LIMIT CYCLE BIFURCATIONS OF THE CLASSICAL LIÉNARD POLYNOMIAL SYSTEM

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Abstract
In this paper, applying a canonical system with field rotation parameters and using geometric properties of the spirals filling the interior and exterior domains of limit cycles, we solve the problem on the maximum number of limit cycles for the classical Liénard polynomial system which is related to the solution of Smale's thirteenth problem.

Key words
Liénard's polynomial system; Smale's thirteenth problem; limit cycle; field rotation parameter; bifurcation.

1 Introduction

Consider polynomial Liénard equations of the form

$$\ddot{x} + f(x)\,\dot{x} + g(x) = 0, \tag{1.1}$$

where $f(x)$ and $g(x)$ are known respectively as the damping and restoring coefficients. In the phase plane, the representation of the Liénard equation (1.1) is given by the dynamical system

$$\dot{x} = y, \quad \dot{y} = -g(x) - f(x)\,y. \tag{1.2}$$

There are many examples in the natural sciences and technology in which this and related systems are applied [Bautin and Leontovich, 1990], [Gasull and Torregrosa, 1999 – Smale, 1998]. Such systems are often used to model either mechanical or biomedical systems, and in the literature, many systems are transformed into Liénard type to aid in the investigations. Recently, e. g., the Liénard system (1.2) has been shown to describe the operation of an optoelectronics circuit that uses a resonant tunnelling diode to drive a laser diode to make an optoelectronic voltage controlled oscillator [Slight, Romeira, Liquan, Figueiredo, Wasige and Ironside, 2008].

In this paper, applying the bifurcation methods developed in [Botelho and Gaiko, 2006 – Gaiko and van Horssen, 2009, IJDSDE], we study a classical Liénard polynomial system with respect to real variables and parameters,

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + (\alpha_0 + \alpha_1\,x + \alpha_2\,x^2 + \ldots \\ &\quad + \alpha_{2k-1}\,x^{2k-1} + \alpha_{2k}\,x^{2k})\,y, \end{aligned} \tag{1.3}$$

with a unique finite singular point at the origin, when we set $g(x) \equiv x$ in system (1.2).

In [Gaiko, 2003, 2008 and 2009], we have already presented a solution of Hilbert's sixteenth problem in the quadratic case of polynomial systems proving that for quadratic systems four is really the maximum number of limit cycles and $(3\!:\!1)$ is their only possible distribution. We have also established some preliminary results on generalizing our ideas and methods to special cubic, quartic and other polynomial dynamical systems. In [Gaiko and van Horssen, 2004], e. g., we have constructed a canonical cubic dynamical system of Kukles type and have carried out the global qualitative analysis of its special case corresponding to a generalized Liénard equation. In particular, it has been shown that the foci of such a Liénard system can be at most of second order and that such system can have at most three limit cycles in the whole phase plane. Moreover, unlike all previous works on the Kukles-type systems, global bifurcations of limit and separatrix cycles using arbitrary (including as large as possible) field rotation parameters of the canonical system have been studied. As a result, a classification of all possible types of separatrix cycles for the generalized Liénard system has been obtained and all possible distributions of its limit cycles have been found. In [Gaiko and van Horssen, 2009, IJBC and IJDSDE], we have completed the global qualitative analysis of a planar Liénard-type dynamical system with a piecewise linear function containing an arbitrary number of dropping sections and approximating an arbitrary polynomial function. In [Botelho and Gaiko, 2006; Broer and Gaiko, 2010], we have carried out the global qualitative analysis of a centrally symmetric cubic system which is used as a learning model of a planar neural network and a quartic dynamical system which models the dynamics of the populations of predators and their prey in a given ecological system, respectively.

In Section 2 of this paper, applying a canonical system with field rotation parameters and using geometric properties of the spirals filling the interior and exterior domains of limit cycles, we solve the problem on the maximum number of limit cycles for system (1.3) which is related to the solution of Smale's thirteenth problem [Smale, 1998].

2 Liénard polynomial system

Consider the Liénard polynomial system (1.3). It is easy to see that system (1.3) has a unique finite singular point: an anti-saddle at the origin. At infinity, system (1.3) for $k \geq 1$ has two singular points: a node at the "ends" of the x-axis and a saddle at the "ends" of the y-axis. For studying the infinite singularities, the methods applied in [Bautin and Leontovich, 1990] for Rayleigh's and van der Pol's equations and also Erugin's two-isocline method developed in [Gaiko, 2003] can be used. Following [Gaiko, 2003], we will study limit cycle bifurcations of (1.3) by means of a canonical system containing only the field rotation parameters of (1.3). The following theorem is valid.

Theorem 2.1. *The Liénard polynomial system (1.3) with limit cycles can be reduced to the canonical form*

$$\begin{aligned} \dot{x} &= y \equiv P, \\ \dot{y} &= -x + (\alpha_0 + x + \alpha_2\, x^2 + \ldots + x^{2k-1} \\ &\quad +\alpha_{2k}\, x^{2k})\, y \equiv Q, \end{aligned} \tag{2.1}$$

where $\alpha_0, \alpha_2, \ldots, \alpha_{2k}$ are field rotation parameters of (1.3).

Proof. Let all parameters of system (1.3) with even indexes vanish,

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + (\alpha_1\, x + \alpha_3\, x^3 + \ldots \\ &\quad +\alpha_{2k-1}\, x^{2k-1})\, y, \end{aligned} \tag{2.2}$$

and consider the corresponding equation

$$\frac{dy}{dx} = F(x, y) \equiv \frac{-x+(\alpha_1\, x+\alpha_3\, x^3+\ldots+\alpha_{2k-1}\, x^{2k-1})\, y}{y}. \tag{2.3}$$

Since $F(-x, y) = -F(x, y)$, the direction field of (2.3) (and the vector field of (2.2) as well) is symmetric with respect to the y-axis. It follows that for arbitrary values of the parameters $\alpha_1, \alpha_3, \ldots, \alpha_{2k-1}$ system (2.2) has a center at the origin and cannot have a limit cycle surrounding this point. Therefore, without loss of generality, all odd parameters of system (1.3) can be supposed to be equal, e. g., to one: $\alpha_1 = \alpha_3 = \ldots = \alpha_{2k-1} = 1$ (see also [Gaiko, 2006]).

To prove that the rest (even) parameters rotate the vector field of (2.1), let us calculate the following determinants:

$$\begin{aligned} \Delta_{\alpha_0} &= PQ'_{\alpha_0} - QP'_{\alpha_0} = y^2 \geq 0, \\ \Delta_{\alpha_2} &= PQ'_{\alpha_2} - QP'_{\alpha_2} = x^2 y^2 \geq 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \Delta_{\alpha_{2k}} &= PQ'_{\alpha_{2k}} - QP'_{\alpha_{2k}} = x^{2k} y^2 \geq 0. \end{aligned}$$

By definition of a field rotation parameter [Gaiko, 2003], for increasing each of the parameters $\alpha_0, \alpha_2, \ldots, \alpha_{2k}$, under the fixed others, the vector field of system (5) is rotated in a positive direction (counterclockwise) in the whole phase plane; and, conversely, for decreasing each of these parameters, the vector field of (2.1) is rotated in a negative direction (clockwise).

Thus, for studying limit cycle bifurcations of (1.3), it is sufficient to consider canonical system (2.1) containing only its even parameters, $\alpha_0, \alpha_2, \ldots, \alpha_{2k}$, which rotate the vector field of (2.1). The theorem is proved. □

By means of canonical system (2.1), let us study global limit cycle bifurcations of (1.3) and prove the following theorem.

Theorem 2.2. *The Liénard polynomial system (1.3) has at most k limit cycles.*

Proof. According to Theorem 2.1, for the study of limit cycle bifurcations of system (1.3), it is sufficient to consider canonical system (2.1) containing only the field rotation parameters of (1.3): $\alpha_0, \alpha_2, \ldots, \alpha_{2k}$.

Let all these parameters vanish:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + (x + x^3 + \ldots + x^{2k-1})\, y. \end{aligned} \tag{2.4}$$

System (2.4) is symmetric with respect to the y-axis and has a center at the origin. Let us input successively the field rotation parameters into this system beginning with the parameters at the highest degrees of x in parentheses and alternating with their signs, see [Gaiko, 2006]. So, begin with the parameter α_{2k} and let, for definiteness, $\alpha_{2k} > 0$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + (x + x^3 + \ldots + x^{2k-1} \\ &\quad +\alpha_{2k}\, x^{2k})\, y. \end{aligned} \tag{2.5}$$

In this case, the vector field of (2.5) is rotated in the positive direction (counterclockwise) turning the origin into a nonrough unstable focus [Bautin and Leontovich, 1990].

Fix α_{2k} and input the parameter $\alpha_{2k-2} < 0$ into (2.5):

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= -x + (x + x^3 + \ldots + \alpha_{2k-2}x^{2k-2} \\ &\quad + x^{2k-1} + \alpha_{2k}\, x^{2k})\, y.\end{aligned} \tag{2.6}$$

Then the vector field of (2.6) is rotated in the opposite direction (clockwise) and the focus immediately changes the character of its stability (since its degree of nonroughness decreases and the sign of the field rotation parameter at the lower degree of x in parentheses changes) generating a stable limit cycle. Under further decreasing α_{2k-2}, this limit cycle will expand infinitely, not disappearing at infinity (because of the parameter α_{2k} at the higher degree of x).

Denote the limit cycle by Γ_1, the domain outside the cycle by D_1, the domain inside the cycle by D_2 and consider logical possibilities of the appearance of other (semi-stable) limit cycles from a "trajectory concentration" surrounding the origin. It is clear that, under decreasing the parameter α_{2k-2}, a semi-stable limit cycle cannot appear in the domain D_2, since the focus spirals filling this domain will untwist and the distance between their coils will increase because of the vector field rotation.

By contradiction, we can also prove that a semi-stable limit cycle cannot appear in the domain D_1. Suppose it appears in this domain for some values of the parameters $\alpha^*_{2k} > 0$ and $\alpha^*_{2k-2} < 0$. Return to initial system (2.4) and change the inputting order for the field rotation parameters. Input first the parameter $\alpha_{2k-2} < 0$:

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= -x + (x + x^3 + \ldots + \alpha_{2k-2}x^{2k-2} \\ &\quad + x^{2k-1})\, y.\end{aligned} \tag{2.7}$$

Fix it under $\alpha_{2k-2} = \alpha^*_{2k-2}$. The vector field of (2.7) is rotated clockwise and the origin turns into a nonrough stable focus. Inputting the parameter $\alpha_{2k} > 0$ into (2.7), we get again system (2.6), the vector field of which is rotated counterclockwise. Under this rotation, a stable limit cycle Γ_1 will immediately appear from infinity, more precisely, from a separatrix cycle of the Poincaré circle form containing infinite singularities of saddle and node types [Bautin and Leontovich, 1990]. This cycle will contract, the outside spirals winding onto the cycle will untwist and the distance between their coils will increase under increasing α_{2k} to the value α^*_{2k}. It follows that there are no values of $\alpha^*_{2k-2} < 0$ and $\alpha^*_{2k} > 0$, for which a semi-stable limit cycle could appear in the domain D_1.

This contradiction proves the uniqueness of a limit cycle surrounding the origin in system (2.6) for any values of the parameters α_{2k-2} and α_{2k} of different signs. Obviously, if these parameters have the same sign, system (2.6) has no limit cycles surrounding the origin at all.

Let system (2.6) have the unique limit cycle Γ_1. Fix the parameters $\alpha_{2k} > 0$, $\alpha_{2k-2} < 0$ and input the third parameter, $\alpha_{2k-4} > 0$, into this system:

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= -x + (x + x^3 + \ldots + \alpha_{2k-4}x^{2k-4} \\ &\quad + x^{2k-3} + \ldots + \alpha_{2k}\, x^{2k})\, y.\end{aligned} \tag{2.8}$$

The vector field of (2.8) is rotated counterclockwise, the focus at the origin changes the character of its stability and the second (unstable) limit cycle, Γ_2, immediately appears from this point. Under further increasing α_{2k-3}, the limit cycle Γ_2 will join with Γ_1 forming a semi-stable limit cycle, Γ_{12}, which will disappear in a "trajectory concentration" surrounding the origin. Can another semi-stable limit cycle appear around the origin in addition to Γ_{12}? It is clear that such a limit cycle cannot appear either in the domain D_1 bounded on the inside by the cycle Γ_1 or in the domain D_3 bounded by the origin and Γ_2 because of the increasing distance between the spiral coils filling these domains under increasing the parameter α_{2k-4}.

To prove the impossibility of the appearance of a semi-stable limit cycle in the domain D_2 bounded by the cycles Γ_1 and Γ_2 (before their joining), suppose the contrary, i. e., for some set of values of the parameters, $\alpha^*_{2k} > 0$, $\alpha^*_{2k-2} < 0$, and $\alpha^*_{2k-4} > 0$, such a semi-stable cycle exists. Return to system (8) again and input first the parameters $\alpha_{2k-4} > 0$ and $\alpha_{2k} > 0$:

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= -x + (x + x^3 + \ldots + \alpha_{2k-4}x^{2k-4} \\ &\quad + x^{2k-3} + \alpha_{2k}\, x^{2k})\, y.\end{aligned} \tag{2.9}$$

Both parameters act in a similar way: they rotate the vector field of (2.9) counterclockwise turning the origin into a nonrough unstable focus.

Fix these parameters under $\alpha_{2k-4} = \alpha^*_{2k-4}$, $\alpha_{2k} = \alpha^*_{2k}$ and input the parameter $\alpha_{2k-2} < 0$ into (2.9) getting again system (2.8). Since, by our assumption, this system has two limit cycles for $\alpha_{2k-2} > \alpha^*_{2k-2}$, there exists some value of the parameter, α^{12}_{2k-2} ($\alpha^*_{2k-2} < \alpha^{12}_{2k-2} < 0$), for which a semi-stable limit cycle, Γ_{12}, appears in system (2.8) and then splits into a stable cycle, Γ_1, and an unstable cycle, Γ_2, under further decreasing α_{2k-2}. The formed domain D_2 bounded by the limit cycles Γ_1, Γ_2 and filled by the spirals will enlarge since, on the properties of a field rotation parameter, the interior unstable limit cycle Γ_2 will contract and the exterior stable limit cycle Γ_1 will expand under decreasing α_{2k-2}. The distance between the spirals of the domain D_2 will naturally increase, which will prevent the appearance of a semi-stable limit cycle in this domain for $\alpha_{2k-2} < \alpha^{12}_{2k-2}$.

Thus, there are no such values of the parameters, $\alpha^*_{2k} > 0$, $\alpha^*_{2k-2} < 0$, and $\alpha^*_{2k-4} > 0$, for which system (2.8) would have an additional semi-stable limit cycle. Obviously, there are no other values of the parameters α_{2k}, α_{2k-2}, and α_{2k-4} for which system (2.8) would have more than two limit cycles surrounding the origin. Therefore, two is the maximum number of limit cycles for system (2.8).

Suppose that system (2.8) has two limit cycles, Γ_1 and Γ_2 (this is always possible if $\alpha_{2k} \gg -\alpha_{2k-2} \gg \alpha_{2k-4} > 0$). Fix the parameters α_{2k}, α_{2k-2}, α_{2k-4} and consider a more general system than (2.8) inputting the fourth parameter, $\alpha_{2k-6} < 0$, into (2.8):

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + (x + x^3 + \ldots + \alpha_{2k-6}x^{2k-6} \\ &\quad + x^{2k-5} + \ldots + \alpha_{2k}\, x^{2k})\, y. \end{aligned} \tag{2.10}$$

For decreasing α_{2k-6}, the vector field of (2.10) will be rotated clockwise and the focus at the origin will immediately change the character of its stability generating a third (stable) limit cycle, Γ_3. With further decreasing α_{2k-6}, Γ_3 will join with Γ_2 forming a semi-stable limit cycle, Γ_{23}, which will disappear in a "trajectory concentration" surrounding the origin; the cycle Γ_1 will expand infinitely tending to the Poincaré circle at infinity.

Let system (2.10) have three limit cycles: Γ_1, Γ_2, Γ_3. Could an additional semi-stable limit cycle appear with decreasing α_{2k-6}, after a splitting of which system (2.10) would have five limit cycles around the origin? It is clear that such a limit cycle cannot appear either in the domain D_2 bounded by the cycles Γ_1 and Γ_2 or in the domain D_4 bounded by the origin and Γ_3 because of the increasing distance between the spiral coils filling these domains after decreasing α_{2k-6}. Consider two other domains: D_1 bounded on the inside by the cycle Γ_1 and D_3 bounded by the cycles Γ_2 and Γ_3. As before, we will prove the impossibility of the appearance of a semi-stable limit cycle in these domains by contradiction.

Suppose that for some set of values of the parameters $\alpha^*_{2k} > 0$, $\alpha^*_{2k-2} < 0$, $\alpha^*_{2k-4} > 0$, and $\alpha^*_{2k-6} < 0$, such a semi-stable cycle exists. Return to system (2.4) again, input first the parameters $\alpha_{2k-6} < 0$, $\alpha_{2k-2} < 0$ and then the parameter $\alpha_{2k} > 0$:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + (x + \ldots + \alpha_{2k-6}x^{2k-6} + \ldots \\ &\quad + \alpha_{2k-2}x^{2k-2} + x^{2k-3} + \alpha_{2k}\, x^{2k})y. \end{aligned} \tag{2.11}$$

Fix the parameters α_{2k-6}, α_{2k-2} under the values α^*_{2k-6}, α^*_{2k-2}, respectively. With increasing α_{2k}, the node at infinity will change the character of its stability, the separatrix behaviour of the infinite saddle will be also changed and a stable limit cycle, Γ_1, will immediately appear from the Poincaré circle at infinity [Bautin and Leontovich, 1990]. Fix α_{2k} under the value α^*_{2k} and input the parameter $\alpha_{2k-4} > 0$ into (2.11) getting system (2.10).

Since, by our assumption, (2.10) has three limit cycles for $\alpha_{2k-4} < \alpha^*_{2k-4}$, there exists some value of the parameter α^{23}_{2k-4} $(0 < \alpha^{23}_{2k-4} < \alpha^*_{2k-4})$ for which a semi-stable limit cycle, Γ_{23}, appears in this system and then splits into an unstable cycle, Γ_2, and a stable cycle, Γ_3, with further increasing α_{2k-4}. The formed domain D_3 bounded by the limit cycles Γ_2, Γ_3 and also the domain D_1 bounded on the inside by the limit cycle Γ_1 will enlarge and the spirals filling these domains will untwist excluding a possibility of the appearance of a semi-stable limit cycle there.

All other combinations of the parameters α_{2k}, α_{2k-2}, α_{2k-4}, and α_{2k-6} are considered in a similar way. It follows that system (2.10) has at most three limit cycles. If we continue the procedure of successive inputting the even parameters, $\alpha_{2k-8}, \ldots, \alpha_2, \alpha_0$, into system (2.4), it is possible first to obtain k limit cycles $(\alpha_{2k} \gg -\alpha_{2k-2} \gg \alpha_{2k-4} \gg -\alpha_{2k-6} \gg \alpha_{2k-8} \gg \ldots)$ and then to conclude that canonical system (2.1) (i. e., the Liénard polynomial system (1.3) as well) has at most k limit cycles. The theorem is proved. □

Note that by the change of variables $X = x$ and $Y = y + F(x)$, where $F(x) = \int_0^x f(s)\, \mathrm{d}s$, system (1.2) with $g(x) \equiv x$ is reduced to an equivalent system

$$\dot{X} = Y - F(X), \quad \dot{Y} = -X \tag{2.12}$$

which can be written in the form

$$\dot{x} = y, \quad \dot{y} = -x + F(y) \tag{2.13}$$

or

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + \beta_1\, y + \beta_2\, y^2 + \beta_3\, y^3 + \ldots \\ &\quad + \beta_{2k}\, y^{2k} + \beta_{2k+1}\, y^{2k+1}. \end{aligned} \tag{2.14}$$

Therefore, we can conclude (see also [Gaiko, 2006]) that Theorem 2.2 supports the conjecture of [Lins, de Melo and Pugh, 1977] on the maximum number of limit cycles for the Liénard polynomial system (2.14).

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