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PERFORMANCE EVALUATION OF A SLIDING MODE CONTROLLER IN DISCRETE TIME DOMAIN USING POLYHEDRAL APPROXIMATION METHOD

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Abstract
It is well known that a sliding mode controller can reject external perturbations when it is implemented in the continuous-time domain. However, when a discrete implementation of the continuous time sliding controller is realized, the controller performance can be degraded. The aim of this work is to evaluate a sliding mode controller performance when it is discretely implemented. This evaluation is carried out using the polyhedral approximation method.

Key words
Sliding-mode controller, polyhedral approximation.

1 Introduction

The sliding mode control approach is one of the most efficient tools to design robust controllers to dynamical systems [Utkin, 1977]. This control method design has been reported in extent literature [Edwards and Spurgeon, 1998] since it was first proposed in the 1950's in the ex Soviet Union [Hung, Gao, and Hung, 1993]. The idea of this technique is to realize a discontinuous feedback control law to make the closed-loop system *insensitive* to parametric uncertainty and external perturbations [Edwards and Spurgeon, 1998; Utkin, 1977; Hung, Gao, and Hung, 1993]. Nowadays, this control design has been employed to a wide variety of engineering systems [Perruquetti and Barbot, 2002].

Somehow the control realization is usually implemented on digital computers [Ogata, 1995; Gao, Wang, and Homaifa, 1995; Tang and Misawa, 2000]. But, when the sliding mode controller is digitally programmed, the *insensitivity* property is no longer hold due to the finite time sampling process of the digital devices [Tang and Misawa, 2000]. So, sliding control performance in its discrete-time version requires to be studied. It should be noted that discrete-time sliding mode control cannot be obtained from its continuous time model by means of simple equivalence [Gao, Wang, and Homaifa, 1995], and in some cases, the discrete-time model obtained introduces some kind of bounded perturbation.

On the other hand, to compute the control law, and to analyze stability of the closed-loop system, it is necessary to know the system vector state (or to estimate it from the output and input information of the plant, if possible). That is why the problem of state vector estimation for systems with unknown but bounded disturbances is important in the context of the control theory field.

One of well known techniques of dynamical system state estimation is the Kalman filter in its stochastic approach [Alamo, Bravo, and Camacho, 2005; Matasov, 1998]. Sometimes, the prior statistical information about perturbations cannot be obtained, but it is known that at a sample time they are bounded with some convex sets [Schweppe, 1968; Alamo, Bravo, and Camacho, 2005; Chernousko, 2002; Kurzhanski and Varaiya, 2005; Shiryaev and Podivilova, 2014; Podivilova, 2012]. According to this set-membership approach the estimation procedure consists of the construction of feasible sets, which are guaranteed to contain all possible values of dynamical system state vector at sample times. But exact calculation of feasible sets is not always computationally possible, and to reduce com-

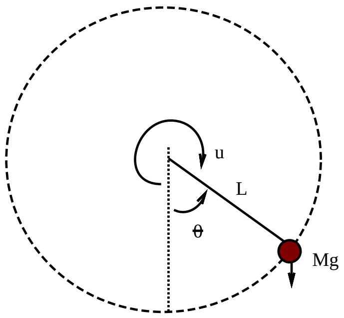


Figure 1. Diagram of a pendulum.

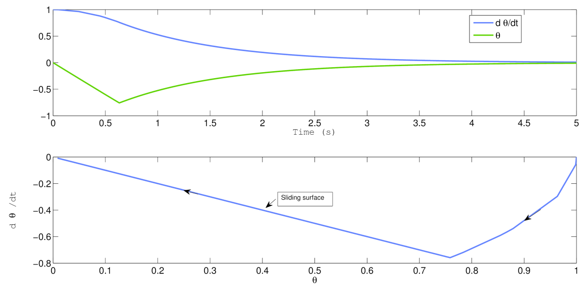


Figure 2. Simulation results: state trajectories (upper plot) and phase portrait (bottom plot).

plexity of the algorithm, the approximation of feasible sets with some canonical geometric sets is applied (although it causes loss of accuracy) [Alamo, Bravo, and Camacho, 2005; Chernousko, 2002; Kurzhanski and Varaiya, 2005; Polyak et al., 2004].

On the contrary, the polyhedral approximation algorithm shows to have some advantages of the existent state vector estimation tools. This method is described here. Then, performance of a sliding mode controller in discrete-time domain applied to the pendulum system is analyzed. On the other hand, other techniques to study robustness of discrete-time controllers can be found, for instance, in [Shiriaev et al., 2001; Fradkov and Furuta, 1996]. However, from the mathematical point of view, the polyhedral approximation method offers a numerical option to study the closed-loop systems robustness in discrete-time domain in terms of the sampling rate.

The rest of the paper is structured as follows. Section 2 describes the continuous-time sliding mode control applied to the well known pendulum system that is also an academic example to introduce control of mechanical devices [Edwards and Spurgeon, 1998]. In Section 3 the discrete model of the sliding mode control is analyzed using the polyhedral method. Section 4 states this polyhedral algorithm tool. Section 5 shows the obtained numerical results. Finally, Section 6 gives the conclusions.

2 Continuous-time Model

Consider the mechanical system of a pendulum (Figure 1). Its normalized equation is [Edwards and Spurgeon, 1998]:

$$\ddot{\theta}(t) = -a_1 \sin(\theta(t)) + u(t), \quad (1)$$

where a_1 is a positive scalar that depends on the system parameters (M, g, L: the mass, the gravity acceleration, and the pendulum length, respectively), and on a scaling factor [Edwards and Spurgeon, 1998]; $\theta(t)$ is the angular displacement (Rad); $u(t)$ is the torque control input (N·m) (see Figure 1). The following sliding mode controller is used[Edwards and Spurgeon, 1998]:

$$u(t) = -sign(s(t)), \quad (2)$$

and

$$s(t) = m\theta(t) + \dot{\theta}(t), \quad (3)$$

where m is a positive controller parameter, and $sign(\cdot)$ is the signum function. It can be ensured that, in finite time, the phase portrait intercepts the sliding surface $L_s = (\theta, \dot{\theta} : s(\theta, \dot{\theta}) = 0)$ and is forced to remain into there according to the following dynamic:

$$\dot{\theta}(t) = -m\theta(t). \quad (4)$$

So, the state-space trajectory of the closed-loop system (1)-(3) converges to the equilibrium point $(\theta, \dot{\theta}) = (0, 0)$, where the nonlinear term $a_1 \sin(\theta(t))$, which can be considered as a disturbance, has been completely rejected [Edwards and Spurgeon, 1998]. Figure 2 shows simulation results using $a_1 = 0.25$, $m = 1$, $\theta(0) = 1$ Rad, and $\dot{\theta}(0) = 0$ Rad/sec.

3 Discrete-time Realization of the Sliding Mode Controller

To obtain a discrete-time version of the closed-loop system (1)-(3), let $w(t) = a_1 \sin(\theta(t))$ be the perturbation on the system. Then, we have:

$$\ddot{\theta}(t) = u(t) + w(t), \quad u(t) = -sign(s(t)). \quad (5)$$

The state-space representation of the above system can be written as:

$$\dot{x} = Ax + Bu + \Gamma w, \quad u(t) = -sign(s(t)), \quad (6)$$

$$y = Cx, \quad (7)$$

where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $\Gamma = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} m & 1 \end{bmatrix}$ and $x = \begin{bmatrix} \theta(t) \\ \dot{\theta}(t) \end{bmatrix}$. In discrete control of continuous time systems, to study stability and robustness, it is necessary to convert the continuous-time state-space equations into the discrete-time ones. This conversion can be done in different ways, but assuming fictitious samplers and fictitious zero-order holding devices [1] [Ogata, 1995]. The conversion is performed using zero-order holding process that assumes that the control inputs are piecewise constant over the sampling period. For a sampling rate of 1 second we obtain the following discrete-time version of the system (6), (7):

$$x_{k+1} = A_D x_k + B_D u_k + \Gamma_D w_k, \tag{8}$$

$$y_{k+1} = C_D x_{k+1}, \quad k = 0, ..., N, \tag{9}$$

where $A_D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B_D = \begin{bmatrix} -0.5 \\ -1 \end{bmatrix}$, $\Gamma_D = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$, $C_D = \begin{bmatrix} m & 1 \end{bmatrix}$, $w_k = -a_1 \sin(\theta_k)$, $u_k = sign(s_k)$, and $s_k = m\theta_k + \dot{\theta}_k$.

System (8), (9) represents the discrete-time model of the system (6), (7), where, obviously, a discretization error appears depending on the sampling time.

Therefore, the objective is to evaluate the sliding mode control performance on this discrete-time version of the continuous-time plant by using polyhedral approximation. The method allows to calculate set estimates basing on the system's output given by (7). Obviously, if sensors are used, the output equation $y(t)$ can be realized as demanded. It is expected that the state trajectory will converge to some region around the equilibrium point. The size of this region can be obtained with our polyhedral algorithm. Thus, the polyhedral approximation method will be applied to analyze the performance of the controller in its discrete-time version for different sampling rates.

4 Dynamical System State Estimation Using Polyhedral Approximation

Given the discrete dynamical system (8), (9), consider the initial state x_0 and uncertain perturbations w_k at a sample time k be bounded with convex sets:

$$x_0 \in \overline{X}_0 : \{-1 \le x_0(1) \le 1, \quad x_0(2) = 1\},$$
$$w_k \in W : \{-a_1 \le w_k \le a_1\}. \tag{10}$$

Let us perform dynamical system state estimation using minimax filter [Shiryaev and Podivilova, 2014; Shiryaev and Podivilova, 2013; Alamo, Bravo, and Camacho, 2005]. It consists of a construction of feasible sets $\overline{X}_k$, which are guaranteed to contain all possible values of dynamical system state vector $x_k \in \overline{X}_k$ at a sample time:

$$X_{k+1/k} = A_D \overline{X}_k + \Gamma_D W + B_D u_k, \tag{11}$$

[1] Zero-order holders are the most frequently used in digital systems.

$$X[y_{k+1}] = \{x \in R^2 | C_D x = y_{k+1}\}, \tag{12}$$

$$\overline{X}_{k+1} = X_{k+1/k} \cap X[y_{k+1}]. \tag{13}$$

Minimax filter (11)-(13) involves performing set operations: linear transformation of sets, Minkowski sum of sets, set intersections. But when the problem dimension increases troubles in performing Minkowski sum operation in real-time occur. It is suggested to construct feasible sets without performing intractable operations of Minkowski sum and intersection. Only system equations and linear inequalities systems describing possible value sets of initial state of the system, disturbances and measurement noises are used [Shiryaev and Podivilova, 2014].

Now allow the sets $\overline{X}_0$, W be described using linear inequality systems:

$$\overline{X}_0 : A_{x_0} x_0 \le b_{x_0}, \quad W : A_w w_k \le b_w. \tag{14}$$

At a sample time all this information is included in one large system of linear inequalities. This system implicitly describes the feasible set. To get an explicit description it is necessary to approximate it with polyhedra of any required form; i.e. with any range of facets. The approximation algorithm consists of solving linear programming problems for all of the given facets with the received linear inequalities system as constraints (see Algorithm 1).

Algorithm 1:
Step 1. State the linear equation system describing dynamical system at a sample time k as:

$$\begin{pmatrix} I & -A & -\Gamma \\ G & 0 & 0 \end{pmatrix} \begin{pmatrix} x_{k+1} \\ x_k \\ w_k \end{pmatrix} = \begin{pmatrix} B_D u_k \\ y_{k+1} \end{pmatrix}. \tag{15}$$

Step 2. Describe the linear inequality system describing bounds as:

$$\begin{pmatrix} 0 & A_{x_k} & 0 \\ 0 & 0 & A_w \end{pmatrix} \begin{pmatrix} x_{k+1} \\ x_k \\ w_k \end{pmatrix} \le \begin{pmatrix} b_{x_k} \\ b_w \end{pmatrix}. \tag{16}$$

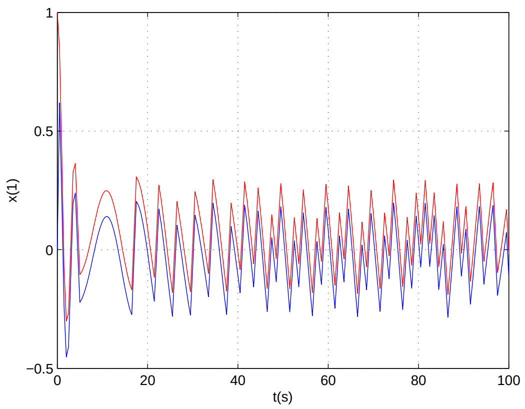
Step 3. Calculate the feasible set as a linear inequality system:

$$\overline{X}_{k+1} \subset X_{k+1} : A_{x_{k+1}} x_{k+1} \le b_{x_{k+1}}, \tag{17}$$

$$< a_i, x > \to \max_{(15),(16)}, \tag{18}$$

and $b_i = < a_i, x^* >$, where a_i – i-th row of matrix $A_{x_{k+1}}$, x^* is the solution of the linear programming problem (18).

To find bounds of coordinates of vector x_k for discrete-time model of a pendulum (8), (9) we will perform approximation of feasible set with rectangles. To estimate the set of the control inputs, the centers of approximating rectangles were chosen. The result of approximation for modelling the process with sample time 0.5s is shown on Figure 3. The Figure 3 shows the bounds of the coordinates $x(1)$ and $x(2)$ of the vector x_k, where $x(1)$ represents the angular position (Rad) and $x(2)$ represents the angular velocity (Rad/sec), both, in discrete-time, of the mechanical system.



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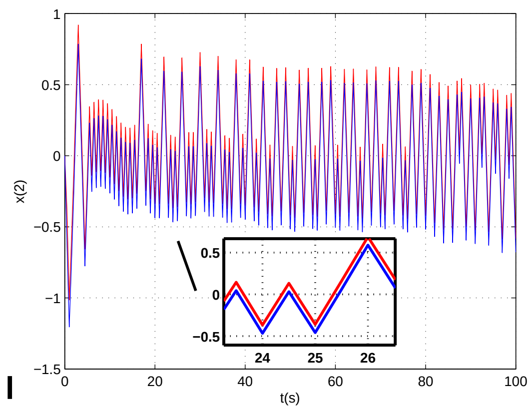
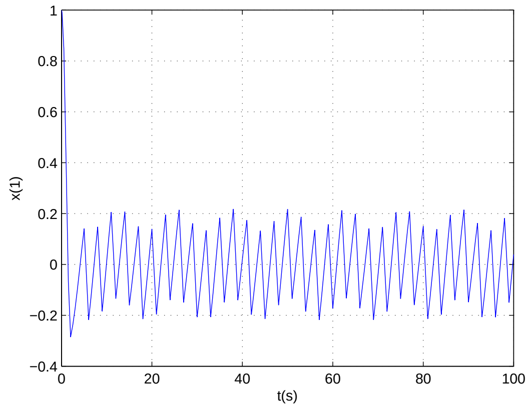


b)

Figure 3. Approximation of feasible sets of x_k in case of discretization with sample time 0.5s (red line corresponds to upper bound, blue line corresponds to lower bound).

Note from (9) that the measurements are received without errors, that is why the tubes of possible values of $x(1)$ and $x(2)$ are quite narrow and these tubes oscillate around equilibrium point. When a smaller sample time is used the amplitude of these oscillations decreases.



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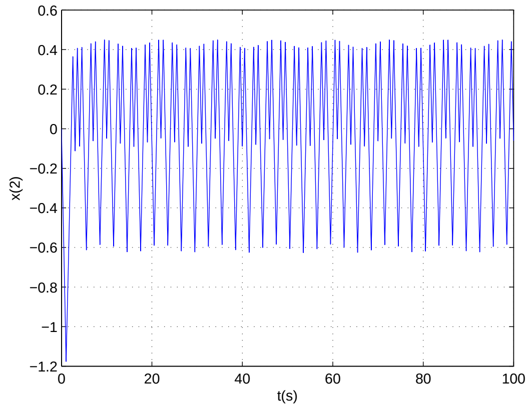


b)

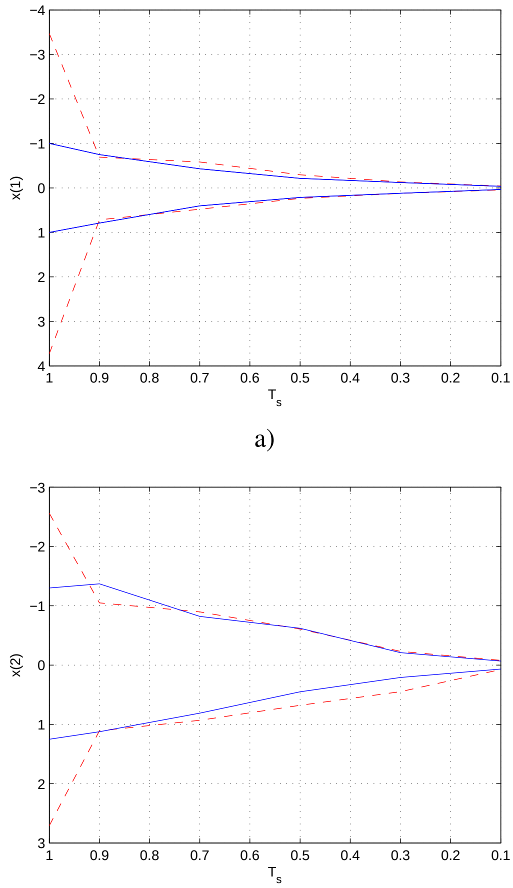
Figure 4. The simulation of continuous-time model with discrete-time controller with sample time 0.5s.

5 Implementation of Discrete-time Controller to Continuous Model

Let us perform the continuous time simulation of the process using discrete-time controller with different sample times using Simulink. The result of the experiment for sample time 0.5s is shown in the Figure 4.

Let us compare the amplitudes of the system state oscillation around the equilibrium point for discrete-time and continuous-time approaches. Maximum and minimum values of the coordinates $x(1)$ and $x(2)$ in both cases were calculated. These values were chosen from the period of the last 20 seconds of the modeling time because for this time we can see some kind of stability of the system. The bounds of values of the coordinates $x(1)$ and $x(2)$ for different sample times are shown on Figure 5.

As we can see from Figure 5, the amplitude of the variables decrease when we decrease the sample time in both discrete-time and continuous-time cases. But



a)

b)

Figure 5. Bounds of values of $x(1)$ and $x(2)$ (dashed line corresponds to discrete-time simulation, solid line corresponds to continuous-time simulation, T_s is the sampling rate in seconds).

for some sample times (for example T_s=0.5s), the result in discrete-time case occurs to be a little worse, but acceptable to predict the system behavior when a discrete-time version of the controller is realized.

6 Conclusion

In this paper we have presented a method to estimate stability of a closed-loop system designed in continuous-time domain (the plant and the controller), when the controller is going to be realized in discrete-time domain. The method is based on guaranteed dynamical system state estimation approach using polyhedral approximation. According to numerical experiments, the proposed method gives an acceptable estimation to predict the behavior of a continuous-time domain controller to the discrete-time field one. With this method, we can avoid to analyze stability (and robustness) of the closed-loop system in discrete-time land. Stability of nonlinear systems in discrete-time domain can be a heavily work compared to its continuous-time version, i.e. the mathematics may become more involved and complicated [Spooner *et al.*, 2002].

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