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EMERGENCE AND STABILITY OF AUTORESONANCE IN NONLINEAR OSCILLATORS

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Abstract

Formation and control of autoresonance (AR) with persistent growth of energy is based on the inherent property of nonlinear systems to stay in resonance when the driving frequency varies in time. However, the mechanism of transitions from bounded oscillations to AR is still an open question. As this paper demonstrates, and in sharp contrast to previous investigations, the emergence of AR from stable bounded oscillations is basically analogous to the transition from small to large oscillations in the time-invariant oscillator driven by an external harmonic excitation with constant frequency. It is shown that AR results from the loss of stability of the so-called limiting phase trajectory (LPT) of small oscillations. A critical parameter, which determines a boundary between small and large oscillations in the time-invariant system, may be considered as a lower threshold of the autoresonance excitation in the oscillator with slowly-varying parameters. The values of bifurcation parameters accounting for small sweeping rate and close to numerical results are obtained from the condition of the capture into resonance.

Key words

Autoresonance, parametric instability, control of oscillations, control of time-varying systems.

1 Introduction

Formation and control of the desired stable dynamics of a nonlinear system is one of the most important objectives of control science. Typically, control may be achieved by a feedback scheme. However, realization of feedback in more complex systems requires careful analysis of the uniqueness and stability of the induced nonlinear state. This leads to the question of whether one can avoid feedback, using a simple programme control. In the AR applications, the answer to this question is positive. The idea is to exploit the inherent feature of nonlinear systems to stay in resonance when the driving frequency varies in time. This self-phase-locking ability and continuous resonant growth of energy due to slow variations of the excitation frequency is usually referred to as the *autoresonance* phenomenon.

After first studies for the purposes of particle acceleration [Veksler, 1944; McMillan, 1945], AR has become a very active field of research. Theoretical approaches, experimental evidence and applications of AR in different fields of natural science, from plasmas to planetary dynamics, are reported in numerous papers listed in [Friedland, 2013]; additional theoretical and computational results can be found in [Chacón, 2005; Ben-David, 2006; Kalyakin, 2008; Witkov, 2010; Blekhman, 2012]; recent advances in this field are discussed, e.g., in [Andersen, 2011; Murch, 2011; Perkin, 2012; Shalibo, 2012; Friedland, 2013].

It was noticed in the early theory of cyclotron [Bohm, 1947] that a physical mechanism behind autoresonance is adiabatic nonlinear phase-locking between the system and the driving signal. But it remains unclear under what conditions the transition from bounded oscillations to AR may occur. The results presented in this work indicate that the emergence of AR is similar to the transition from small (quasi-linear) to large (nonlinear) oscillations in the system with constant parameters and constant excitation frequency [Manevitch, 2011]. We demonstrate that AR occurs due to the loss of stability of the so-called limiting phase trajectory (LPT) of small oscillations. In addition, we derive a relationship between the parameters, which can be interpreted as a lower excitation level capable of producing AR.

The paper is organized as follows. In Sec. II, we introduce the slowly time-dependent Duffing oscillator as a benchmark model. In contrast to previous studies, a system with time-dependent linear stiffness is considered. Then we briefly recall main terms and definitions concerning the LPTs in the system with constant parameters and constant driving frequency. Section III demonstrates numerical results, which confirm the similarity of the occurrence of autoresonance in the slowly time-dependent Duffing model to the transitions from small (quasi-linear) to large (nonlinear) oscillations in the system with constant parameters and constant excitation frequency. It is shown that a critical parameter, which determines a boundary between small and large oscillations in the time-invariant system, may be treated as a lower threshold of autoresonance in the oscillator with slowly-varying parameters. Furthermore, a bifurcation parameter, numerically obtained in [Grosfeld, 2002; Marcus, 2004; Friedland, 2008] is inacceptable in the problem examined in this paper. Stability of autoresonance is studied in Sec. IV.

It is important to note that the Duffing system is chosen for illustrative purposes and qualitative features of the results hold true for oscillators with different nonlinearities.

2 Model

The equation of motion of the time-dependent Duffing oscillators is given by

$$\frac{d^2u}{dt^2}+(1-\varepsilon\zeta(t,\varepsilon))u+8\varepsilon\alpha u^3=2\varepsilon F\cos\theta(t,\varepsilon) \quad (1)$$

where $\zeta(t,\varepsilon) = s_1+\varepsilon\beta_1 t$, $\theta(t,\varepsilon) = 1+\varepsilon s_2 t + \varepsilon^2\beta_2 t^2/2$, $\varepsilon > 0$ is a small parameter of the system. The initial conditions $u=0, v=du/dt$ at $t=0$ determine the so-called limiting phase trajectory (LPT) corresponding to the maximum possible energy transfer from the source of energy to the oscillator [Manevitch, 2011].

Asymptotic solutions of Eq. (1) for small ε are constructed with the help of the multiple scale method [Kevorkian, 1996]. To this end, we introduce a pair of the complex-conjugate variables φ and φ^*

$$\varphi=(v+iu)e^{-it}, \varphi^*=(v-iu)e^{it}. \quad (2)$$

As in previous works, [Manevitch, 2011, 2013; Kovaleva, 2012], the function φ is constructed in the form of the asymptotic series

$$\varphi(t,\tau_1)=\varphi_0(\tau_1)+\varepsilon\varphi_1(t,\tau_1)+...,\tau_1=\varepsilon t,$$

with the slow term $\varphi_0(\tau_1)$ satisfying the equation

$$\frac{d\varphi_0}{d\tau_1}+i[(s_1+\beta_1\tau_1)-3\alpha|\varphi_0^2|]\varphi_0= \quad (3)$$
$$=-iFe^{i\tau_1(s_2+\beta_2\tau_1/2)},\quad \varphi_0(0)=0.$$

Let $s=s_1+s_2$, $\beta=\beta_1+\beta_2$. We first assume that $s>0$, $\alpha>0$. The change of variable $\varphi_0(\tau_1) = \lambda\psi e^{i\tau_1(s_2+\beta_2\tau_1/2)}$ and the rescaling

$$\tau=s\tau_1, \lambda=\frac{s^{1/2}}{(3\alpha)^{1/2}}, f=\frac{F}{s\lambda}=\frac{3\alpha^{1/2}}{s^{3/2}}F \quad (4)$$

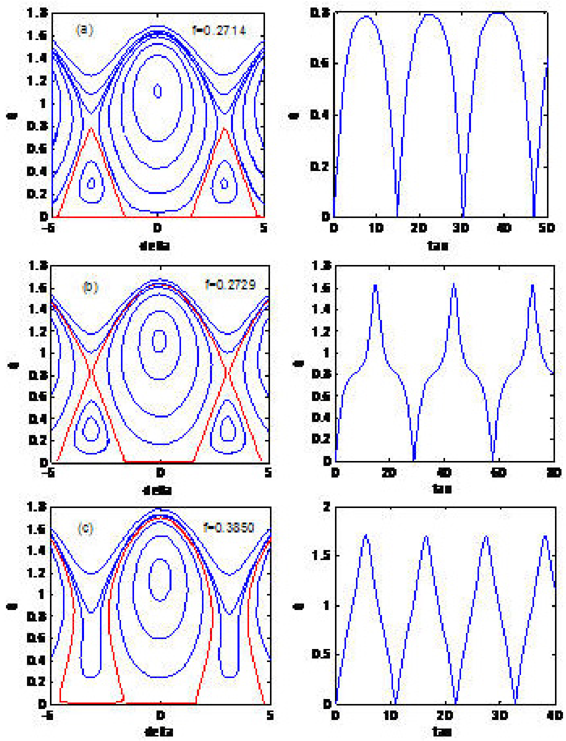

Figure 1. Phase portraits and envelopes $a(\tau)$ for systems with weak (a), moderate (b), and strong (c) nonlinearity.

transform Eq. (3) into the following two-parameter equation for the slowly-varying envelope ψ :

$$\frac{d\psi}{d\tau}+i(1+\beta\tau-|\psi|^2)\psi=-if,\psi(0)=0. \quad (5)$$

In the next step, the polar representation $\psi = ae^{i\Delta}$ transforms Eq. (5) into the real-valued system

$$\frac{da}{d\tau}=-f\sin\Delta, \quad (6)$$
$$\frac{d\Delta}{d\tau}=-(1+\beta\tau)-a^{-1}f\cos\Delta$$

with initial conditions $a(0)=0, \Delta(0)=-\pi/2$ that correspond to the LPT of system (6). From Fig. 1, it is seen that this trajectory represents an outer boundary for a set of trajectories encircling the stationary points of system (6).

System (6) represents a nonlinear analog of the Landau-Zener (LZ) equations of quantum tunnelling [Zobay, 2000; Liu, 2002; Kovaleva, 2011; Manevitch, 2011, 2013]. A direct analysis shows that Eqs. (6) may have both bounded and unbounded solutions (Fig. 2).

If $s<0, \alpha<0$, then the rescaling

$$\tau=|s|\tau_1, \lambda=\frac{|s|^{1/2}}{|3\alpha|^{1/2}}, f=\frac{F}{s\lambda}=\frac{3|\alpha|^{1/2}}{|s|^{3/2}}F \quad (7)$$

converts Eq. (3) into the complex-valued equation

$$\frac{d\psi}{d\tau} - i(1 + \beta\tau - |\psi|^2)\psi = -if, \psi(0) = 0. \quad (8)$$

The representation $\psi = ae^{-i\Delta}$ converts Eq. (8) to the real-valued system similar to (6), with the change Δ to $-\Delta$. In what follows we study in detail the case $s > 0, \alpha > 0$.

In order to clarify the initiation of the unbounded modes from the LPT, we first consider a *time-invariant analog* of system (6), in which $\beta = 0$.

It was earlier shown [Manevitch, 2011] that there exist two critical values f_1 and f_2, which define the boundaries between different types of the dynamical behavior. If $f < f_1$ or $f_1 < f < f_2$, then there exists a stable centre on each of the axes $\Delta = -\pi$ and $\Delta = 0$, and an intermediate unstable hyperbolic point on the axis $\Delta = -\pi$ (Figs. 1a, 1b); if $f \geq f_2$, then there exists only the stable centre on the axis $\Delta = 0$ (Fig. 1c). The conditions $f < f_1, f_1 < f < f_2$, and $f > f_2$ characterize *quasi-linear*, *moderately nonlinear*, and *strongly nonlinear* systems, respectively.

A boundary between small and large oscillations corresponds to

$$f_1 = \sqrt{\frac{2}{27}} \approx 0.2721. \quad (9)$$

It was demonstrated [Manevitch, 2011] that at $f = f_1$ the LPT of small oscillations coalesces with the separatrix going through the homoclinic point on the axis $\Delta = -\pi$. This implies that the transition from small to large oscillations occurs through the loss of stability of the LPT of small oscillations. At $f = f_2$, where

$$f_1 = \frac{2}{\sqrt{27}} \approx 0.3849, \quad (10)$$

the stable center on the axis $\Delta = -\pi$ vanishes through the coalescence with the homoclinic point, and only a single stable center remains on the axis $\Delta = 0$ (Fig. 1c). Figure 1 clearly demonstrates the "limiting" property of the LPTs in the time-invariant system. It is seen that the LPT represents an outer boundary for a set of closed trajectories encircling the stable center in the phase plane (Δ, a).

3 Numerical Results

In this section, we present the numerical results that help understand the dynamics of system (6) with nonzero sweeping rate. The system with $\alpha > 0$ (hard nonlinearity), $s > 0$ and $0 < |\beta| << 1$ is considered. Two possibilities are discussed: (i) periodic oscillations shown in Fig. 1 are transformed into bounded

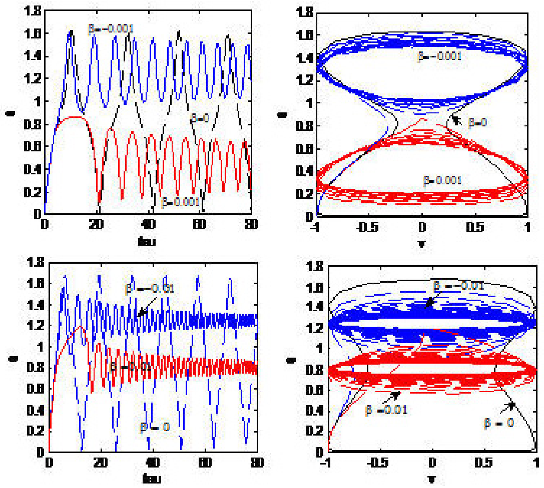

Figure 2. Plots of $a(\tau)$ and phase portraits of system (6) with $f = 0.2799$ and $\beta = \pm 0.001$(top row); $f = 0.3412$ and $\beta = \pm 0.01$ (bottom row). Plots of the LPTs corresponding to $\beta = 0$ are shown for comparison.

oscillations with amplitude $a(\tau)$ approaching to a constant value a_0 as $\tau \to \infty$ (Fig. 2); (ii) periodic oscillations are transformed into growing (autoresonance) oscillations (Fig. 3).

Figure 2 demonstrates the results of numerical simulations in the case of $f > f_1$, for which the conservative system ($\beta = 0$) exhibits large oscillations. It is obvious that, if $\beta > 0$, the instant detuning $1+\beta\tau$ increases with time, thereby biasing the system into the domain of small oscillations, while the negative rate $\beta < 0$ decreases detuning and thus biases the system into the domain of large oscillations. This effect is clearly seen in Fig. 2.

The phase portraits (right column) show that in the first half-cycle of oscillations the shape of the bottom (red) orbit ($\beta > 0$) is similar to the LPT of small oscillations (Fig. 1a), whereas the top (blue) orbit ($\beta < 0$) becomes close to the LPT of large oscillations (Fig. 1b) of the time-invariant analogue of system (6) ($\beta = 0$). The projection of the trajectory $a(\tau)$ onto the phase plane (a, v) represents the spiral orbit with an attracting focus $a = a_0, v = 0$, where a_0 =lim$a(\tau)$ as $\tau \to \infty$. The analysis of bounded oscillations as well as the calculation of the limiting value a_0 is suggested in [Kovaleva, 2012; Manevitch, 2011, 2013].

Figure 3 depicts the occurrence of AR from stable bounded oscillations under changes of the parameter f. It is seen that the shape of the first half-cycle of small oscillations corresponding to $\beta > 0$ is consistent with the LPT of quasi-linear periodic oscillations (Fig. 1a), while the first half-cycle of autoresonance is similar to the half-cycle of the LPT in the system with moderate nonlinearity (Fig. 1b); then the shape of oscillations takes the saw-tooth form typical for the system with strong nonlinearity (Fig. 1c). This means that the transition from bounded to unbounded oscillations in the system with slow positive sweep occurs in

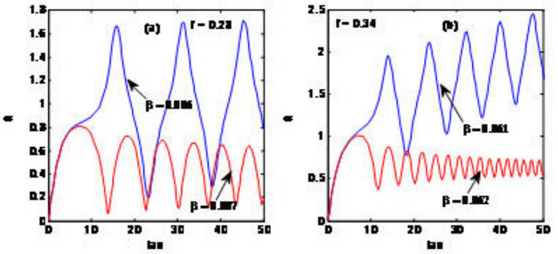

Figure 3. Transition from bounded oscillations to AR.

the same way as the transition from small to large oscillations in the system with constant parameters, that is, due to destruction of the LPT of small quasi-linear oscillations.

If the parameter f is close to the critical value f_1, then the transition from bounded oscillations to AR takes place under very slow sweep. For instance, $0.006 < \beta < 0.007$ at $f \approx 0.28$ (Fig. $3a$); in this case, the difference between f and f_1 is less than 2.8%. On the other hand, for $f = 0.34$, the critical rate $0.061 < \beta < 0.062$ (Fig. $3b$) ; the difference between f and f_1 is about 20%. This implies that the inequality

$$f > f_1 \tag{11}$$

can be interpreted as the *necessary condition* of the AR excitation.

The transformations $\Psi = |\beta|^{-1/4}\psi$, $\tilde{\tau} = |\beta|^{1/2}(\tau + 1/\beta)$, $\mu = |\beta|^{-3/4} f$ reduces Eq. (5) to the form

$$\frac{d\Psi}{d\tilde{\tau}} + i(\tilde{\tau} - |\Psi|^2)\Psi = -i\mu, \;\; \Psi(\tilde{\tau}_0) = 0, \tag{12}$$

with the zero initial condition at $\tilde{\tau}_0 = |\beta^{-1/2}|\text{sign}\beta$. It was found numerically [Grosfeld, 2002; Marcus, 2004; Friedland, 2008] that in this system autoresonance occurs if $\mu > \mu_{th} = 0.41$. At earlier work [Grosfeld, 2002; Marcus, 2004] the threshold $\mu_{th} = 0.41$ was treated as independent of $\tilde{\tau}_0$ but a more thorough study [Friedland, 2008] demonstrated that for $\tilde{\tau}_0 > 0$ or, by definition, $\beta > 0$, the threshold μ_{th} grows significantly when $\tilde{\tau}_0$ increases, or $|\beta|$ decreases. Thus, even if we omit a discussion of applicability of a numerically found threshold for a large class of physical problems, we should underline that the threshold $\mu_{th} = 0.41$ is unusable in the problem under consideration wherein the effect of small values of $\beta > 0$ is examined.

4 Stability of Autoresonance in the Adiabatic System

In the literature (e.g., [Fajans, 2001]), the analysis of AR is built on the assumption of small phase deviations from the stationary state of the system with "frozen" detuning. However, from Fig. $4d$ it is seen that, although in the phase-locking regime the phase

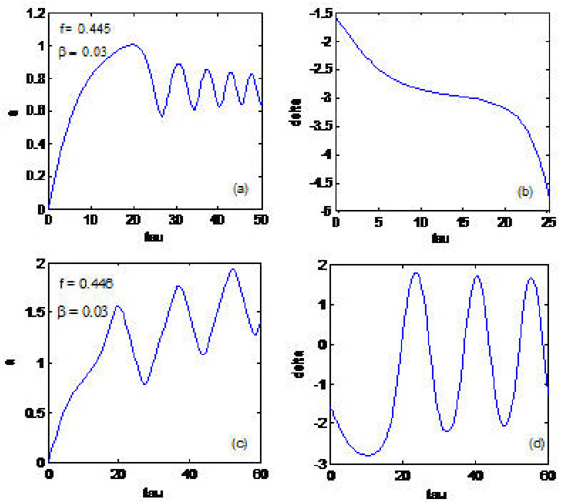

Figure 4. Plots of $a(\tau)$ and $\Delta(\tau)$ in tunneling ($f = 0.445$, $\beta = 0.03$) and autoresonance ($f = 0.446$, $\beta = 0.03$).

converges to $\Delta = 0$ as $\tau \to \infty$, in the large initial interval it oscillates about $\Delta = 0$ with bounded but not small amplitude $|\Delta| < \pi$. Motivated by the numerical results, this paper does not invoke assumptions of small phase deviations from quasi-stationary states.

We explore the phase-locking regime with the help of the standard procedure of the resonance analysis [Arnold, 2006]. For this purpose, we introduce the parameters $\mu = \varepsilon^{1/2}, t_k = \mu^k\tau, \beta = \mu^2\gamma, f = \mu^2\Phi$ and then rewrite system (6) as

$$\begin{aligned} \frac{da}{dt_1} &= -\mu\Phi\sin\Delta, \\ \mu\frac{d\Delta}{dt_1} &= -(1+\gamma t_2) + a^2 - a^{-1}\mu^2\Phi\sin\Delta. \end{aligned} \tag{13}$$

If the system is captured in the neighborhood of the resonance manifold, then one can set

$$-(1+\gamma t_2) + a^2 = \mu p. \tag{14}$$

Considering p as a new variable and ignoring higher-order terms, we obtain the system

$$\begin{aligned} \frac{d\Delta}{dt_1} &= \frac{\partial H}{\partial p} = p, \\ \frac{dp}{dt_1} &= -\frac{\partial H}{\partial \Delta} = -[\gamma + 2(1+\gamma t_2)^{1/2}\Phi\sin\Delta], \end{aligned} \tag{15}$$

generated by the Hamiltonian $H = \frac{1}{2}p^2 + U(\Delta, t_2)$, where

$$U(\Delta, t_2) = \gamma\Delta - 2(1+\gamma t_2)^{1/2}\Phi\cos\Delta. \tag{16}$$

The function H can be interpreted as energy of the pendulum with tilt angle Δ and velocity p; U expresses the potential energy of the pendulum; $\gamma > 0$ plays the role of a constant moment, which tilts the potential well and changes the equilibrium position of the pendulum; the coefficient γt_2 expresses the change of the pendulum length.

Phase locking persists if the system remains captured into the time-dependent well of the potential $U(\Delta, t_2)$ for any t_2, that is:

(i) the parameter γ is small enough to ensure the well-shaped potential with a distinctive minimum;

(ii) initial energy is small enough to avoid the escape from the well.

While the first condition (i) was discussed earlier [Fajans, 2001, Marcus, 2004], the second condition (ii) was not taken into consideration. Here, both conditions are discussed.

In order to give a formal description of condition (i), we use the arguments similar to that ones from [Marcus, 2004] but omit the assumptions of small deviations from quasi-stationary states. It follows from (15) that the term γt_2 enlarges the depth of the potential well at each time instant t_2. This means that the well-shaped potential at $t_2 = 0$ ensures the well-shaped potential for any $t_2 > 0$. The generating conservative potential is written as

$$U(\Delta) = \gamma\Delta - 2\Phi\cos\Delta. \tag{17}$$

Potential (16) has a minimum if the equation

$$\frac{\partial U_0}{\partial \Delta} = \gamma + 2\Phi\sin\Delta = 0 \tag{18}$$

is solvable. The condition of solvability holds only if $\gamma \leq 2\Phi$, or, in the previous notations,

$$\beta \leq 2f. \tag{19}$$

A similar condition was derived in [Fajans, 2001]. It is important to note that inequality (18) always holds for sufficiently small rates β.

We now formulate condition (ii). To this end, we consider a conservative analog of system (12)

$$\begin{aligned} \frac{d\Delta}{dt_1} &= \frac{\partial H_0}{\partial p} = p, \\ \frac{dp}{dt_1} &= -\frac{\partial H_0}{\partial \Delta} = -(\gamma + 2\Phi\sin\Delta), \end{aligned} \tag{20}$$

where $H_0(\Delta, p) = \frac{1}{2}p^2 + U_0(\Delta)$. It follows from (13) that the initial conditions for system (18) are defined as $\mu p = -1, \Delta = -\pi/2$.

Let the potential U_0 attain its minimum and maximum at $\Delta = d_1$ and $\Delta = d_2$, respectively. It follows from (17) that the neighboring to $\Delta = -\pi/2$ points $d_{1,2}$ are defined as $d_1 = -d$, $d_2 = -\pi + d$, where $d = -\arcsin(\beta/2f)$. Hence, the minimum and the maximum of U_0 are given by

$$\begin{aligned} U_0(d_1) &= -\gamma d - 2\Phi\cos d, \\ U_0(d_2) &= \gamma(\pi - d) + 2\Phi\cos d. \end{aligned} \tag{21}$$

Since $\Delta = d_2$ corresponds to the potential barrier, then the maximum admissible energy H_{bar} at which the system remains trapped into the well corresponds to the attainment of the barrier with velocity $p = 0$; this yields $H_{bar} = U_0(d_2)$. This implies that the conservative system (19) stays trapped into the well if the initial energy does not exceed $U_0(d_2)$, that is,

$$[1/(2\mu^2) + U_0(-\pi/2)] \leq U_0(d_2).$$

If we recall that $\beta = \mu^2\gamma$, $f = \mu^2\Phi$, we get

$$1 \leq -\beta(\pi - d) + 4f\cos d. \tag{22}$$

If $|\beta/2f| << 1$, then inequality (21) is reduced to the simpler form

$$\beta < \beta_{cr} = (4f - 1)/\pi. \tag{23}$$

We now compare conditions (18) and (22). It is important to note that inequality (18) always holds for sufficiently small β and, in general, roughly estimates admissible values of the rate β. Indeed, it follows from Fig. 3 that for $f = 0.28$ autoresonance occurs at $\beta = 0.006$, while condition (18) gives $\beta < 0.56$. At the same time, from (22) we find that $\beta \leq \beta_{cr} = 0.015$ if $f = 0.28$. Although β_{cr} is still greater than the computational result $\beta = 0.006$, it significantly improves an admissible interval compared to condition (18).

5 Conclusions

In this paper the origin and stability of autoresonance were examined using the concept of limiting phase trajectories (LPT). It was shown that, in contrast to previous investigations, the emergence of AR from stable bounded oscillations is similar to the transition from small to large oscillations in the time-invariant oscillator driven by an external harmonic excitation with constant frequency. It is demonstrated that AR results from the loss of stability of the so-called limiting phase trajectory separating the domains of small and large oscillations. In addition, it has been shown that AR stability is closely connected with the permanent capture into resonance in the system with slowly-varying parameters. Note that the considered model of the Duffing oscillator was chosen only for illustrative purposes. The obtained results can be extended to more general cases of the arrays of nonlinear oscillators.

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