On the Robust Control of Chaos in Cournot Economic Model with Complementary Goods

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Abstract
In this paper a robust controller is designed for stabilizing unstable fixed points of an uncertain chaotic economics system. The robust control scheme is presented and the performance of the proposed approach is examined by applying it to simple Cournot dynamic model with nonlinear reaction functions of by assuming complements goods. Simulation results show the effectiveness and feasibility of the method for chaos control in uncertain chaotic markets.

Key words
Chaos, Chaos Control, Quasi-sliding mode, Chaotic Economics Systems

1 Introduction

So many phenomena take place in real world which can no longer be illustrated with the aids of simple models. Economical systems are one of the kinds. It is unanimously accepted that economy belongs to very complex systems which usually take the form of being either stochastic or deterministic. It is due to the fact that the linear models could no longer anticipate the ongoing events of the system; hence the economists started developing complex systems for macroeconomic and microeconomic theories.
Goodwin probably was the first economist to realize the importance of nonlinear mechanism for the economical systems [Y. Takeuchi and T. Yamamura 2004] as one of his works; Goodwin enhanced and reduced the deficiencies of the Keynesian fiscal policies. Kaldor model is a very distinguished one since he presented a model in which the capital stock changes are caused by past investment decisions [Kaldor N. 1940]. The main characteristic of the model is the nonlinearity of investment and saving function. Also Kalecki introduced a lag connected with a time delay needed for new capital to be installed [Kalecki M. 1935]. Up to now, many researchers spending time on the matter [Majumdar, M., T. Mitra and K. Nishimura, 2000] for overview of nonlinear dynamic theory in economics, [K. Ishiyama and Y. Saiki 2005] for their quantitative analysis of Keynes-Goodwin macroeconomic growth model, [M. Szydlowski and A. Krawiec 2006] for investigating the stability problem of the Kaldor-Kalecki business cycle and so on.
Macroeconomic division pays attention to global agendas, such as business cycle, market mechanisms, and economical growth and in custom all the issues related to the micro milieus [Day, R. 1994]. However on the other hand the microeconomic models deal with the interrelationships of the inner entities of a bigger economical environment. The demeanor of the entities forms the overall behavior of the system. Thus the nexus between the entities would be a worth noting matter.
The economical systems due to their complex dynamics usually evince chaotic manners. Researches on chaotic behavior and its effect on economical systems have attracted a great deal of research interests in recent years. For the first time, states that chaotic dynamics can exist in an economics model [R.H. Strotz, J.C. Mcanulty and J.B. Naines 1953]. Also, some researches show that even oligopolistic markets, which first proposed by Cournot in 1838, may exhibit chaotic behavior under certain conditions [M. Kopel 1997 and T. Puu 2000]. In another work [D. Rand 1978], the author shows that a Cournot duopoly model with non-monotonic reaction functions may give rise to chaotic dynamic. [T. Puu 1991]

presents a microeconomic foundation which supports uni-modal reaction function. He shows that a meaningful uni-modal reaction function can be developed under assumptions of the profit maximization with the hyperbolic market demand and linear cost function. A simple Cournot dynamic model with nonlinear reaction function is presented by assumption of complements goods [A. Matsumoto and Y. Nonaka March 2004]. It states that when the nonlinearity gets strong, the model shows chaotic fluctuations.

Due to the chaotic manners in economical systems, the controllability of such systems would be a matter to deal with. The efficiency and the physical meanings of the controllers on the system is an open question for researchers, despite the ongoing developments. Whether the outcome of a chaotic system would be more beneficial than the simple one is a problem, which by no means can be elucidated nor be explained in details. Eliminating the chaotic behavior by means of controlling systems is an interesting subject in chaos theory. In the recent years, chaos control has been used in many theoretical and engineering problems and many papers have been published over the past two decades about controlling chaos in systems ranging from economics to biology with various methods of control. The first document in control of chaotic systems, named OGY, returns to 1990s [E. Ott, C. Grebogi JA. York 1990 and T. Shinbort, E. Ott, C. Grebogi, J. Yorke 1990]. In this scheme, an analytical method using low energy control signals was introduced for stabilizing Unstable Periodic Orbits (UPO) in a chaotic attractor. The main disadvantage of this method is its dependency to time of filling the chaotic attractor by trajectories. This method was successfully applied for chaos control in some economic systems [E. Ahmed, S.Z. Hassan 2000 and H.Z. Agiza 1999] Although the OGY method is well understood from the theoretical point of view, its experimental implementations are seriously limited by the fact that all the quantities needed to calculate the values of a system control parameter are not directly given in an experimental data chain and to perform the control, one needs to apply a complex data analysis. In contrast to the OGY method, the chaos control introduced by Pyragas, based on delayed feedback control, can be easily applied to experimental system when the equations of motions are not known [K. Pyragas 1992]. This approach also uses the recurrence of chaotic trajectories. Pyragas method has been extensively employed for stabilizing periodic solutions of nonlinear systems [G. Chen, X. Yu 1999 and Arecchi F.T., Boccaletti S., Ciofini M., and Meucci R. 1998]. Stabilizing the first order fixed point of the Cournot duopoly via delayed feedback control has been investigated in [Chen L, Chen G. 2007].

One can easily use the OGY or the Pyragas method to design stabilizing controllers while the governing equations of such systems are exactly known. However, determining an exact governing equation for a system is not possible as the parameters always have uncertainties. Thus, using a robust control strategy in such systems seems to be vital. Sliding mode control, a famous nonlinear robust method, initially developed for continuous time systems [Khalil H 2001]. The modified method, called the quasi-sliding mode control, was presented for adapting the sliding mode method to use in discrete time systems [X. Yu and S. Yu 2000, D. Munoz and D. Sbarbro 2000]. The main advantage of the new strategy is its ability to control chaos when the actual parameters of market are not available or have some uncertainties.

In this paper the effect of a robust chaos controller on an economic model has been fathomed. The chaotic dynamic of the system itself has been explained fully in details by A. Matsumoto and Y. Nonaka [A. Matsumoto and Y. Nonaka March 2004]. A quasi-sliding mode controller is designed in order to stabilize the system on its first periodic orbit assuming some uncertainties in the system. Simulation results show that the proposed techniques can be successfully implemented for chaos suppression in the system, even when the system is subjected into uncertainties.

2 The Economic Model

The dynamical system model of Cournot Model with Complementary Goods can be obtained based on the model suggested in A. Matsumoto and Y. Nonaka March 2004. This model explains the ongoing dynamical behavior between two-market economy with complementary goods *x* and *y*. On the demand side, the inverse demands for two markets are given below:

$$\begin{aligned} p_1(x,y) &= \alpha_1{}^2 - \beta_1 x + (\gamma y)^2 \\ p_2(x,y) &= \alpha_2{}^2 - \beta_2 y + (\gamma x)^2 \end{aligned} \tag{1}$$

where *x* and *y* are market prices of the goods and the coefficients are non-negative ones. The profit can be formulated with the aids of above price equations and production cost functions. The production cost functions are formulated as:

$$\begin{aligned} C_1(x,y) &= c_1 xy \\ C_2(x,y) &= c_2 xy \end{aligned} \tag{2}$$

Hence the profit functions can be formulated as:

$$\begin{aligned} \Pi_1(x,y) &= p_1(x,y)x - c_1 xy \\ \Pi_2(x,y) &= p_2(x,y)y - c_2 xy \end{aligned} \tag{3}$$

Firm 1 maximizes $\Pi_1(x,y)$ with respect *x* and so does firm 2 maximizes its profit with respect to *y*. The condition for profit maximization is the equality between the marginal revenue and the marginal cost. Solving the condition for its output, each firm derives a reaction function relating it to its complementary good. The reaction functions are:

$$\begin{aligned} r_1(y) &= \arg\max_x \Pi_1(x,y) \\ r_2(x) &= \arg\max_y \Pi_2(x,y) \end{aligned} \tag{4}$$

and hence,

$$r_1(y) = \frac{\alpha_1{}^2 - c_1 y + (\gamma_1 y)^2}{2\beta_1}$$
$$r_2(x) = \frac{\alpha_2{}^2 - c_2 x + (\gamma_2 x)^2}{2\beta_2} \quad (5)$$

As it can be seen the reaction functions takes on various shapes depending on the values of sale and production externalities, in another word depending on γ_i and c_i ; $i = (1,2)$. At this point the sales and production externalities functions are chosen in order to maintain the characteristics of the convexity of the reaction functions. The set values of parameters are as followings:

$$c_i = 2\alpha_i\gamma_i$$
$$\gamma_1 = \alpha \;\&\; \gamma_2 = \beta \quad (6)$$
$$\alpha_1 = \alpha - 1, \alpha_2 = 1 \;\&\; \beta_1 = \beta_2 = \frac{1}{2}$$

We are interested in the dynamic interactions between the two firms. To consider the dynamic of the process, the variables are lagged. This lag can be best described with the aids of 2-dimensional logistic maps. Using the above set values for parameters, the logistic maps are described by the followings:

$$H(x_t, y_t) = \begin{cases} x_{t+1} = (\alpha y_t - \alpha + 1)^2 \\ y_{t+1} = (\beta x_t - 1)^2 \end{cases} \quad (7)$$

To have economically meaningful system, the domain of parameters should be restricted [A. Matsumoto and Y. Nonaka March 2004]. Thus, when the parameters of the system is restricted to

$$A \equiv \{(\alpha, \beta) \mid 0 \le \alpha \le 2 \,\text{and}\, 0 \le \beta \le 2\} \quad (8)$$

H is a two dimensional transformation of the unit set. This parameter restriction prevents trajectory divergence in the system.

Also it is known that, under certain conditions for the values of α and β, the above described system exhibits chaotic behavior.

3 Chaos Control

In this section, first a Quasi-sliding mode control scheme is presented. Then, by using the proposed method, the robust control of such system is investigated. Finally, some simulation is performed to validate the controller.

3.1 Quasi-Sliding Mode Control

In this sub-section, the concept of sliding mode control in discrete systems, known as the quasi-sliding mode, is presented based on the scheme introduced in [D. Munoz and D. Sbarbro 2000].

Consider the discrete nonlinear single-input single-output (SISO) plant:

$$x_{t+1} = f_t(x_t, \ldots, x_{t-n}, u_{t-1}, \ldots, u_{t-b}) + g_t(x_t, \ldots, x_{t-c}, u_{t-1}, \ldots, u_{t-d})u_t \quad (9)$$

Where x_{t+1} is the output, u_t is the control input, $f_t(.)$ and $g_t(.)$ are smooth nonlinear functions of past values of the input and output, and the constants *n*, *b*, *c*, and *d* are all positive integers.

For this system, it is assumed that the function $g_t(.)$ is bounded away from zero. Let us assume that a model for (9)exists, in the form of

$$\hat{x}_{t+1} = \hat{f}_t(x_t, \ldots, x_{t-n}, u_{t-1}, \ldots, u_{t-b}) + \hat{g}_t(x_t, \ldots, x_{t-c}, u_{t-1}, \ldots, u_{t-d})u_t \quad (10)$$

where $\hat{f}_t(.)$ and $\hat{g}_t(.)$ are estimated functions of $f_t(.)$ and $g_t(.)$, respectively. The switching function is defined as a linear combination of the past tracking errors as:

$$S_t = e_t + \alpha_1 e_{t-1} + \ldots + \alpha_p e_{t-p} \quad (11)$$

The coefficients $\alpha_1, \ldots, \alpha_p$ in (11) are selected to make the switching function a stable linear combination of the past tracking errors. In other words, all of the roots of the polynomial $z^p + \alpha_1 z^{p-1} + \cdots + \alpha_p$ should lie inside the unit circle. One can always select α_1 such that $|\alpha_1| < 1$, and here it is assumed that $|\alpha_1| < 1$. This way, after a finite period of time, the tracking error will converge towards zero.

A fixed point of (9) is denoted by x^f and satisfies the following relation;

$$x_{t+1}^f = x_t^f = f_t(x_t^f, x_{t-1}^f, \ldots, x_{t-n}^f, 0, \ldots, 0) \quad (12)$$

Thus, the tracking error is defined as:

$$e_t = x_t - x_t^f \quad (13)$$

In contrary to the continuous systems, "sliding mode" cannot be achieved for discrete systems and hence we have to try reaching a quasi-sliding mode. Some concepts regarding this matter can be found in [H.Z. Agiza 1999]. A system is said to be in a quasi-sliding mode when the dynamics of S_k meets the following set of conditions:

I) Starting from any state, the S_k sequence moves toward the quasi-sliding surface, defined by $S_k=0$, and crosses it in a finite period of time.

II) Once the surface is crossed by the first time, the S_k value changes around the surface in a zigzag way.

III) The zigzag motion is stable and stays inside a fixed band.

Conditions I, II, and III can be mathematically expressed as:

$$S_t(S_{t+1} - S_t) < 0 \quad (14)$$

and

$$\text{if } \operatorname{sgn}(S_{t+1}) = -\operatorname{sgn}(S_t) \Rightarrow \begin{cases} \operatorname{sgn}(S_{t+2}) = \operatorname{sgn}(S_t) \\ |S_{t+1}| < \xi \end{cases} \quad (15)$$

where ξ is the fixed bound mentioned in the third condition.

Let us consider that, in closed loop, the switching function will exhibit the following behavior [D. Munoz and D. Sbarbro 2000]:

$$S_{t+1} = (x_{t+1} - \hat{x}_{t+1}) - \varepsilon.\operatorname{sgn}(S_t), \; \varepsilon > 0 \quad (16)$$

Starting from (16) and using (9), and (11) one can determine that the input signal must be:

$$u_t = -\frac{1}{\hat{g}_t}\left(\left(\hat{f}_t - x_{t+1}^f\right) + \alpha_1 e_t + \ldots + \alpha_p e_{t-p+1} + \varepsilon.\operatorname{sgn}(S_t)\right) \quad (17)$$

We can now readily check if such behavior for the switching function can guarantee the conditions in (14) and (15). Inserting (16) right into (14) we get:

$$\begin{aligned} S_t(S_{t+1} - S_t) &= S_t\left((x_{t+1} - \hat{x}_{t+1}) - \varepsilon.\operatorname{sgn}(S_t) - S_t\right) \\ &\le |x_{t+1} - \hat{x}_{t+1}||S_t| - \varepsilon|S_t| \end{aligned} \quad (18)$$

hence, it is sufficient to take

$$\varepsilon = \eta . \chi \tag{19}$$

where η is an arbitrary constant larger than unity and χ is an upper bound for the modeling error,

$$\chi > \left| x_{t+1} - \hat{x}_{t+1} \right| , \qquad \forall t \tag{20}$$

The first condition in (15) is also easy to check. The expression for S_{t+2} can be directly derived from (16).

$$S_{t+2} = \left(x_{t+2} - \hat{x}_{t+2} \right) - \varepsilon . \operatorname{sgn}\left(S_{t+1} \right) \tag{21}$$

Considering that S_t has just crossed the sliding surface, $\operatorname{sgn}(S_{t+1}) = -\operatorname{sgn}(S_t)$, thus:

$$S_{t+2} = \left(x_{t+2} - \hat{x}_{t+2} \right) + \varepsilon . \operatorname{sgn}\left(S_t \right) \tag{22}$$

Taking into account the conditions in (19) and (20), the first condition in (15) is guaranteed. The second requirement in (15) needs the consideration of the closed loop system in which one gets the following representation for the closed-loop dynamics of the switching surface:

$$\begin{aligned} S_{t+1} = {} & \left(1 - \frac{g_t}{\hat{g}_t} \right) \alpha_1 S_t + \left(f_t - \frac{g_t}{\hat{g}_t} \hat{f}_t \right) \\ & + \left(1 - \frac{g_t}{\hat{g}_t} \right) \vartheta - \left(1 - \frac{g_t}{\hat{g}_t} \right) x_{t+1}^f - \varepsilon \frac{g_t}{\hat{g}_t} \operatorname{sgn}\left(S_t \right) \end{aligned} \tag{23}$$

where,

$$\vartheta = (\alpha_2 - \alpha_1^2) e_{t-1} + \cdots + (\alpha_p - \alpha_1 \alpha_{p-1}) e_{t-p+1} - \alpha_1 \alpha_p e_{t-p} \tag{24}$$

The only recurrent terms in S_t are the first and the last terms on the right hand side. The latter is a bounded function of S_t, so considering that $\left| \alpha_1 \right| < 1$, the switching function will remain stable only if :

$$\left| 1 - \frac{g_t}{\hat{g}_t} \right| < 1 \tag{25}$$

3.2 Control Implementation

In the previous sub-section we derive the The implementation of nonlinear control techniques for suppressing chaotic dynamics in the system will be achieved by adding a feedback control forcing signal to the second state of the Eq. (7). The new governing equation including the form of control force, u , can be written as:

$$\begin{cases} x_{t+1} = (\alpha y_t - \alpha + 1)^2 \\ y_{t+1} = (\beta x_t - 1)^2 + u_t \end{cases} \tag{26}$$

By substituting the second relation of Eq. (26) in the first one, the SISO format of the equation is obtained as,

$$y_{t+1} = (\beta(\alpha y_{t-1} - \alpha + 1)^2 - 1)^2 + u_t \tag{27}$$

This input force can be assumed as the compensated policy of the firm 2 to control over the chaos in the market. The second firm inaccuracy of the other firms' production prediction led to unstable patterns. This may alter the profit maximization process of the firm which is described by the nonlinear model. Thus, a feedback force should be added in order to stabilize the system whenever needed.

Comparing Eq. (26) with Eq. (9) one may obtain,

$$f_t = (\beta(\alpha y_{t-1} - \alpha + 1)^2 - 1)^2 , g_t = 1 \tag{28}$$

Assume that the exact value of α and β are unknown and their estimated value are denoted by $\hat{\alpha}$ and $\hat{\beta}$. One may set the following relations to show the bounds of uncertainties:

$$\left| \alpha - \hat{\alpha} \right| < \varepsilon_\alpha \quad , \quad \left| \beta - \hat{\beta} \right| < \varepsilon_\beta \tag{29}$$

where ε_α and ε_β are the upper limits of the uncertainties.

Defining the tracking error as,

$$e_t = y_t - y_t^f \tag{30}$$

one may obtain the sliding surface as,

$$S_t = e_t + \alpha_1 e_{t-1} \tag{31}$$

Considering Eq. (20) one may write,

$$\begin{aligned} \chi & > \left| f_{t+1} - \hat{f}_{t+1} \right| \\ & = \left| (\beta(\alpha y_{t-1} - \alpha + 1)^2 - 1)^2 - (\hat{\beta}(\hat{\alpha} y_{t-1} - \hat{\alpha} + 1)^2 - 1)^2 \right| \end{aligned} \tag{32}$$

One may easily see that,

$$\begin{aligned} & \left| f_{t+1} - \hat{f}_{t+1} \right| \le \\ & \left| (\left(\hat{\beta} + \varepsilon_\beta \right)((\left(\hat{\alpha} + \varepsilon_\alpha \right)\left(y_{t-1} - 1 \right) + 1)^2 - 1)^2 - (\hat{\beta}(\hat{\alpha} y_{t-1} - \hat{\alpha} + 1)^2 - 1)^2 \right| \end{aligned} \tag{33}$$

Thus, we may set,

$$\chi = \left| (\left(\hat{\beta} + \varepsilon_\beta \right)((\left(\hat{\alpha} + \varepsilon_\alpha \right)\left(y_{t-1} - 1 \right) + 1)^2 - 1)^2 - (\hat{\beta}(\hat{\alpha} y_{t-1} - \hat{\alpha} + 1)^2 - 1)^2 \right| \tag{34}$$

So, the control action, according to Eq. (17) can be obtained as,

$$u_t = -\left(\left(\hat{f}_t - y_t^f \right) + \alpha_1 e_t + \chi \eta . \operatorname{sgn}\left(S_t \right) \right) \tag{35}$$

As $g_t = 1$ the Eq. (25) representing the stability criterion for switching function, is already satisfied.

Remark: Although the stability of the proposed control technique has been proved theoretically, there are some technical problems such as strong chattering in implementation of control law due to use of sign function in Eq. (17). To overcome this problem one can use the saturation function instead of sign function:

$$sat\left(S/\phi \right) = \begin{cases} S/\phi & \left| S/\phi \right| < 1 \\ sign\left(S/\phi \right) & otherwise \end{cases} \tag{36}$$

where φ is a positive small number. In this case, some steady state error is generated which implies that there exists some error between the stabilized trajectory and the actual one. By decreasing φ to zero, the mentioned error will converge to zero.

4 Simulation Results

The above described control scheme is now used to control the states of a chaotic economic system. The aforementioned economic system in Eq. (7) exhibits chaotic behavior with the following parameters [A. Matsumoto and Y. Nonaka March 2004]:

$$\beta = 1.75, \alpha = 1.1 \tag{37}$$

In Fig. (1) a bifurcation diagram of the system with α=1.1 is illustrated.

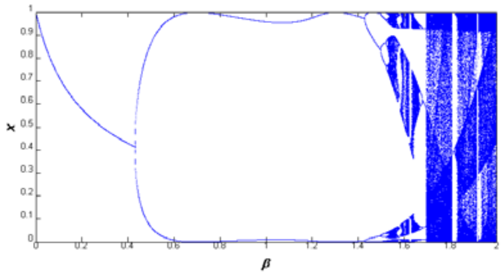

Figure 1. Bifurcation Diagram with $\alpha = 1.1$

Using mathematical and simulation, authors in [A. Matsumoto and Y. Nonaka March 2004] showed that for some values of α and β, the long-run average profit of the system is higher in chaotic region than its stationary points. For example, they show that when the value of α is fixed to 1.1, the long-run average profit of firm 1 is higher than the corresponding stationary profit when the value of β is larger than about 1.7, while the long-run average profit of firm 2 is less than its corresponding stationary point. This will continue until the value of β reaches 1.8, where the long-run average of firm 2 is become higher than its stationary point.

Form such example, one may see that the profit of the biggest firm (firm 2) is less than its corresponding stationary profit in a bound of $\beta = [1.7, 1.8]$, while the firm 1 is exceeding its stationary profit. Thus, Firm 2, to hold its position in the market, should apply a control force to compensate such "Relative Loss". Moreover, in a bound of $\beta = [1.7, 1.8]$ both firms is in the chaos region and their long-run profit is not a "profitable" situation for both.

All this, will convince a manager to apply a private control strategy to this system in order to "re-maximize" the profit when needed. By considering some uncertainties in the model parameters, the method proposed is applied to the system.

Regarding Eq. (29), the following uncertainties are considered in the simulation:

$$\varepsilon_\alpha = 0.1\hat{\alpha} \qquad \varepsilon_\beta = 0.1\hat{\beta} \tag{38}$$

where $\hat{\alpha} = 1.1$ and $\hat{\beta} = 1.75$. Also α_1 is assumed to be 0.1.

Also, it is to be noted that the fixed point of the system can be found by solving the following Equation:

$$y_t^f = (\beta(\alpha y_t^f - \alpha + 1)^2 - 1)^2 \tag{39}$$

For above parameters, one may find only a meaningful solution for such equation as:

$$y_t^f = 0.474324 \tag{40}$$

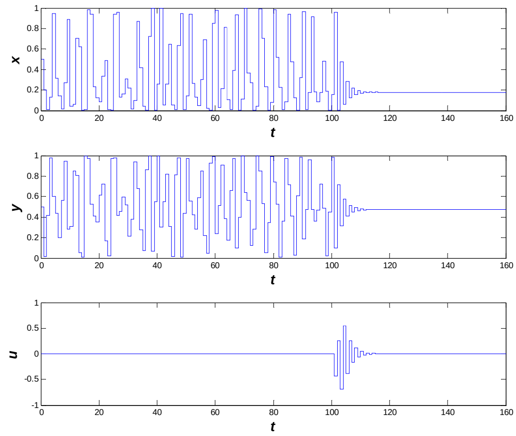

Figure 2. Simulation results for uncertain parameters

The results for this uncertain system are shown in Fig. (2). It is observed that, the fixed point of the system is stabilized and the control action converge to zero.

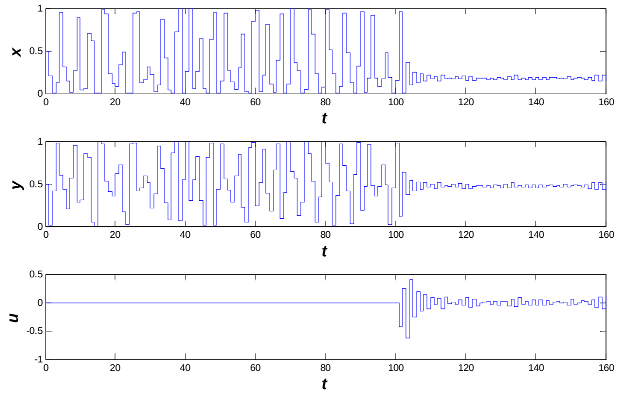

Figure 3. Simulation results for uncertain parameters with a 10% noise in feedback states

Also, in this case, the robustness of the designed controller against the output noise is checked for 10% noise added to the measured value of x. In both simulations the control signal starts after 100 time-step.

It is to be noted that the amplitude of the control signal is sufficiently small and converges to zero in a finite small duration of time, and the system is strongly robust to disturbances and measurement noise.

5 Conclusion

In this paper, a method of robust control is applied to eliminate chaos in an economic model on stabilizing a fixed point of the system. The quasi-sliding mode control is applied to the system by considering some uncertainties in the parameters. Also the robustness of the system is checked for possible measurement noise in the feedback states. This technique is successfully implemented to stabilize the fixed points of the system. Simulation results show that the presented algorithm may be successfully applied to chaotic complementary goods markets in order to guarantee higher profit and/or regular markets.

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