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NONLINEAR MODAL PROPERTIES OF DISCRETE SYSTEMS WITH PIECEWISE RESTORING FORCE

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Abstract

Nonlinear modal properties of vibrating systems with nonsmooth (piecewise linear) characteristics typical of a beam with a breathing crack are investigated. The system is non linearizable and exhibits the peculiar feature of a number of nonlinear normal modes (NNMs) greater than the degrees of freedom. A parametric analysis of the NNMs has been performed for a wide range of the damage parameter: the influence of damage on the nonlinear frequencies has been investigated and bifurcations characterized by the onset of superabundant modes with or without internal resonance, have been revealed.

Key words

Nonlinear normal modes, piecewise linear systems, bifurcations, internal resonance.

1 Introduction

The classical modal analysis in linear dynamics can be extended to nonlinear systems by introducing the concept of nonlinear normal modes (NNMs). According to Rosemberg [Rosemberg, 1962; Vakakis, 1996] a NNM of an undamped system is defined as a synchronous periodic oscillation where all generalized coordinates of the system reach their extreme values or pass through the zeros simultaneously. To make this definition suitable for nonsmooth systems, it is necessary to include in the definition of NNMs also the periodic motions in which all generalized coordinates vibrate *not necessarily* in a synchronous way [Chati et al., 1997; Vestroni et al., 2006]. The NNMs are very important because, in analogy to linear theory, resonance in forced systems typically occurs in the neighborhood of the frequencies of NNMs of the free systems. Hence, knowledge of the normal modes of a nonlinear system can provide valuable insight on the position of its resonances, a feature of considerable engineering importance. Moreover, since the number of normal modes of a nonlinear system may exceed its degrees of freedom (*superabundant NNMs*), certain forced resonances are essentially nonlinear and have no analogies in linear theory; in such cases a linearisation of the system either might not be possible, or might not provide all the possible resonances that can be realized. The present paper is devoted to analyze the nonlinear modal characteristics of 2-DOF mechanical systems with piecewise linear restoring force. These oscillators, Fig. 1, are roughly representative of an asymmetrically cracked cantilever beam vibrating in bending and hence exhibiting a bilinear stiffness depending on whether the crack is open or closed. This is a widely studied example of continuous piecesewise smooth system (PSS): in other words, the phase space is divided into different regions and the system vector field is the same in the adjacent regions separated by the boundary whereas its Jacobian changes. The system is non linearizable and for specific damage values exhibits a number of NNMs greater than the degrees of freedom; since the nonlinearity is concentrated at the origin, its nonlinear frequencies are independent of the energy level and uniquely depend on the damage parameter [Chati et al., 1997; Vestroni et al., 2006]. A contribution of the paper is represented by the investigation of the onset of superabondant modes caused by internal resonance: it will be shown that the shape of these nonlinear normal modes is very different than that of modes on fundamental branch and that their influence on the forced response is very significant.

2 System model

The investigated system, Fig. 1, consists of a 2-DOF oscillator with a linear and a piecewise linear spring. Two cases have been considered: Model A) the linear spring of constant k_2 connects the two masses m_1 and m_2, whilst the piecewise linear spring with undamaged stiffness k_1 and reduced stiffness εk_1 ($0 \leq \varepsilon < 1$) connects m_1 to the ground (Fig. 1a); Model B) the piecewise linear spring connects the two masses, whilst the linear one connects m_1 to the ground (Fig. 1b). When ε=0 the two models are reduced to the same linear system.

2.1 Governing equations

By assuming:

$$y_1 = x_1, y_2 = \dot{x}_1,\ y_3 = x_2, y_4 = \dot{x}_2 \qquad (1)$$

the equations of motion read as:

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = \left(-(k_{bil} + k_2) y_1 + k_2 y_3\right)/m_1 \\ \dot{y}_3 = y_4 \\ \dot{y}_4 = \left(k_2 y_1 - k_2 y_3\right)/m_2 \end{cases} \qquad (2)$$

for Model A, and:

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = \left(-(k_{bil} + k_1) y_1 + k_{bil} y_3\right)/m_1 \\ \dot{y}_3 = y_4 \\ \dot{y}_4 = \left(k_{bil} y_1 - k_{bil} y_3\right)/m_2 \end{cases} \qquad (3)$$

for Model B. The piecewise linear stiffness, Fig. 1c, is given by:

$$k_{bil} = k_i(1 - H\varepsilon),\ H = \begin{cases} 1 & \eta \geq 0 \\ 0 & \eta < 0 \end{cases} \qquad (4)$$

where $k_i=k_1$, $\eta=y_1$ (Model A); $k_i=k_2$, $\eta=y_3-y_1$ (Model B).

These models are then numerically analyzed for their NNMs in configuration space with procedures based on continuation techniques and Poincaré maps; the relevant bifurcations are studied as a function of damage parameter ε. In the following it is assumed that $k_1=k_2=k$ and $m_1=m_2=m$.

3 Normal Modes

3.1 Linear Normal Modes (LNM)

For ε=0 the system is linear and exhibits the two LNMs $\mathbf{u}_\mathbf{x}$ and $\mathbf{u}_\mathbf{y}$ the modal lines of which are reported in Fig. 2. Both lines pass through the origin and the frequency ratio is given by:

$$\frac{\omega_y}{\omega_x} = \frac{1}{2}\left(3 + \sqrt{5}\right) \cong 2.62 \qquad (4)$$

3.2 Nonlinear Normal Modes (NNM)

As observed in Sect. 1, the NNMs are a generalization of LNMs: extending the classical Rosenberg's definition we define Nonlinear Normal Mode any (non-necessarily synchronous) periodic motion of the system.

For ε>0 the system becomes nonlinear and the two models exhibit a different dynamical behaviour, however some common basic features concerning the NNMs can be recognized. *i)* The two LNMs (ε=0) become NNMs for ε>0 with frequencies ω_1, ω_2 independent of energy level and, for fixed values of m and k, uniquely depending on ε.

ii) Both ω_1 and ω_2 decrease for increasing ε and the frequency-damage law can be approximately determined in closed form [Chati et al., 1997, Casini and Vestroni, 2007]. *iii*) The frequency ratio ω_2/ω_1 is independent of m and k [Casini and vestroni, 2007]. In particular, the frequency ratio-damage laws in Model A, Fig. 3a, and in Model B, Fig. 3b, are significantly different giving reasons for the different dynamic scenarios described in next Section. *iv*) When ε varies, first and second NNM may interact generating (n:m) internal resonances [Pak, 2006]: this occurs when the *nonlinear* frequencies ω_1 and ω_2 are nearly commensurate i.e. $n\omega_1 \cong m\omega_2$.

4 Evolution of NNMs

The evolution of NNMs and the onset of superabundant modes are studied as a function of ε. The Poincaré maps with surface of section η=0 and y_2>0 are illustrated. NNMs are also described by means of the relevant modal line in the configuration plane.

4.1 Bifurcations generated by modal interaction

Figures 4 report for both modes the values of the damage parameter corresponding to bifurcations caused by modes 1 and 2 interaction: black circles represent (n:1) internal resonances, gray circles (n:2) and white circles (n:3), (n:4), (n:5) internal resonances. Each internal resonance produces a specific structural change in Poincaré maps. By comparing Figures 4a and 4b it can be observed that, due to the different frequency ratio-damage laws, Model B exhibits internal resonances for larger values of ε; moreover some internal resonances exhibited by Model A are not possible in Model B and *vice-versa*: for instance Model B exhibits a (5:2) internal resonance which cannot be realized in Model A where ω_2/ω_1 is always larger than 2.62>5/2.

Figures 5 and 6 refer to bifurcations occurring in (n:1) internal resonance for Model A; in Model B these are qualitatively similar but, as already noticed, take place for larger ε.

Figure 5 reports the period-damage plot with the various branches of (n:1) periodic solutions: the period of Mode 2 (black curve) is almost constant, whereas the period of Mode 1 significantly increases with ε. There is a sequence of higher periodic NNMs bifurcating from the backbone of Mode 1, called *tongues*. Each tongue takes place in the neighborhood of a (n:1) internal resonance. Enlargement in Fig. 5 and Figures 6 refer to the case (3:1): at $\varepsilon_{3:1}$ one stable and one unstable superabundant NNM (**C**) generated by a cyclic-fold bifurcation appear. In particular, figures 6 report the Poincaré maps with the relevant modal line in the configuration plane.

Figures 7 refer to the bifurcation in the neighborhood of the (7:2) internal resonance: the first mode (**A**) loses its stability and bifurcates in a NNM (**B**) with period doubling. Furthermore a second NNM (**C**) appears and approaches the unstable NNM as long as the first mode recovers the stability. Qualitatively similar behaviour is exhibited by the other (n:2) bifurcations: however for larger values of ε, the bifurcated modal curves are more complicated and the windows around $\varepsilon_{n:2}$ becomes narrower.

Qualitatively different changes in the Poincaré maps are produced by (n:3,4,5…) resonances. For instance, figures 8 refer to the case (8:3): unlike the case (n:2), the first mode (**A**) is always stable but two pairs of stable and unstable NNMs (**B**, **C**) appear and disappear; the frequency content in **B** and **C** is characterized by two main frequencies the ratio of which is exactly 8-to-3.

4.2 Bifurcations with no internal resonance

Both models show some bifurcations unrelated to internal resonances. An interesting case is revealed in Model B where the second mode loses its stability and a period-doubling occurs as shown in Figs. 9. As it can be seen in the Poincaré maps of Figs. 9, besides periodic (single point) and quasi-periodic solutions (closed curve) also chaotic orbit are revealed.

4.3 Remarks

a) The modal curves with frequency ω_1 are bent in several waves while the ones with frequency ω_2 are nearly straight; significant drift increasing with ε is exhibited by the second modal curves [Vestroni et al., 2006]. b) The occurrence of (n:m) internal resonance generates a structural change in the Poincaré map and the onset of one or more superabundant NNMs in which the ratio of frequency is exactly n/m. c) Superabundant NNMs are possible also for weak nonlinearities; for instance in Model A, superabundant NNMs due to a (21:8) resonance arise around $\varepsilon_{21:8}$=0.029. d) Structural changes occur in the Poincaré map for varying ε. It is worth noticing that some global bifurcations are characterized by the onset of pairs of stable and unstable superabundant NNMs while other bifurcations are due to the change in stability of the two modes: in this context it has been found that for both models the first mode loses and recovers its stability several times and the second mode is always stable in Model A whereas becomes unstable for high damage in Model B.

5 Conclusions

A 2-DOF piecewise smooth oscillator, representative of a cracked beam, has been studied in free oscillations. A parametric analysis of the NNMs has been performed for a wide range of the damage parameter: the influence of damage on the nonlinear frequencies has been investigated and bifurcations characterized by the onset of superabundant modes have been revealed. The fundamental branches of the two modes, and their stability are then evaluated. The bifurcated branches are followed by a numerical procedure based on continuation method and the stable superabundant modes are determined via direct integration. Particular attention has been devoted to the study nonlinear modal interaction producing global changes in the Poincarè maps.

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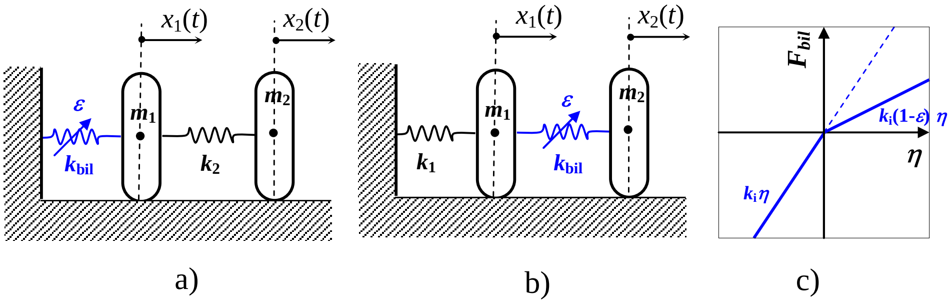

Figure 1. System model: a) model A; b) model B; c) piecewise restoring force.

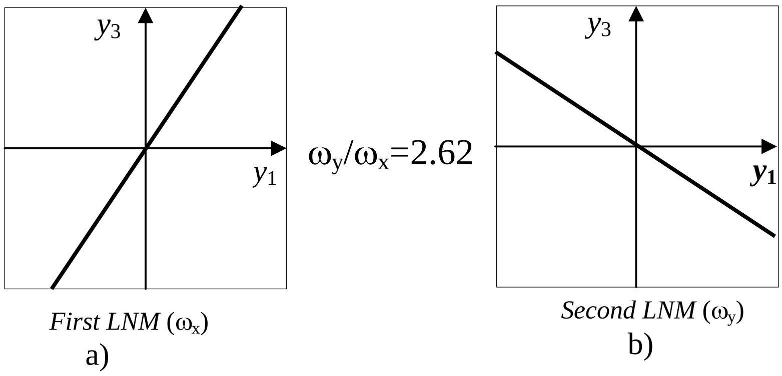

Figure 2. Linear Normal Modes (LNM) for ε=0: a) in-phase; b); out-of-phase mode.

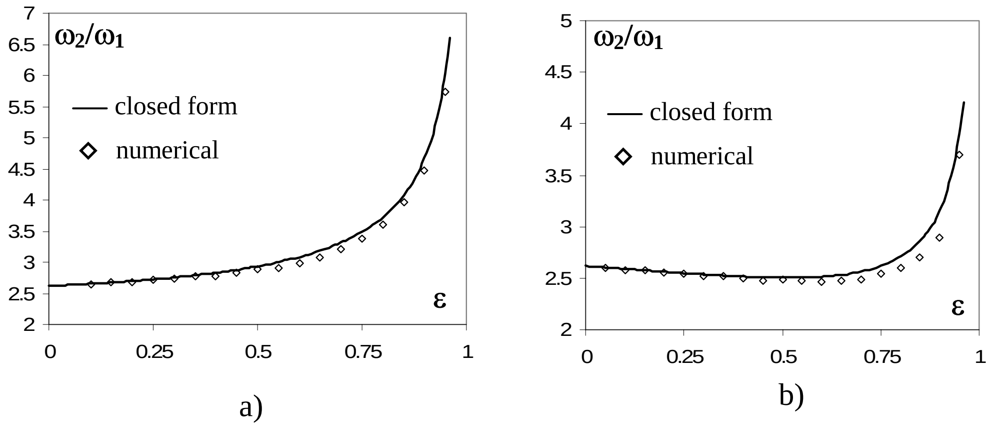

Figure 3. ω_2/ω_1 ratio-damage plot: a) model A; b); model B.

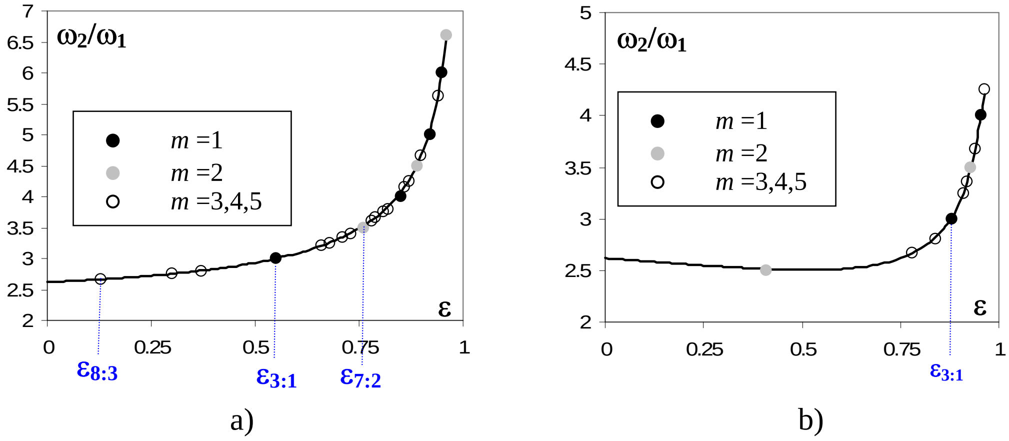

Figure 4. Modes 1 and 2 internal resonances (n:m): a) model A; b); model B.

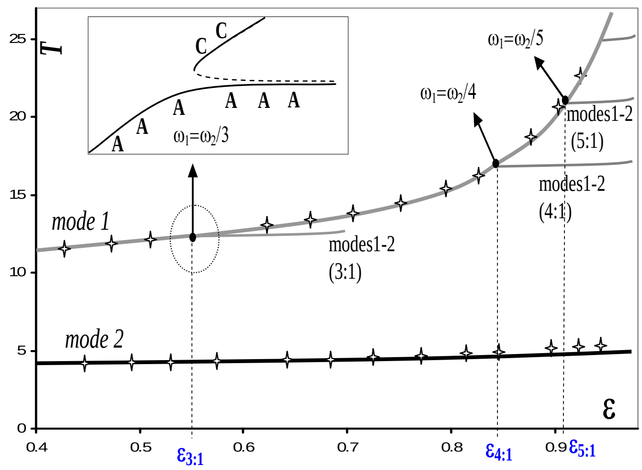

Figure 5. Model A, period-damage plot of NLMs 1 and 2: (n:1) internal resonance.

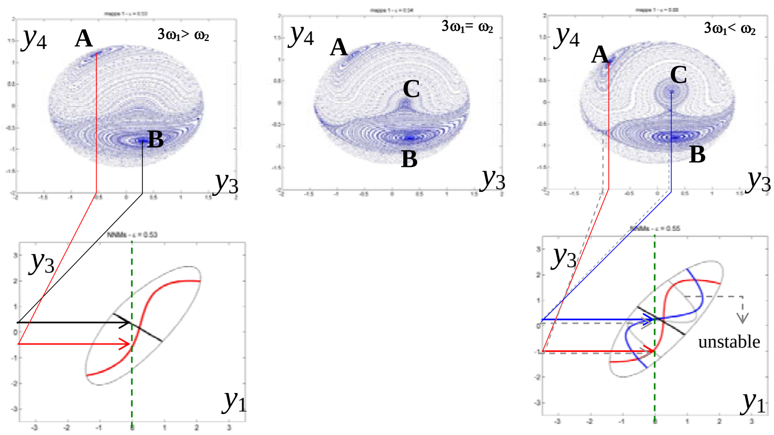

Figure 6. Model A: cyclic-fold bifurcation caused by a (3:1) internal resonance around $\varepsilon_{3:1}$=0.54

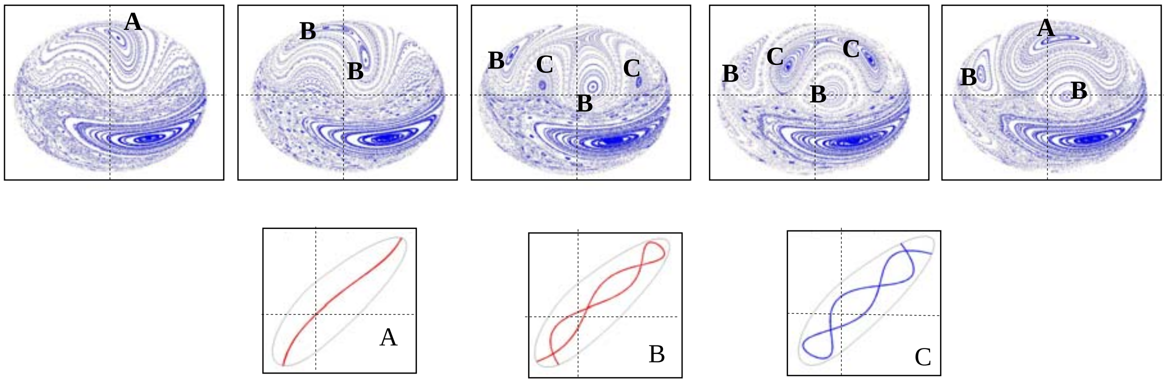

Figure 7. Model A, (7:2) internal resonance near $\varepsilon_{7:2}$=0.7585: period-doubling.

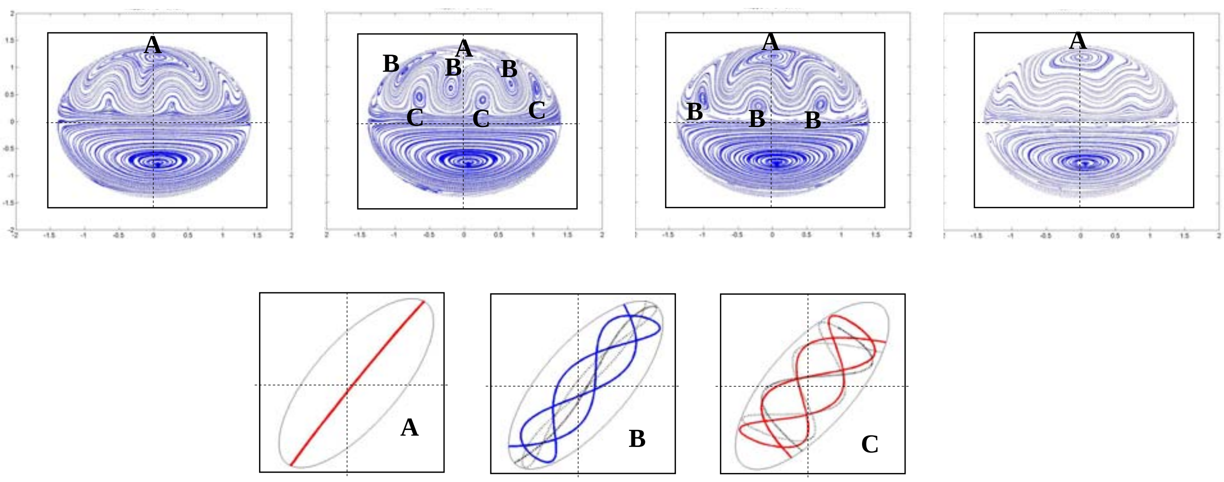

Figure 8. Model A, (8:3) internal resonance near $\varepsilon_{8:3}$=0.138.

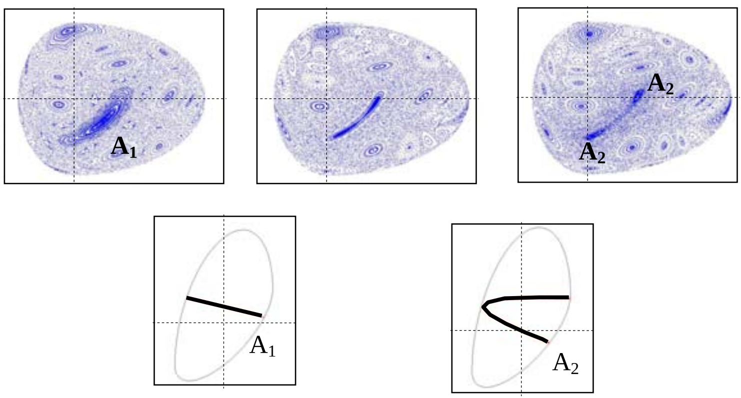

Figure 9. Model B: stability loss and period doubling of mode 2 around ε=0.828.