\right], \\ i = 1,\dots,m, \end{gathered} \tag{34}$$

$$\begin{gathered} V_k^{(2)}(T) = \varphi_k^{(T)} - \frac{2a}{\lambda_k l}\left[\varphi_T(0) - (-1)^k \varphi_T(l)\right], \\ \dot{V}_k^{(2)}(T) = \psi_k^{(T)} - \frac{2a}{\lambda_k l}\left[\psi_T(0) - (-1)^k \psi_T(l)\right]. \end{gathered} \tag{35}$$

where $F_k^{(s)}(t)$, $V_k^{(s)}(t)$, $\varphi_k^{(i)}$, $\psi_k^{(0)}$, and $\psi_k^{(T)}$, $s = 1,2$, denote the Fourier coefficients of the functions $F(x,t)$, $V(x,t)$, $\varphi_i(x)$, $\psi_0(x)$, and $\psi_T(x)$, respectively.

The general solution to equation (27) and its derivative have the form:

$$\begin{gathered} V_k^{(s)}(t) = V_k^{(s)}(0)\cos\lambda_k t + \\ + \frac{1}{\lambda_k}\dot{V}_k^{(s)}(0)\sin\lambda_k t + \\ + \frac{1}{\lambda_k}\int_0^t F_k^{(s)}(\tau)\sin\lambda_k(t-\tau)d\tau, \\ \dot{V}_k^{(s)}(t) = -\lambda_k V_k^{(s)}(0)\sin\lambda_k t + \\ + \dot{V}_k^{(s)}(0)\cos\lambda_k t + \\ + \int_0^t F_k^{(s)}(\tau)\cos\lambda_k(t-\tau)d\tau, \ s = 1,2. \end{gathered} \tag{36}$$

Given the initial, intermediate and final conditions, we obtain from (36) that the functions $F_k(\tau)$ for each k must satisfy the following integral relations:

$$\begin{gathered} \int_0^T F_k^{(s)}(\tau)\sin\lambda_k(T-\tau)d\tau = \tilde{C}_{1k}^{(s)}(T), \\ \int_0^T F_k^{(s)}(\tau)\cos\lambda_k(T-\tau)d\tau = \tilde{C}_{2k}^{(s)}(T), \\ \int_0^{t_i} F_k^{(s)}(\tau)\sin\lambda_k(t_i-\tau)\,d\tau = \tilde{C}_{1k}^{(s)}(t_i), \\ i = 1,\dots,m, \ s = 1,2, \end{gathered} \tag{37}$$

where the following two notations are introduced:

$$
\begin{gathered}
\tilde{C}_{1k}^{(s)}(T) = \lambda_k V_k^{(s)}(T) - \\
-\lambda_k V_k^{(s)}(0)\cos\lambda_k T - \dot{V}_k^{(s)}(0)\sin\lambda_k T, \\
\tilde{C}_{2k}^{(s)}(T) = \dot{V}_k^{(s)}(T) + \\
+\lambda_k V_k^{(s)}(0)\sin\lambda_k T - \dot{V}_k^{(s)}(0)\cos\lambda_k T, \\
\tilde{C}_{1k}^{(s)}(t_i) = \lambda_k V_k^{(s)}(t_i) - \\
-\lambda_k V_k^{(s)}(0)\cos\lambda_k t_i - \dot{V}_k^{(s)}(0)\sin\lambda_k t_i, \\
i = 1, \ldots, m, \ s = 1, 2.
\end{gathered}
\tag{38}
$$

In the case of the problem with displacement of the left end and the fixed right end, substituting the expression of the function $F_k^1(t)$ from (28) into relations (37) and integrating it by parts, subject to consistency conditions (9), we obtain that the functions $\dot{\mu}(t)$ for each k must satisfy the following integral relations:

$$
\begin{gathered}
\int_0^T \dot{\mu}(\tau)\cos\lambda_k(T-\tau)\,d\tau = C_{1k}^{(1)}(T), \\
\int_0^T \dot{\mu}(\tau)\sin\lambda_k(T-\tau)\,d\tau = C_{2k}^{(1)}(T), \\
\int_0^T \dot{\mu}(\tau)h_k^{(1)}(\tau)\,d\tau = C_{1k}^{(1)}(t_1), \\
\cdots \\
\int_0^T \dot{\mu}(\tau)h_k^{(m)}(\tau)\,d\tau = C_{1k}^{(1)}(t_m),
\end{gathered}
\tag{39}
$$

where

$$
\begin{gathered}
C_{1k}^{(1)}(T) = \frac{\psi_0(0)}{\lambda_k}\sin\lambda_k T - \frac{l}{2a}\tilde{C}_{1k}^{(1)}(T), \\
C_{2k}^{(1)}(T) = \frac{\psi_T(0)}{\lambda_k} - \frac{\psi_0(0)}{\lambda_k}\cos\lambda_k T + \frac{l}{2a}\tilde{C}_{2k}^{(1)}(T), \\
C_{1k}^{(1)}(t_i) = \frac{\psi_0(0)}{\lambda_k}\sin\lambda_k t_i - \frac{l}{2a}\tilde{C}_{1k}^{(1)}(t_i), \\
h_k^{(i)}(\tau) = \begin{cases} \cos\lambda_k(t_i-\tau), & 0 \le \tau \le t_i, \\ 0, & t_i < \tau \le T, \end{cases} \\
i = 1, \ldots, m.
\end{gathered}
\tag{40}
$$

To make integral relations (39) more readable, we introduce the following notations:

$$
\begin{gathered}
\bar{H}_k^{(1)}(\tau) = \begin{pmatrix} \cos\lambda_k(T-\tau) \\ \sin\lambda_k(T-\tau) \\ h_k^{(1)}(\tau) \\ \cdots \\ h_k^{(m)}(\tau) \end{pmatrix}, \\
C_k^{(1)}(t_1, \ldots, t_m, T) = \begin{pmatrix} C_{1k}^{(1)}(T) \\ C_{2k}^{(1)}(T) \\ C_{1k}^{(1)}(t_1) \\ \cdots \\ C_{1k}^{(1)}(t_m) \end{pmatrix}.
\end{gathered}
\tag{41}
$$

Next, for the case of the two-end displacement problem, substituting the expression of the function $F_k^2(t)$ from (29) into relations (37) and integrating it by parts, subject to consistency conditions (10), we obtain that the functions $\dot{\mu}(t)$ and $\dot{\nu}(t)$ for each k must satisfy the following integral relations:

$$
\begin{gathered}
\int_0^T \dot{\mu}(\tau)\cos\lambda_k(T-\tau)\,d\tau - \\
-(-1)^k \int_0^T \dot{\nu}(\tau)\cos\lambda_k(T-\tau)\,d\tau = C_{1k}^{(2)}(T), \\
\int_0^T \dot{\mu}(\tau)\sin\lambda_k(T-\tau)\,d\tau - \\
-(-1)^k \int_0^T \dot{\nu}(\tau)\sin\lambda_k(T-\tau)\,d\tau = C_{2k}^{(2)}(T), \\
\int_0^T \dot{\mu}(\tau)h_k^{(i)}(\tau)\,d\tau - \\
-(-1)^k \int_0^T \dot{\nu}(\tau)h_k^{(i)}(\tau)\,d\tau = C_{1k}^{(2)}(t_i),
\end{gathered}
\tag{42}
$$

where

$$
\begin{gathered}
C_{1k}^{(2)}(T) = \frac{1}{\lambda_k}\left[-\frac{\lambda_k l}{2a}\tilde{C}_{1k}(T) - (-1)^k\psi_0(l)\sin\lambda_k T + \right. \\
\left. +\psi_0(0)\sin\lambda_k T\right], \\
C_{2k}^{(2)}(T) = \frac{1}{\lambda_k}\left[\frac{\lambda_k l}{2a}\tilde{C}_{2k}^{(2)}(T) + (-1)^k\psi_0(l)\cos\lambda_k T + \right. \\
\left. +\psi_T(0) - (-1)^k\psi_T(l) - \psi_0(0)\cos\lambda_k T\right], \\
C_{1k}^{(2)}(t_i) = \frac{1}{\lambda_k}\left[-\frac{\lambda_k l}{2a}\tilde{C}_{1k}(t_i) - (-1)^k\psi_0(l)\sin\lambda_k t_i + \right. \\
\left. +\psi_0(0)\sin\lambda_k t_i\right], \ i = 1, \ldots, m.
\end{gathered}
\tag{43}
$$

Note that expressions $\tilde{C}_{1k}^{(s)}(T)$, $\tilde{C}_{2k}^{(s)}(T)$, $\tilde{C}_{1k}^{(s)}(t_i)$ and $h_{1k}^{(i)}(\tau)$ are given in (38) and (40).

To make integral relations (42) more readable, we introduce the following notations:

$$\bar{H}_k^{(2)}(\tau) = \begin{pmatrix} \cos\lambda_k(T-\tau) & (-1)^{k+1}\cos\lambda_k(T-\tau) \\ \sin\lambda_k(T-\tau) & (-1)^{k+1}\sin\lambda_k(T-\tau) \\ h_k^{(1)}(\tau) & (-1)^{k+1}h_k^{(1)}(\tau) \\ \dots & \dots \\ h_k^{(m)}(\tau) & (-1)^{k+1}h_k^{(m)}(\tau) \end{pmatrix},$$

$$C_k^{(2)}(t_1,\dots,t_m,T) = \begin{pmatrix} C_{1k}^{(2)}(T) \\ C_{2k}^{(2)}(T) \\ C_{1k}^{(2)}(t_1) \\ \dots \\ C_{1k}^{(2)}(t_m) \end{pmatrix}. \tag{44}$$

In practice, several first n harmonics of vibrations are usually selected and the problem of control synthesis is solved by the methods of control theory of finite-dimensional systems. Consequently, in what follows our constructions will be in line with this approach.

Then, given introduced notations (41) and (44), relations (39) and (42) for the first n harmonics will be written as follows:

$$\int_0^T H_n^{(s)}(\tau)\, U_n^{(s)}(\tau)d\tau = \eta_n^{(s)},\ s=1,2. \tag{45}$$

where $U_n^1(\tau) = \dot{\mu}_n^1(\tau) = \dot{\mu}(\tau)$,

$$U_n^{(2)}(\tau) = \begin{pmatrix} \dot{\mu}_n^{(2)}(\tau) \\ \dot{\nu}_n^{(2)}(\tau) \end{pmatrix} = \begin{pmatrix} \dot{\mu}(\tau) \\ \dot{\nu}(\tau) \end{pmatrix},$$

$$H_n^{(s)}(\tau) = \begin{pmatrix} \bar{H}_1^{(s)}(\tau) \\ \bar{H}_2^{(s)}(\tau) \\ \dots \\ \bar{H}_n^{(s)}(\tau) \end{pmatrix},$$

$$\eta_n^{(s)} = \begin{pmatrix} C_1^{(s)}(t_1,\dots,t_m,T) \\ C_2^{(s)}(t_1,\dots,t_m,T) \\ \dots \\ C_n^{(s)}(t_1,\dots,t_m,T) \end{pmatrix} \tag{46}$$

with dimensions $H_n^{(s)}(\tau) - (n(m+2)\times s)$, $\eta_n^{(s)} - (n(m+2)\times 1)$.

Thus, integral conditions (39) and (42) obtained for Problem 1_0 and Problem 2_0, respectively, are represented by condition (45).

From (45) it follows that the first n harmonics of system (27) with conditions (30)–(32) or (33)–(35) are completely controllable if and only if for any vector $\eta_n^{(s)}$ (46) one can find a control $U_n^{(s)}(t)$, $t\in[0,T]$, satisfying condition (45).

5 Solution to the Problems

Note that the left-hand part of condition (45) is a linear operation generated by the control function $U_n^{(s)}(\tau)$ on the time interval $[0, T]$, and functionals (7) or (8) are norms of some space L_2.

Optimal control problems with integral condition (45) given functional (7) or (8) can be treated as a moment problem, and a solution to these problems should be sought with the use of the moment problem [Krasovsky, 1968]. To solve the finite-dimensional (for $k = 1,2,\dots,n$) moment problem for Problem 1 with functional (7) and integral conditions (39) (or (3.19) for $s=1$) one needs to find the values $p_k^{(1)}$, $q_k^{(1)}$, $\gamma_{ik}^{(1)}$, $k=1,\dots,n$, $i=1,\dots,m$, linked by the condition

$$\sum_{k=1}^{n}\Big[p_k^{(1)}C_{1k}^{(1)}(T) + q_k^{(1)}C_{2k}^{(1)}(T) + \\ + \sum_{i=1}^{m}\gamma_{ik}^{(1)}C_{1k}^{(1)}(t_i)\Big] = 1, \tag{47}$$

for which

$$(\rho_{1n})^2 = \min_{(47)} \int_0^T (h_{1n})^{(2)}(\tau)d\tau, \tag{48}$$

where

$$h_{1n}(\tau) = \sum_{k=1}^{n}\Big[p_k^{(1)}\cos\lambda_k(T-\tau) + \\ + q_k^{(1)}\sin\lambda_k(T-\tau) + \sum_{i=1}^{m}\gamma_{ik}^{(1)}h_k^{(i)}(\tau)\Big]. \tag{49}$$

To determine the values $p_k^{(1)0}$, $q_k^{(1)0}$, $\gamma_{ik}^{(1)0}$, $k=1,\dots,n$, that minimize (48) and satisfy (47), we introduce the function

$$f_{1n} = \int_0^T \big(h_{1n}(\tau)\big)^2 d\tau + \beta_{1n}\Big[\sum_{k=1}^{n}\Big(p_k^{(1)}C_{1k}^{(1)}(T) + \\ + q_k^{(1)}C_{2k}^{(1)}(T) + \sum_{i=1}^{m}\gamma_{ik}^{(1)}C_{1k}^{(1)}(t_i)\Big) - 1\Big],$$

where β_{1n} is the undetermined Lagrange multiplier. Based on this method, we calculate the derivatives of the function f_{1n} with respect to $p_k^{(1)}$, $q_k^{(1)}$, $\gamma_{ik}^{(1)}$, $k = 1,\dots,n$, $i=1,\dots,m$, set them to zero, given notation (49), (40), and add condition (47) to the obtained equations to arrive at the closed system $2n+mn+1$ of algebraic equations with respect to the same unknown quantities $p_k^{(1)}$, $q_k^{(1)}$, $\gamma_{ik}^{(1)}$, $k=1,\dots,n$, $i=1,\dots,m$, and β_{1n}:

$$\sum_{j=1}^{n}\left[a_{jk}^{(1)}p_j^{(1)}+b_{jk}^{(1)}q_j^{(1)}+\sum_{\alpha=1}^{m}c_{jk}^{(1\alpha)}\gamma_{\alpha j}^{(1)}\right]= -\frac{\beta_{1n}}{2}C_{1k}^{(1)}(T),$$
$$\sum_{j=1}^{n}\left[d_{jk}^{(1)}p_j^{(1)}+e_{jk}^{(1)}q_j^{(1)}+\sum_{\alpha=1}^{m}f_{jk}^{(1\alpha)}\gamma_{\alpha j}^{(1)}\right]= -\frac{\beta_{1n}}{2}C_{2k}^{(1)}(T),$$
$$\sum_{j=1}^{n}\left[a_{jk}^{(1i)}p_j^{(1)}+b_{jk}^{(1i)}q_j^{(1)}+\sum_{\alpha=1}^{m}g_{jk}^{(1\alpha i)}\gamma_{\alpha j}^{(1)}\right]= -\frac{\beta_{1n}}{2}C_{1k}^{(1)}(t_i), \tag{50}$$
$$\sum_{k=1}^{n}\left[p_k^{(1)}C_{1k}^{(1)}(T)+q_k^{(1)}C_{2k}^{(1)}(T)+ \sum_{i=1}^{m}\gamma_{ik}^{(1)}C_{1k}^{(1)}(t_i)\right]=1,$$
$$k=1,\dots,n,\ i=1,\dots,m,$$

where

$$a_{jk}^{(1)}=\int_0^T\cos\lambda_j(T-\tau)\cos\lambda_k(T-\tau)\,d\tau,$$
$$b_{jk}^{(1)}=\int_0^T\sin\lambda_j(T-\tau)\cos\lambda_k(T-\tau)\,d\tau,$$
$$c_{jk}^{(1\alpha)}=\int_0^T h_j^{(\alpha)}(\tau)\cos\lambda_k(T-\tau)\,d\tau,$$
$$d_{jk}^{(1)}=\int_0^T\cos\lambda_j(T-\tau)\sin\lambda_k(T-\tau)\,d\tau,$$
$$e_{jk}^{(1)}=\int_0^T\sin\lambda_j(T-\tau)\sin\lambda_k(T-\tau)\,d\tau, \tag{51}$$
$$f_{jk}^{(1\alpha)}=\int_0^T h_j^{(\alpha)}(\tau)\sin\lambda_k(T-\tau)\,d\tau,$$
$$a_{jk}^{(1i)}=\int_0^T\cos\lambda_j(T-\tau)h_k^{(i)}(\tau)\,d\tau,$$
$$b_{jk}^{(1i)}=\int_0^T\sin\lambda_j(T-\tau)h_k^{(i)}(\tau)\,d\tau,$$
$$g_{jk}^{(1\alpha i)}=\int_0^T h_j^{(\alpha)}(\tau)h_k^{(i)}(\tau)\,d\tau.$$

Let the quantities $p_k^{(1)0}$, $q_k^{(1)0}$, $\gamma_{ik}^{(1)0}$, $k=1,\dots,n$, $i=1,\dots,m$, and β_{1n}^0 be the solution to the closed system of algebraic equations (50). Then, by (49), (48), we have

$$h_{1n}^0(\tau)=\sum_{k=1}^{n}\left[p_k^{(1)0}\cos\lambda_k(T-\tau)+ +q_k^{(1)0}\sin\lambda_k(T-\tau)+\sum_{i=1}^{m}\gamma_{ik}^{(1)0}h_k^{(i)}(\tau)\right], \tag{52}$$
$$(\rho_{1n}^0)^2=\int_0^T\left(h_{1n}^0(\tau)\right)^2d\tau.$$

Optimal functions $\dot\mu_n^{(1)0}(\tau)$ for any $n=1,2,\dots$ are represented in the form

$$\dot\mu_n^{(1)0}(\tau)=\frac{1}{(\rho_{1n}^0)^2}h_{1n}^0(\tau).$$

Then it follows that

$$\mu_n^{(1)0}(t)=\frac{1}{(\rho_{1n}^0)^2}\int_0^t h_{1n}^0(\tau)d\tau+S_1,\ t\in[0,\,T], \tag{53}$$

where S_1 is the constant of integration. Given that $\mu_n^{(1)0}(0)=S_1$, from consistency conditions (9) we have $S_1=\varphi_0(0)$.

Thus, the function of the optimal displacement function of the left end of the string, or the optimal boundary control $\mu_n^{(1)0}(\tau)$, $\tau\in[0,\,T]$, by equations (40), (52), and (53), is represented as:
for $t_{j-1}\le t<t_j$, $j=1,2,\dots,m$, $t_0=0$:

$$\mu_n^{(1)0}(t)= =\frac{1}{(\rho_{1n}^0)^2}\sum_{k=1}^{n}\frac{1}{\lambda_k}\left[F_{jk}^{(1)}\left(p_k^{(1)0},q_k^{(1)0},\gamma_{jk}^{(1)0},\lambda_k,T,t_i,t\right)+ + G_{jk}^{(1)}\left(p_k^{(1)0},q_k^{(1)0},\gamma_{jk}^{(1)0},\lambda_k,T,t_i\right)\right]+\varphi_0(0), \tag{54}$$

where

$$F_{jk}^{(1)}\left(p_k^{(1)0},q_k^{(1)0},\gamma_{jk}^{(1)0},\lambda_k,T,t_i,t\right)= =-p_k^{(1)0}\sin\lambda_k(T-t)+q_k^{(1)0}\cos\lambda_k(T-t)- -\sum_{i=1}^{j}\gamma_{ik}^{(1)0}\sin\lambda_k(t_i-t),$$
$$G_{jk}^{(1)}\left(p_k^{(1)0}q_k^{(1)0},\gamma_{jk}^{(1)0},\lambda_k,T,t_i\right)= =p_k^{(1)0}\sin\lambda_kT-q_k^{(1)0}\cos\lambda_kT+\sum_{i=1}^{j}\gamma_{ik}^{(1)0}\sin\lambda_kt_i,$$

and for $t_m \le t \le t_{m+1} = T$:

$$\mu_n^{(1)0}(t) = \frac{1}{(\rho_{1n}^0)^2} \sum_{k=1}^{n} \frac{1}{\lambda_k} \Big[-p_k^{(1)0} \sin \lambda_k (T-t) + \\ + q_k^{(1)0} \cos \lambda_k (T-t) + p_k^{(1)0} \sin \lambda_k (T-t_m) - \\ - q_k^{(1)0} \cos \lambda_k (T-t_m) \Big] + \varphi_0(0). \tag{55}$$

Thus, the solution to Problem 1 is represented by equations (54) and (55).

To solve the finite-dimensional (given $k = 1, 2, \ldots, n$) moment problem for Problem 2 with functional (8) and integral conditions (42) (or (45) for $s = 2$) one needs to find the quantities p_k^2, q_k^2, γ_{ik}^2, $k = 1, 2, \ldots, n$, $i = 1, \ldots, m$, linked by the condition

$$\sum_{k=1}^{n} \Big[p_k^{(2)} C_{1k}^{(2)}(T) + q_k^{(2)} C_{2k}^{(2)}(T) + \\ + \sum_{i=1}^{m} \gamma_{ik}^{(2)} C_{1k}^{(2)}(t_i) \Big] = 1, \tag{56}$$

for which

$$(\rho_{2n})^2 = \min_{(56)} \int_0^T \left[h_{1n}^2(\tau) + h_{2n}^2(\tau) \right] d\tau, \tag{57}$$

where

$$h_{1n}(\tau) = \sum_{k=1}^{n} \Big[p_k^{(2)} \cos \lambda_k (T-\tau) + \\ + q_k^{(2)} \sin \lambda_k (T-\tau) + \sum_{i=1}^{m} \gamma_{ik}^{(2)} h_k^{(i)}(\tau) \Big], \\ h_{2n}(\tau) = \sum_{k=1}^{n} (-1)^{k+1} \Big[p_k^{(2)} \cos \lambda_k (T-\tau) + \\ + q_k^{(2)} \sin \lambda_k (T-\tau) + \sum_{i=21}^{m} \gamma_{ik}^{(2)} h_k^{(i)}(\tau) \Big]. \tag{58}$$

Similarly, solving the problem of minimizing functional (57), subject to condition (56), we find the required quantities $p_k^{(2)0}$, $q_k^{(2)0}$, $\gamma_{ik}^{(2)0}$, $k = 1, 2, \ldots, n$, $i = 1, \ldots, m$. Then, from (57) and (58) we have:

$$(\rho_{2n}^0)^2 = \int_0^T \Big[\left(h_{1n}^0(\tau) \right)^2 + \left(h_{2n}^0(\tau) \right)^2 \Big] d\tau, \\ h_{1n}^0(\tau) = \sum_{k=1}^{n} \Big[p_k^{(2)0} \cos \lambda_k (T-\tau) + \\ + q_k^{(2)0} \sin \lambda_k (T-\tau) + \sum_{i=1}^{m} \gamma_{ik}^{(2)0} h_k^{(i)}(\tau) \Big], \\ h_{2n}^0(\tau) = \sum_{k=1}^{n} (-1)^{k+1} \Big[p_k^{(2)0} \cos \lambda_k (T-\tau) + \\ + q_k^{(2)0} \sin \lambda_k (T-\tau) + \sum_{i=1}^{m} \gamma_{ik}^{(2)0} h_k^{(i)}(\tau) \Big]. \tag{59}$$

Thus, the functions of optimal string ends displacement, i.e., the optimal boundary controls $\mu_n^{(2)0}(t)$ and $\nu_n^{(2)0}(t)$, $\tau \in [0, T]$, by equations (40) and (59), are represented as:
for $t_{j-1} \le t < t_j$, $j = 1, 2, \ldots, m$, $t_0 = 0$:

$$\mu_n^{(2)0}(t) = \frac{1}{(\rho_{2n}^0)^2} \times \\ \times \sum_{k=1}^{n} \frac{1}{\lambda_k} \Big[F_{jk}^{(2)} \left(p_k^{(2)0}, q_k^{(2)0}, \gamma_{jk}^{(2)0}, \lambda_k, T, t_i, t \right) + \\ + G_{jk}^{(2)} \left(p_k^{(2)0}, q_k^{(2)0}, \gamma_{jk}^{(2)0}, \lambda_k, T, t_i \right) \Big] + \varphi_0(0), \\ \nu_n^{(2)0}(t) = \frac{1}{(\rho_{2n}^0)^2} \times \\ \times \sum_{k=1}^{n} \frac{(-1)^{k+1}}{\lambda_k} \Big[F_{jk}^{(2)} \left(p_k^{(2)0}, q_k^{(2)0}, \gamma_{jk}^{(2)0}, \lambda_k, T, t_i, t \right) + \\ + G_{jk}^{(2)} \left(p_k^{(2)0}, q_k^{(2)0}, \gamma_{jk}^{(2)0}, \lambda_k, T, t_i \right) \Big] + \varphi_0(l), \tag{60}$$

where

$$F_{jk}^{(1)} \left(p_k^{(1)0}, q_k^{(1)0}, \gamma_{jk}^{(1)0}, \lambda_k, T, t_i, t \right) = \\ = -p_k^{(1)0} \sin \lambda_k (T-t) + q_k^{(1)0} \cos \lambda_k (T-t) - \\ - \sum_{i=1}^{j} \gamma_{ik}^{(1)0} \sin \lambda_k (t_i - t), \\ G_{jk}^{(1)} \left(p_k^{(1)0}, q_k^{(1)0}, \gamma_{jk}^{(1)0}, \lambda_k, T, t_i \right) = \\ = p_k^{(1)0} \sin \lambda_k T - q_k^{(1)0} \cos \lambda_k T + \sum_{i=1}^{j} \gamma_{ik}^{(1)0} \sin \lambda_k t_i,$$

and for $t_m \le t \le t_{m+1} = T$:

$$\mu_n^{(2)0}(t) = \frac{1}{(\rho_{2n}^0)^2} \sum_{k=1}^{n} \frac{1}{\lambda_k} \Big[-p_k^{(2)0} \sin \lambda_k (T-t) + \\ + q_k^{(2)0} \cos \lambda_k (T-t) + p_k^{(2)0} \sin \lambda_k (T-t_m) - \\ - q_k^{(2)0} \cos \lambda_k (T-t_m) \Big] + \varphi_0(0), \\ \nu_n^{(2)0}(t) = \frac{1}{(\rho_{2n}^0)^2} \sum_{k=1}^{n} \frac{(-1)^{k+1}}{\lambda_k} \Big[-p_k^{(2)0} \sin \lambda_k (T-t) + \\ + q_k^{(2)0} \cos \lambda_k (T-t) + p_k^{(2)0} \sin \lambda_k (T-t_m) - \\ - q_k^{(2)0} \cos \lambda_k (T-t_m) \Big] + \varphi_0(l). \tag{61}$$

The solution to Problem 2 is represented by equations (60) and (61).

Thus, having explicit forms of the optimal boundary control function $\mu_n^{(s)0}(t)$ and $\nu_n^{(s)0}(t)$, $t \in [0,T]$, $s = 1, 2$, we can construct the corresponding deflection function $Q_n^{(s)0}(x,t)$. Substituting the obtained expressions for the optimal controls $\mu_n^{(s)0}(t)$ and $\nu_n^{(s)0}(t)$, $\tau \in [0,T]$, $s = 1, 2$ into (28) and (29), and substituting the expression obtained for $F_k^{(s)0}(t)$ into (36), we obtain the function $V_k^{(s)0}(t)$, $t \in [0,T]$, $s = 1, 2$, $k = 1, \dots, n$. Then, from (26) we have

$$V_n^{(s)0}(x,t) = \sum_{k=1}^{n} V_k^{(s)0}(t) \sin \frac{\pi k}{l} x, \tag{62}$$

where

$$V_k^{(s)0}(t) = V_k^{(s)}(0) \cos \lambda_k t + \frac{1}{\lambda_k} \dot{V}_k^{(s)}(0) \sin \lambda_k t + \\ + \frac{1}{\lambda_k} \int_0^t F_k^{(s)0}(\tau) \sin \lambda_k (t-\tau) d\tau,$$

and from (15) and (16) it follows that functions $W_n^{(s)0}(x,t)$, $s = 1, 2$, take the form:

$$\begin{gathered} W_n^{(1)0}(x,t) = (1 - \frac{x}{l}) \mu_n^{(1)0}(t), \\ W_n^{(2)0}(x,t) = \\ = \left[\nu_n^{(2)0}(t) - \mu_n^{(2)0}(t)\right] \frac{x}{l} + \mu_n^{(2)0}(t). \end{gathered} \tag{63}$$

Then, by (11), given (62) and (63), for Problem 1 and Problem 2, respectively, we have

$$\begin{gathered} Q_n^{(1)0}(x,t) = \sum_{k=1}^{n} V_k^{(1)0}(t) \sin \frac{\pi k}{l} x + \\ + (1 - \frac{x}{l}) \mu_n^{(1)0}(t), \\ Q_n^{(2)0}(x,t) = \sum_{k=1}^{n} V_k^{(2)0}(t) \sin \frac{\pi k}{l} x + \\ + \left[\nu_n^{(2)0}(t) - \mu_n^{(2)0}(t)\right] \frac{x}{l} + \mu_n^{(2)0}(t). \end{gathered} \tag{64}$$

Thus, for the first n harmonics, the optimal string deflection functions $Q_n^{(1)0}(x,t)$ and $Q_n^{(2)0}(x,t)$ are represented by formula (64).

6 Example (with a Computational Experiment)

Next, we illustrate the above for Problem 1 for $m = 1$. Suppose that for an intermediate instant of time t_1 ($0 = t_0 < t_1 < t_2 = T$) the values of deflection of the points (shape) of the string is specified in the form

$$Q(x,t_1) = \varphi_1(x), \ 0 \le x \le l. \tag{65}$$

Applying the approach detailed above, we construct the optimal boundary control $\mu_n^0(t)$ for $n = 1$ (hence, $k = 1$). In this case, to determine the values of the quantities p_1, q_1, γ_{11}, and β_1, by (50) and (51) ($i = 1$, $\alpha = 1$), we have the following system of algebraic equations:

$$\begin{gathered} a_{11} p_1 + b_{11} q_1 + c_{11}^{(1)} \gamma_{11} = -\frac{\beta_1}{2} C_{11}(T), \\ d_{11} p_1 + e_{11} q_1 + f_{11}^{(1)} \gamma_{11} = -\frac{\beta_1}{2} C_{21}(T), \\ a_{11}^{(1)} p_1 + b_{11}^{(1)} q_1 + g_{11}^{(11)} \gamma_{11} = -\frac{\beta_1}{2} C_{11}(t_1), \\ C_{11}(T) p_1 + C_{21}(T) q_1 + C_{11}(t_1) \gamma_{11} = 1, \end{gathered} \tag{66}$$

where

$$a_{11} = \int_0^T \cos \lambda_1 (T-\tau) \cos \lambda_1 (T-\tau)\, d\tau = \\ = \frac{T}{2} + \frac{1}{4\lambda_1} \sin 2\lambda_1 T,$$

$$b_{11} = d_{11} = \int_0^T \cos \lambda_1 (T-\tau) \sin \lambda_1 (T-\tau)\, d\tau = \\ = \frac{1}{2\lambda_1} \ \sin^2 \lambda_1 T,$$

$$e_{11} = \int_0^T \sin \lambda_1 (T-\tau) \sin \lambda_1 (T-\tau)\, d\tau = \\ = \frac{T}{2} - \frac{1}{4\lambda_1} \sin 2\lambda_1 T,$$

$$a_{11}^{(1)} = c_{11}^{(1)} = \int_0^T h_1^{(1)}(\tau) \cos \lambda_1 (T-\tau)\, d\tau = \\ = \frac{1}{2\lambda_1} \ \sin \lambda_1 t_1 \cos \lambda_1 T + \frac{t_1}{2} \cos \lambda_1 (T - t_1),$$

$$b_{11}^{(1)} = f_{11}^{(1)} = \int_0^T h_1^{(1)}(\tau) \sin \lambda_1 (T-\tau)\, d\tau = \\ = \frac{1}{2\lambda_1} \ \sin \lambda_1 t_1 \ \sin \lambda_1 T + \frac{t_1}{2} \sin \lambda_1 (T - t_1),$$

$$g_{11}^{(11)} = \int_0^T h_1^{(1)}(\tau)\, h_1^{(1)}(\tau)\, d\tau = \\ = \frac{t_1}{2} + \frac{1}{4\lambda_1} \sin 2\lambda_1 t_1.$$

In this section, to make the notation more readable we omit the superscript $C_{11}(T)$, $C_{21}(T)$, $C_{11}(t_1)$, which stands for Problem 1, i.e., $s = 1$.

Below we present the results of calculations for a specific example. We test the performance of the proposed approach by comparing the behavior of the string deflection function and its derivative to the corresponding functions that are given. For the sake of definiteness, assume that $t_1 = \frac{2l}{a}$, $T = \frac{4l}{a}$, $\lambda_1 = \frac{a\pi}{l}$, and, respectively,

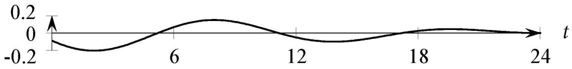

Figure 1. Graph of the function $V_1^0(t)$.

$t_1\lambda_1 = 2\pi$, $T\lambda_1 = 4\pi$, $\lambda_1(T - t_1) = 2\pi$. Assume that $a = \dfrac{1}{6}$, $l = 1$. For the chosen values of a, l we have $t_1 = 12, T = 24, \lambda_1 = \dfrac{\pi}{6}$. Let the following initial state be specified for $t = 0$:

$$\varphi_0(x) = 0.5x^3 - 0.4x^2 - 0.1x,$$
$$\psi_0(x) = 0.4x^2 - 0.4x,$$

for $t_1 = 12$, an intermediate state is given:

$$\varphi_1(x) = 0.2x^2 - 0.2x,$$

and the following final states are given for $T = 24$:

$$\varphi_T(x) = 0,\ \psi_T(x) = 0.$$

The coefficients of the Fourier series for the above functions $\varphi_0\,(x)$, $\psi_0\,(x)$, $\varphi_1\,(x)$, $\varphi_T\,(x)$, $\psi_T\,(x)$ for $n = 1$ are defined by the following relations:

$$\varphi_1^{(0)} = -\frac{14}{5\pi^3},\ \psi_1^{(0)} = -\frac{16}{5\pi^3},\ \varphi_1^{(1)} = -\frac{8}{5\pi^3},$$
$$\varphi_1^{(T)} = 0,\ \psi_1^{(T)} = 0.$$

The following relations hold for the selected functions:

$$\varphi_0(0) = \varphi_0(1) = \psi_0(0) = \psi_0(1) = \varphi_1(0) =$$
$$= \varphi_1(1) = \varphi_T(0) = \psi_T(0) = \varphi_T(1) = \psi_T(1) = 0,$$

that satisfy the consistency conditions.

From (30)–(32) we have

$$V_1(0) = -\frac{14}{5\pi^3},\ \dot{V}_1(0) = -\frac{16}{5\pi^3},\ V_1(t_1) = -\frac{8}{5\pi^3},$$
$$V_1(T) = 0,\ \dot{V}_1(T) = 0,$$
$$C_{11}(T) = -\frac{7}{5\pi^2},\ C_{11}(t_1) = -\frac{3}{5\pi^2},\ C_{21}(T) = \frac{48}{5\pi^3}.$$

Therefore, solving the system of algebraic equations obtained by (50), we have

$$p_1^0 = -\frac{20\pi^4}{25\pi^2 + 1152},\ q_1^0 = \frac{120\pi^3}{25\pi^2 + 1152},$$
$$\gamma_1^0 = \frac{5\pi^4}{25\pi^2 + 1152},\ \beta_1^0 = -\frac{300\pi^6}{25\pi^2 + 1152}.$$

Then, by equations (54), (55), we determine the displacement function $\mu_1^0(t)$, i.e., the optimal boundary control, which has the following explicit form:

$$\mu_1^0\,(t) = -\frac{24}{5\pi^4} - \frac{3}{5\pi^3}\sin\frac{\pi}{6}t + \frac{24}{5\pi^4}\cos\frac{\pi}{6}t,$$
$$t \in [0, 12),$$
$$\mu_1^0\,(t) = -\frac{24}{5\pi^4} - \frac{4}{5\pi^3}\sin\frac{\pi}{6}t + \frac{24}{5\pi^4}\cos\frac{\pi}{6}t,$$
$$t \in [12, 24].$$

The boundary energy integral (7) takes the following value:

$$\int_0^{24}\left[\left(\mu_1^0\,(t)\right)'\right]^2 dt \approx 0.0097.$$

We obtain the following explicit expressions for the functions $V_1^0(t)$, $t \in [0, 24]$:

$$\begin{gathered}
V_1^0(t) = \left(-\frac{14}{5\pi^3} + \frac{t}{10\pi^3}\right)\cos\frac{\pi}{6}t+ \\
+\left(-\frac{99}{5\pi^4} + \frac{4t}{5\pi^4}\right)\sin\frac{\pi}{6}t,\ t \in [0, 12), \\
V_1^0(t) = \left(-\frac{48}{15\pi^3} + \frac{2t}{15\pi^3}\right)\cos\frac{\pi}{6}t+ \\
\left(-\frac{100}{5\pi^4} + \frac{4t}{5\pi^4}\right)\sin\frac{\pi}{6}t,\ t \in [12, 24].
\end{gathered} \tag{67}$$

Figure 1 presents the graph of the function $V_1^0(t)$, $t \in [0, 24]$. It shows that the forced vibration represented by equation (27), whose solution is represented by formula (67), as a result of damping satisfies the final condition.

Under the influence of the constructed optimal boundary control $\mu_1^0(t)$ the corresponding optimal string deflection function is determined by equation (64) and has the following explicit form:

$$\begin{gathered}
Q_1^0(x,t) = \left[\left(-\frac{14}{5\pi^3} + \frac{t}{10\pi^3}\right)\cos\frac{\pi}{6}t+\right. \\
\left.+\left(-\frac{99}{5\pi^4} + \frac{4t}{5\pi^4}\right)\sin\frac{\pi}{6}t\right]\sin\pi x+ \\
+(1-x)\left(-\frac{24}{5\pi^4} - \frac{3}{5\pi^3}\sin\frac{\pi}{6}t + \frac{24}{5\pi^4}\cos\frac{\pi}{6}t\right), \\
t \in [0, 12), \\
Q_1^0(x,t) = \left[\left(-\frac{48}{15\pi^3} + \frac{2t}{15\pi^3}\right)\cos\frac{\pi}{6}t+\right. \\
\left.+\left(-\frac{100}{5\pi^4} + \frac{4t}{5\pi^4}\right)\sin\frac{\pi}{6}t\right]\sin\pi x+ \\
+(1-x)\left(-\frac{24}{5\pi^4} - \frac{4}{5\pi^3}\sin\frac{\pi}{6}t + \frac{24}{5\pi^4}\cos\frac{\pi}{6}t\right), \\
t \in [12, 24].
\end{gathered}$$

Hence, from the obtained expressions it follows that at the initial, intermediate and final instants of time ($t =$

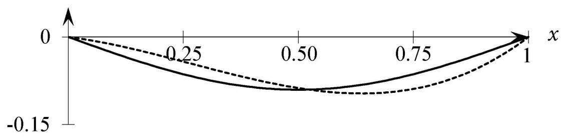

Figure 2. Graphs of the functions $Q_1^0(x,0)$ (dotted line) and $\varphi_0(x)$ (solid line).

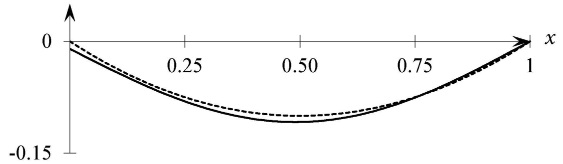

Figure 3. Graphs of the functions $\dot{Q}_1^0(x,0)$ (dotted line) and $\psi_0(x)$ (solid line).

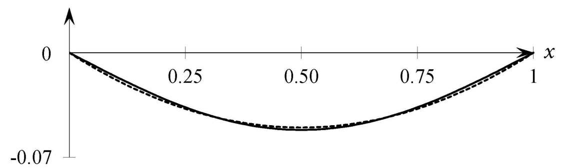

Figure 4. Graphs of the functions $Q_1^0(x,12)$ (dotted line) and $\varphi_1(x)$ (solid line).

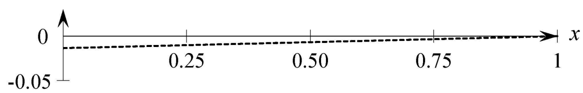

Figure 5. Graph of the function $\dot{Q}_1^0(x,24)$.

$0,\ 12,\ 24)$ the explicit expressions of functions $Q_1^0(x,t)$ and $\dot{Q}_1^0(x,t)$ have the following form:

$$Q_1^0(x,0) = -\frac{14}{5\pi^3}\sin\pi x,$$
$$\dot{Q}_1^0(x,0) = -\frac{16}{5\pi^3}\sin\pi x - \frac{1}{10\pi^2}(1-x),$$
$$Q_1^0(x,12) = -\frac{8}{5\pi^3}\sin\pi x,$$
$$Q_1(x,24) = 0,$$
$$\dot{Q}_1^0(x,24) = -\frac{2}{15\pi^2}(1-x).$$

The graphical representations of functions $Q_1^0(x,0)$ and $\varphi_0(x)$, $\dot{Q}_1^0(x,0)$ and $\psi_0(x)$, $Q_1^0(x,12)$ and $\varphi_1(x)$, $\dot{Q}_1^0(x,24)$ are shown in Figs. 2–5.

To compare the deviation of the constructed functions from the given ones and to analyze the obtained results, we introduce the following notations:

$$\varepsilon(x,t_j) = \left|Q_1^0(x,t_j) - \varphi_j(x)\right|,$$
$$\mathrm{E}(x,t_j) = \max_{0\le x\le 1}\varepsilon(x,t_j),$$
$$\widehat{\varepsilon}\ (x,t_\delta) = \left|\dot{Q}_1^0(x,t_\delta) - \psi_\delta(x)\right|,$$
$$\widehat{\mathrm{E}}\ (x,t_\delta) = \max_{0\le x\le 1}\widehat{\varepsilon}\ (x,t_\delta),$$

where $j = 0,1,2$, $\delta = 0,2$ (the instant of time T corresponds to $t_j = t_\delta = t_2$, i.e., $j = \delta = 2$). The following values were obtained when evaluating the discrepancy between the above functions:

$$\mathrm{E}(x,0) \approx 0.02696,$$
$$\int_0^1 \varepsilon\,(x,\,0)\,dx \approx 0.01569,$$
$$\widehat{\mathrm{E}}\ (x,0) \approx 0.01013,$$
$$\int_0^1 \widehat{\varepsilon}\ (x,\,0)\,dx \approx 0.00506,$$
$$\widehat{\mathrm{E}}\ (x,12) \approx 0.00021,$$
$$\int_0^1 \widehat{\varepsilon}\ (x,12)\,dx \approx 0.00124,$$
$$\widehat{\mathrm{E}}\ (x,24) \approx 0.01351,$$
$$\int_0^1 \widehat{\varepsilon}\ (x,24)\,dx \approx 0.00675.$$

Thus, for $n = 1$, the above calculations and comparative analysis of the results showed that the behavior of the string deflection functions and their derivatives under the influence of the constructed optimal boundary control was close enough to the given functions. Therefore, we conclude that the proposed constructive approach can be used for practical applications since the absolute and integral values of the maximum deviation prove quite acceptable.

7 Conclusion

We have contributed a constructive method for deriving an optimal boundary control of the oscillation process of a homogeneous string with given intermediate conditions on the values of the deflection function. The quality criterion is the integral of the boundary energy specified over the entire time interval. The results obtained here can be instrumental in constructing optimal boundary controls of vibration processes in physical and engineering systems.

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