PERFORMANCE ANALYSIS OF HARMONICALLY FORCED NONLINEAR SYSTEMS

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Abstract: The paper deals with the performance analysis of harmonically forced nonlinear systems of Lur'e type.

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1. INTRODUCTION

It is well known that any solution of a stable linear time-invariant (LTI) system with a harmonic input converges to a unique harmonic limit solution that depends only on the input and not on the initial conditions. Nonlinear systems with such a property are referred to as convergent systems. Solutions of the convergent systems "forget" their initial conditions and after some transient time depend on the system input that can be a command or reference signal. For LTI systems such a harmonic limit solution can be easily derived while a similar problem for nonlinear systems is hard to tackle. Some recent results in this field can be found, e.g. in (Jönsson, *et. al.*, 2003).

In this paper we consider the harmonic linearization (Rozenwasser, 1969; Khalil, 2002) of nonlinear systems of Lur'e type. We formulate a frequency domain condition which ensures that the harmonic linearization is well-posed and derive upper bounds for its approximation error. We say that the procedure of harmonic linearization is well-posed if for a given amplitude and frequency of the excitation signal, the corresponding algebraic harmonic balance equation has a unique solution for a class of nonlinearities described by incremental sector condition. It turns out that this problem can be tackled with the frequency inequality of the incremental circle criterion verified for the frequency of the external excitation.

2. HARMONIC LINEARIZATION OF LUR'E SYSTEMS

Consider the following system of differential equations

$$\begin{cases} \dot{x} = Ax - B\phi(y) + Fu \\ y = Cx + Du \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}$ is the output, ϕ is a continuous scalar function and the matrices A, B, C, D, F are of corresponding dimensions. We will assume that the nonlinear function ϕ satisfies the following incremental sector condition

$$0 \leq \frac{\phi(y_1) - \phi(y_2)}{y_1 - y_2} \leq \mu \tag{2}$$

for some positive μ. We do not consider the a special case $\mu = +\infty$ here, but focus only on finite μ. A more general result that includes also $\mu = +\infty$ can be derived from our results using the standard methods of absolute stability theory.

Suppose $u(t)$ is a periodic function of time with period T and system (1) has a unique T-periodic solution $\bar{x}(t)$.

Let us approximate nonlinear system (1) by a linear system

$$\begin{cases} \dot{\xi} = A\xi - BK\zeta + Fu \\ \zeta = C\xi + Du \end{cases} \tag{3}$$

If the matrix $A - BKC$ does not have eigenvalues on the imaginary axis then for the periodic input $u(t)$ the system has a unique periodic solution $\bar{\xi}(t)$. Let us choose the gain K to minimize the following criterion

$$J := \frac{1}{T}\int_0^T [\phi(\bar{\zeta}(t)) - K\bar{\zeta}(t)]^2 dt,$$

where $\bar{\zeta}(t) = C\bar{\xi}(t) + Du(t)$. This optimal gain can be derived from the condition

$$\frac{dJ}{dK} = 0$$

and is given by

$$K^* = \left(\int_0^T \bar{\zeta}^2(t)dt\right)^{-1} \int_0^T \phi(\bar{\zeta}(t))\bar{\zeta}(t)dt.$$

Now, suppose the input $u(t)$ is a harmonic function of period $T = 2\pi/\omega$

$$u(t) = b\sin\omega t, \tag{4}$$

so the output $\bar{\zeta}$ is given by

$$\bar{\zeta}(t) = a\sin(\omega t + \psi), \quad a > 0 \tag{5}$$

with some ψ.

In this case the optimal gain is given as a function of the amplitude a of the output $\bar{\zeta}(t)$:

$$K(a) = \frac{1}{1\pi a}\int_0^{2\pi} \phi(a\sin\theta)\sin\theta d\theta.$$

If ϕ is an odd function, $K(a)$ is its describing function. Examples of calculations of $K(a)$ for various ϕ can be found in many textbooks on the describing function method. From now on we assume that ϕ is an odd function and, consequently,

$$K(a) = \frac{2}{\pi a}\int_0^{\pi} \phi(a\sin\theta)\sin\theta d\theta.$$

Consider system (3). Let $s = \frac{d}{dt}$. Then one can write

$$\begin{aligned}\bar{\zeta}(t) = &-C(sI_n - A)^{-1}BK(a)\bar{\zeta} \\ &+(C(sI_n - A)^{-1}F + D)u(t).\end{aligned}$$

Since the amplitude of $\bar{\zeta}(t)$ is a, the following relation is valid if A does not have pure imaginary eigenvalues $\pm i\omega$

$$|1 + K(a)G(i\omega)|^2 a^2 = |C(i\omega I_n - A)^{-1}F + D|^2 b^2 \tag{6}$$

where $G(i\omega) = C(i\omega I_n - A)^{-1}B$. The equation (6) is referred to as the *harmonic balance equation.* For this equation one can pose the following problem: given $b > 0, \omega > 0$, does equation (6) have a unique positive real solution $a(b,\omega)$? If so, then substituting $K(a(b,\omega))$ in (3) instead of K one can easily compute the solution $\bar{\xi}(t)$ and then to consider the question of how accurate the solution $\bar{\xi}(t)$ approximates the solution $\bar{x}(t)$.

Before we study equation (6) let us characterize the function $K(a)$. If the nonlinear function ϕ satisfies either the *sector* or the *incremental sector* condition, it is possible to characterize the function $K(a)$ as given by the following results.

Lemma 1. Assume that for all $y \in \mathbb{R}$, $y \neq 0$ there is a $\mu > 0$ such that

$$0 \leq \frac{\phi(y)}{y} \leq \mu$$

Then

$$0 \leq K(a) \leq \mu, \quad \forall a \geq 0.$$

Proof: See (Khalil, 2002), page 285. □

By analogy,

Lemma 2. Assume that for all $y_1, y_2 \in \mathbb{R}$, $y_1 \neq y_2$ there is a $\mu > 0$ such that

$$0 \le \frac{\phi(y_1) - \phi(y_2)}{y_1 - y_2} \le \mu$$

Then

$$0 \le \frac{K(a_1)a_1 - K(a_2)a_2}{a_1 - a_2} \le \mu, \quad \forall a_1, a_2 \ge 0,\ a_1 \ne a_2. \tag{7}$$

Proof: Denote

$$L_K := \frac{K(a_1)a_1 - K(a_2)a_2}{a_1 - a_2}$$

Then

$$\begin{aligned} L_K &= \frac{1}{a_1 - a_2}\left(\frac{2}{\pi}\int_0^\pi \phi(a_1 \sin\theta)\sin\theta d\theta \right. \\ &\quad \left. - \frac{2}{\pi}\int_0^\pi \phi(a_2 \sin\theta)\sin\theta d\theta\right) \\ &= \frac{2}{\pi}\int_0^\pi \frac{(\phi(a_1\sin\theta) - \phi(a_2\sin\theta))\sin^2\theta d\theta}{a_1\sin\theta - a_2\sin\theta} \\ &\le \frac{2\mu}{\pi}\int_0^\pi \sin^2\theta d\theta = \mu \end{aligned}$$

The left inequality is proven in the same way. □

Now consider again equation (6). The following result is valid.

Theorem 3. Suppose the matrix A does not have pure imaginary eigenvalues $\pm i\omega$ and the following frequency inequality

$$\text{Re } G(i\omega) > -\frac{1}{\mu} \tag{8}$$

is fulfilled. Then for any function $K(a)$ satisfying (7) and for any $b > 0$ there is a unique positive real solution $a(b,\omega)$ of equation (6).

Conversely, if

$$\text{Re } G(i\omega) < -\frac{1}{\mu},$$

then there is a function $K(a)$ satisfying (7) such that equation (6) has multiple distinct positive real solutions a for some $b > 0$.

Proof: Consider the left hand side of equation (6)

$$\pi(a) = |a + K(a)aG(i\omega)|^2$$

The idea of the proof is to show that if the frequency inequality (8) holds then $\pi(a)$ is a strictly increasing function. The sector condition (7) implies that $\pi(a)$ is a (Lipschitz) continuous function. Since $\pi(0) = 0$ and $\pi(\infty) = \infty$, existence and uniqueness of the positive real solution $a(b,\omega)$ of (6) follow.

For the sake of simplicity, we will prove the theorem under assumption that $K(a)a$ is a differentiable function of a. The general case can be deduced from (7) after some ε-δ work, for example, taking into account that

$$\begin{aligned} 0 &\le \liminf_{a_1 \to a_2} \frac{K(a_1)a_1 - K(a_2)a_2}{a_1 - a_2} \\ &\le \limsup_{a_1 \to a_2} \frac{K(a_1)a_1 - K(a_2)a_2}{a_1 - a_2} \le \mu. \end{aligned}$$

Differentiating $\pi(a)$ with respect to a yields

$$\begin{aligned} \frac{\pi(a)'}{a} &= (1 + (Ka)'G)(1 + KG^*) \\ &\quad + (1 + KG)(1 + (Ka)'G^*) \\ &\ge 2\left(1 + (K + (Ka)')\,\text{Re}G \right. \\ &\quad \left. + K(Ka)'[\text{Re}(G)]^2\right) \end{aligned} \tag{9}$$

Taking into account (7), it follows that $0 \le K(a) \le \mu$ and $0 \le (K(a)a)' \le \mu$. Together with (8) it implies that the quadratic expression (9) is positive.

To prove the second part of the theorem, notice that

$$\frac{\pi(a)'}{a} = 2\text{Re}\left[(1 + (Ka)'G)(1 + KG^*)\right]$$

and therefore if we choose the function $K(a)$ and a point a_0 such that $(K(a_0)a_0)'$ is sufficiently close to μ, while $K(a_0)$ is sufficiently close to zero (or a way around), the derivative of π becomes strictly negative for such a $K(a_0)$. However, $\pi(0) = 0$ and $\pi(\infty) = \infty$, therefore, one can choose $b \ge 0$ so that equation (6) has multiple distinct positive real solutions. □

It is worth noting that the frequency inequality from the theorem hypothesis is the same inequality imposed by the incremental circle criterion yet verified only for the frequency of the external excitation.

The previous results allow one to complete the procedure of harmonic linearization for system (1). Indeed, if the frequency condition holds, there is a unique positive real solution $a(b,\omega)$, given

b, ω. Then substituting $K(a(b,\omega))$ in (3) gives a system linear in ξ. For such a system one can calculate the unique periodic solution $\bar{\xi}(t)$ using only algebraic calculations. Like in the standard describing function method, one can expect that $\bar{\xi}(t)$ is sufficiently close to $\bar{x}(t)$. In the next section we derive a bound that estimates this difference in L_2-norm.

3. ACCURACY OF HARMONIC LINEARIZATION

In this section we study how accurate the procedure of harmonic linearization is. As before, to formulate the problem statement, we assume that for a given harmonic input $u(t) = b \sin \omega t$ the system (1) has a unique $2\pi/\omega$-periodic solution $\bar{x}(t)$. Together with system (1) consider the output

$$\bar{z}(t) = H\bar{x}(t), \quad z \in \mathbb{R} \tag{10}$$

with an appropriate matrix H. We further assume that the frequency inequality (8) holds and thus, according to the results of the previous section there is a unique positive real solution $a(b,\omega)$ of the harmonic balance equation (6). The problem addressed here is to find an upper bound for

$$\left(\frac{\omega}{2\pi} \int_0^{2\pi/\omega} \left[\bar{z}(t) - \bar{\eta}(t) \right]^2 dt \right)^{\frac{1}{2}}$$

with

$$\bar{\eta}(t) = H\bar{\xi}(t). \tag{11}$$

Let e be the difference $\bar{x} - \bar{\xi}$. Then

$$\begin{aligned} \dot{e} &= Ae - B\left[\phi(\bar{y}) - \phi(\bar{\zeta})\right] + B\Delta(t) \\ \bar{y} &= C\bar{x} + Du \\ \bar{\zeta} &= C\bar{\xi} + Du \end{aligned} \tag{12}$$

where

$$\Delta(t) = K(a(b,\omega))\bar{\zeta}(t) - \phi(\bar{\zeta}(t))$$

Substituting (5) in the previous expression gives

$$\Delta(t) = K(a(b,\omega)) a \sin(\omega t + \psi) - \phi(a \sin(\omega t + \psi))$$

Let

$$v(a) = \left(\frac{1}{2\pi} \int_0^{2\pi} \left[\frac{2}{\pi} \int_0^{\pi} \phi(a \sin\theta) \sin\theta d\theta \cdot \sin\vartheta \right. \right. \\ \left. \left. - \phi(a \sin\vartheta) \right]^2 d\vartheta \right)^{\frac{1}{2}}.$$

This integral can be calculated using the same technique as in calculation of $K(a)$. Notice that

$$v(a(b,\omega)) = \left(\frac{\omega}{2\pi} \int_0^{2\pi/\omega} \Delta^2(t) dt \right)^{\frac{1}{2}}$$

where $a(b,\omega)$ is the solution of (6).

If one makes a technical assumption that the pair (A,B) is controllable and (A,C) is observable and A does not have pure imaginary eigenvalues then the fulfillment of the frequency condition (8) for all $\omega \in \mathbb{R}$ implies that for any $\varepsilon > 0$ there is a symmetric matrix $P = P^\top$ such that

$$e^\top P \left(Ae - B\left[\phi(\bar{y}) - \phi(\bar{\zeta})\right] \right) \le -\varepsilon e^\top e$$

Taking the derivative of the quadratic form

$$V = \frac{e^\top P e}{2}$$

along the solutions of (12) yields

$$\dot{V} \le -\varepsilon e^\top e + e^\top P B \Delta(t)$$

Completing the squares with an appropriate ε one can write

$$\dot{V} \le -(\bar{z} - \bar{\eta})^2 + \gamma^2 \Delta^2(t)$$

with some finite $\gamma > 0$. Integrating the last inequality from 0 to $2\pi/\omega$ and using periodicity of $V(e(t))$ one gets

$$\left(\frac{\omega}{2\pi} \int_0^{2\pi/\omega} \left[\bar{z}(t) - \bar{\eta}(t) \right]^2 dt \right)^{\frac{1}{2}} \le \gamma v(a(b,\omega)). \tag{13}$$

To find the best possible (smallest) γ one can pose and numerically solve the following optimization problem:

Problem 4. Minimize γ such that

i) P is symmetric

ii)

$$\begin{pmatrix} A^\top P + PA + H^\top H & -PB + \frac{\mu}{2} C^\top & PB \\ -B^\top P + \frac{\mu}{2} C & -1 & 0 \\ B^\top P & 0 & -\gamma^2 \end{pmatrix} \le 0$$

Now we can summarize the above arguments in the following result.

Theorem 5. Consider system (1, 2, 4) under the following assumptions

- (A, B) is controllable and (A, C) is observable.
- matrix A does not have eigenvalues on the imaginary axis.
- the frequency inequality (8) is satisfied for all $\omega \in \mathbb{R}$.

Along with system (1, 2, 4) consider its approximation (3, 4) with $K = K(a(b, \omega))$ and $a(b, \omega)$ being the unique positive real solution of the harmonic balance equation (6). Let γ be the solution to Problem 4. Then there is a unique $2\pi/\omega$-periodic solution $\bar{x}(t)$ of (1,4) and the estimate (13) holds for $\bar{z}$ and $\bar{\eta}$ defined in (10), (11).

To complete the proof one has to show that there is a unique $2\pi/\omega$-periodic solution. That follows from the frequency domain inequality via contraction mapping argument.

The previous result requires fulfillment of the incremental circle criterion (frequency domain inequality) which is a sufficient condition that for a given $b > 0$ and $\omega > 0$ the corresponding coefficient of harmonic linearization is uniquely determined. The frequency domain inequality is also a necessary condition in the sense of Theorem 3: it ensures solvability of the algebraic harmonic balance equation for the *class* of functions $K(\cdot)$ satisfying condition (7). However it is possible that for a *given* nonlinearity ϕ the corresponding harmonic balance equation has a unique positive real solution $a(b, \omega)$ while the frequency domain inequality does not hold. In this case it is still possible to estimate the accuracy of the method of harmonic linearization. We hope that an LMI-based procedure of checking frequency-domain inequality for a (semi)-finite range of frequencies can be derived with a recent generalization of Kalman-Yakubovich-Popov lemma due to A. Fradkov (Fradkov, 2006).

It is important to note that the procedure given by Theorem 5 is numerically efficient. Once the incremental gain γ is found by solving Problem 4, the limit solution of the linear approximation and the upper bound on the error of this solution (13) can be easily computed for various pairs of (b,ω) from a certain domain of interest.

The previous theorem can be further generalized if one takes into account that Δ is a T-periodic signal and its Fourier transform does not contain the first harmonic.

Denote

$$\rho_1 := \sup_{k=3,5,\ldots} |C(ik\omega I_n - A + \frac{\mu}{2}BC)^{-1}B|$$

$$\rho_2 := \sup_{k=3,5,\ldots} |H(ik\omega I_n - A + \frac{\mu}{2}BC)^{-1}B|$$

Theorem 6. Consider system (1, 2, 4) under the following assumptions

- (A, B) is controllable and (A, C) is observable.
- the harmonic balance equation has a unique positive real solution $a(b, \omega)$
- $\rho_1 \mu < 2$
- ϕ is an odd function

Along with system (1, 2, 4) consider its approximation (3, 4) with $K = K(a(b, \omega))$ and $a(b, \omega)$ being the unique positive real solution of the harmonic balance equation (6). Let γ be defined as

$$\gamma = \frac{2\rho_2}{2 - \mu\rho_1}$$

Then there is a unique $2\pi/\omega$-periodic solution $\bar{x}(t)$ of (1,4) and the estimate (13) holds for $\bar{z}$ and $\bar{\eta}$ defined in (10), (11).

4. ILLUSTRATIVE EXAMPLE

Consider system (1) with

$$A = \begin{bmatrix} 0 & 0 \\ -K_i & -K_iK_a \end{bmatrix}, \quad B = \begin{bmatrix} -1 \\ K_p - K_iK_a \end{bmatrix},$$

$$C = [0 \quad 1], \quad F = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

with positive K_i, K_p, f and nonnegative K_a and saturation nonlinearity

$$\phi(y) = \mathrm{sign(y)} \min\{1, y\}$$

This system corresponds to a PI-controlled integrator with saturation and anti-windup (if $K_a >$

0). The describing function of the saturation nonlinearity is given by

$$K(a) = \begin{cases} 1, & a \le 1 \\ \dfrac{2}{\pi}\left(\sin^{-1}\left(\dfrac{1}{a}\right) + \dfrac{1}{a}\sqrt{1-\dfrac{1}{a^2}}\right), & a > 1 \end{cases}$$

The square root of the left hand side of the harmonic balance equation $\sqrt{\pi(a)}$ as a function of a is depicted in Fig. 1 with $K_i = 20, K_p = 10, K_a = 0, \omega = 1$ (no anti-windup). The shape of this curve

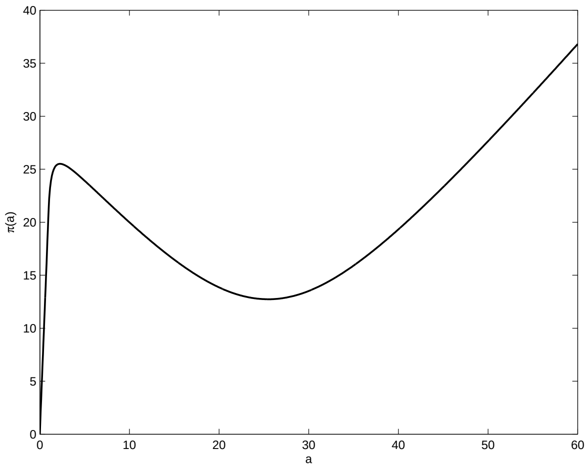

Fig. 1. The square root of the left hand side of the harmonic balance equation $\sqrt{\pi(a)}$ versus a, $K_a = 0$.

suggests that for relatively small b the system has one periodic solution and then, with increase of b the system has three periodic solutions, and then, again if b further increases, the system again has only one periodic solution. This quantitative conclusion is supported by numerical simulation.

It can be proved (van den Berg *et. al.*, 2006) that if $K_a K_p > 1$ (with anti-windup) then the system has exponentially stable (yet not quadratically) periodic solution for arbitrary b, ω. This can be illustrated with Fig. 2 - the function $\pi(a)$ becomes monotonically increasing for $K_a = 1/K_p = 0.1, \omega = 0.4$.

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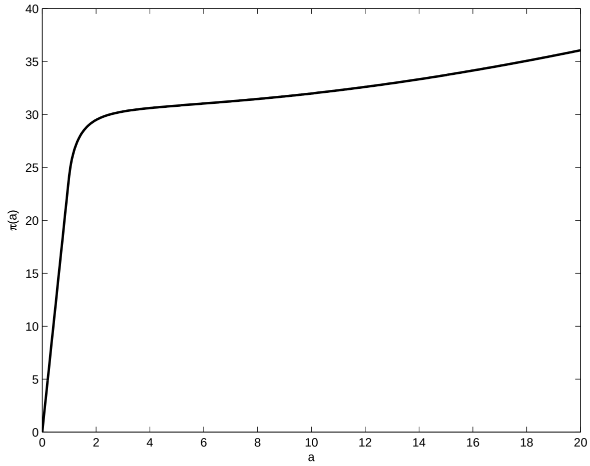

Fig. 2. The square root of the left hand side of the harmonic balance equation $\sqrt{\pi(a)}$ versus a, $K_a = 1/K_p$.

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