EMPIRICAL DETERMINATION OF THE FREQUENCIES OF AN ALMOST PERIODIC TIME SERIES

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ABSTRACT

This paper is concerned with the problem of determination of the finite or countable set $\Lambda = \{\lambda_1, \lambda_2, ...\}$ of frequencies belonging to an almost periodic signal $\{x_k\}$. We seek a simple finite computational method in which a finite set $\Lambda_n = \{\lambda_1^{(n)}, \lambda_2^{(n)}, ..., \lambda_{K_n}^{(n)}\}$ of estimators of frequencies is produced at each stage n from the finite observation $\{x_0, \ldots, x_{2n}\}$ of the sequence. We also want Λ_n converges to Λ but yet each Λ_n is not too big.

We provide a method based on the local maxima of weighted discrete Fourier transform of the finite observation of the almost periodic sequence. The produced estimators are taken from a finite grid $\{2\pi j/(Cn) : j = 0, \ldots, Cn-1\}$ of $[0, 2\pi)$. They converge to the true frequencies with rate $\mathscr{O}(n^{-1})$. We first consider that the signal is observed without noise, and secondly with an additive noise. Then this method is adapted to an almost periodically correlated signal also called almost cyclostationnary signal.

1. INTRODUCTION

Many man-made signals, even natural ones, exhibit periodicities. So in applies studies, the detection of the frequencies from data is essential. To give an example in signal theory, determination of the frequencies is often used for identification problems (e.g. speech recognition [13]). Many other applications can be found in communication theory, climatology, econometry, to name but a few (see e.g. [2, 6, 12]).

The evolution of signal processing device entails a permanent need to perform more efficient methods for detecting the presence of frequencies and estimating these frequencies [5]. Thus there is a large amount of works on frequency estimation or testing problem. In particular for trigonometric polynomial signals with additive noise (see e.g. the survey [1] as well as [4, 8, 10, 11, 14, 15, 16] for some of the more recent papers). In recent papers, He [9, 10] has proposed a method based on threshold for the local maxima of the discrete Fourier transform of a signal with a finite number of frequencies. The comparison of these methods with the strong local maxima introduced below is out of the scope of this paper.

The aim of this paper is the determination of the set of frequencies of an almost periodic sequence, using a simple finite (few time consuming and with lower complexity) computational method based on the local maxima of the discrete Fourier transform (or more precisely a weighted version of the discrete Fourier transform). An estimation procedure is presented and the asymptotic properties of the estimates are studied. Here we are not concerned with the testing problem.

Now we describe the goal of our method. For that we introduce the notion of almost periodic sequence.

Almost periodic sequence. A sequence $\{x_k\}$ of complex numbers is called almost periodic (in the sense of Bohr) if for every $\varepsilon > 0$ the set $\{\tau : \sup_k |x_{k+\tau} - x_k| < \varepsilon\}$ is *relatively dense*, or in more modern terms, has *bounded gaps*. Almost periodic sequences share many properties of almost periodic functions [3]. For example, the Fourier coefficient

$$a(\lambda) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} x_k \exp(-i\lambda k)$$

exists for every $\lambda \in \mathbb{T} \sim \mathbb{R}/2\pi \sim [0, 2\pi)$ and the set of frequencies $\Lambda = \{\lambda \in \mathbb{T} : a(\lambda) \neq 0\}$ associated to the almost periodic sequence $\{x_k\}$ is at most countable. The frequencies λ and coefficients $a(\lambda)$, $\lambda \in \Lambda$, are uniquely determined by the almost periodic sequence $\{x_k\}$ and for this reason it is said that each almost periodic sequence has an associated (unique) Fourier series,

$$x_k \sim \sum_{\lambda\in\Lambda} a(\lambda) \exp(i\lambda k)$$

where $\sim$ is not to be taken as equality except under additional assumptions. For example, any trigonometric polynomial function is almost periodic and its set of frequencies Λ is finite. For any countable family $\{\lambda_\nu\}$ taken from $\mathbb{T}$, and any family $\{a_\nu\}$ of complex numbers such $\sum_\nu |a_\nu| < \infty$, then an almost periodic the sequence $\{x_k\}$ is defined by

$$x_k = \sum_\nu a_\nu \exp(i\lambda_\nu k).$$

Aim : Frequency determination. Suppose we are given the sequence $\{x_k\}$ one element at a time; in other words, we are given the finite sequences $\{x_0, \ldots, x_{2n}\}$ for $n = 1, 2, \ldots$. Now if we are told, a priori, that some λ belongs to Λ, then the sequence

$$a_n(\lambda) = \frac{1}{2n} \sum_{k=1}^{2n} x_k \exp(-i\lambda k)$$

converges to $a(\lambda) \neq 0$, and if $\lambda \notin \Lambda$ then $\lim_n a_n(\lambda) = 0$.

Given the practical constraint of finite computations, we wish to determine Λ by a limit of a sequence of operations, each of which involves only a finite number of calculations.

At each computation stage n we compute a finite set of frequencies

$$\Lambda_n \triangleq \{\lambda_1^{(n)}, \lambda_2^{(n)}, ..., \lambda_{K_n}^{(n)}\}$$

taken from a finite grid $\{2\pi j/(Cn) : j = 0, \dots, Cn-1\}$, using only the finite subsequence $\{x_0, \dots, x_{2n}\}$. The positive integer constant C is accurately chosen in the following. Next we require that

(i) $\lim_n \Lambda_n = \Lambda$, meaning that for every $\lambda \in \Lambda$ there is a subsequence $\{\lambda_{n_j}\}$ with $\lambda_{n_j} \in \Lambda_{n_j}$ and $\lim_j \lambda_{n_j} = \lambda$, and any convergent sequence $\{\lambda_n\}$ taken from the sets $\{\Lambda_n\}$ ($\lambda_n \in \Lambda_n$ for any n) converges only to an element of Λ.

(ii) Λ_n is not too big in the sense that convergent sequences taken from the sets $\{\Lambda_n\}$ converge only to elements of Λ. In other words, Λ is the limiting points of $\cup_n \Lambda_n$.

We note that the latter property is needed because the finite set $\Lambda_n = \{2\pi j/(Cn) : j = 0, \dots, Cn-1\}$ (a set we can specify without computing anything) satisfies the property (i) but yet is too big in the sense that a convergent subsequence to any $\lambda \in \mathbb{T}$ can be found; i.e., $\overline{\cup \Lambda_n} = \mathbb{T}$.

Moreover, two difficulties appear here. First we do not observe the complete sequence $\{x_k\}$, but only a finite subsequence $\{x_1, \dots, x_n\}$. Secondly we do not study $a_n(\lambda)$ for λ varying in $\mathbb{T}$, but only in a finite grid.

For the non-random case we construct an algorithm for determining Λ_n that gives us the desired convergence. For this purpose we select what we call the *strong local maxima* of a weighted version $a_n^{(w)}(\cdot)$ of the discrete Fourier transform $a_n(\cdot)$. As weighting sequence $\{w_k^{(n)}\}$, we use Bartlett (triangular) kernel, for the strong shape of the maximum of its discrete Fourier transform, and for its tractability in Fourier analysis. With this kernel, we obtain a better localization of the true frequencies, and we limit the number of false estimates. Other weighting kernels can be used (e.g. Parzen kernel), however for sake of simplicity we present our study with Bartlett kernel.

Whenever the set Λ of frequencies of the sequence $\{x_k\}$ is finite, that is we have a trigonometric polynomial, we can isolate each frequency, and the convergence of Λ_n is obtained from the properties of the weighting kernel.

Whenever the set Λ is not finite, but countable, we can no longer isolate each frequency, and we can obtain a sequence $\{\lambda_{\nu_n}^{(n)}\}$ of strong local maxima which converges to a point λ which is not a frequency of $\{x_k\}$. Thus we need to study the behavior of the sequence $\{a_n^{(w)}(\lambda_{\nu_n}^{(n)})\}$ to decide whether the limit point λ is a true frequency of the sequence $\{x_k\}$.

Moreover we obtain that any frequency $\lambda \in \Lambda$ can be estimated by a sequence of strong local maxima with a rate of convergence of order $\mathscr{O}(n^{-1})$.

Next we suppose now that randomness is added to the problem. First, rather than observing the almost periodic sequence $\{x_k\}$, we observe the noisy sequence

$$y_k = x_k + z_k$$

where $\{x_k\}$ is a non-random almost periodic sequence and $\{z_k\}$ is an zero-mean random sequence with some asymptotic independence. For simplicity of exposition we assume here that the sequence $\{z_k\}$ is ρ-mixing. As before, at each stage we compute a finite number of frequencies which are now random variables

$$\{\lambda_1^{(n)}(\omega), \lambda_2^{(n)}(\omega), \dots, \lambda_{K_n}^{(n)}(\omega)\} = \Lambda_n(\omega)$$

from $Y_n(\omega) = \{y_j(\omega) = x_j + z_j(\omega),\ j = 1, 2, \dots, n\}$. We show that if $\sup_k \mathrm{E}\{z_k^4\} < \infty$, then the same algorithm that solves the non-random problem produces $\lim_n \Lambda_n(\omega) = \Lambda$ with probability one. The rate of convergence is $\mathscr{O}(n^{-1})$.

Finally, we apply the algorithm for the determination of the frequencies of an almost periodically correlated random sequence [7] also called almost cyclostationnary signal [6]. Now the frequencies are hidden in the covariance kernel of the random sequence.

Notation. From now on we consider $\mathbb{T} \triangleq \mathbb{R}/2\pi \sim [0, 2\pi)$ with the metric $\mathrm{dist}(\lambda', \lambda'') = \min\{|\lambda' - \lambda'' + 2k\pi| : k \in \mathbb{Z}\}$.

2. STRONG LOCAL MAXIMA ALGORITHM

The basic approach to obtaining the convergence of Λ_n to Λ is to compute $a_n^{(w)}(\lambda)$ on a uniform grid Π_n and then to place any $\lambda_j^{(n)} \in \Pi_n$ in the set Λ_n whenever an acceptance criterion is satisfied. In this paper we investigate how well the algorithm based on strong local maxima gives the desired convergence.

2.1 Bartlett kernel

We observe $x_0, \dots, x_{2n}$, and determine the collection of amplitudes using *weighted* Fourier coefficient estimator

$$a_n^{(w)}(\lambda) \triangleq \frac{1}{n} \sum_{k=0}^{2n} x_k w_{n-k}^{(n)} \exp(-i\lambda k)$$

where $\{w_k^{(n)}\}$ is the *Bartlett* (triangular) weight sequence given by $w_k^{(n)} = 1 - |k|/n$ for $|k| < n$ and $w_k^{(n)} = 0$ otherwise. The discrete Fourier transform of $\{w_k^{(n)}\}$ is

$$W_n(\lambda) \triangleq \sum_{k=-n}^{n} w_k^{(n)} \exp(-i\lambda k) = \frac{\sin^2(n\lambda/2)}{n \sin^2(\lambda/2)}.$$

Note that the function $W_n(\cdot)$ is non-negative continuous on $\mathbb{R}$ and $W_n(0) = n$. It satisfies $W_n(\lambda) \le W_n(2\pi/(Cn))$ for any λ such that $\mathrm{dist}(\lambda, 0) \ge 2\pi/(Cn)$, $C \ge 2$. This function is decreasing on $[0, 2\pi/n]$ and

$$\frac{W_n(\pi/(Cn))}{W_n(0)} \sim \left(\frac{\sin(\pi/(2C))}{\pi/(2C)}\right)^2$$

as $n \to \infty$, for any $C \ne 0$. Thus for any $0 < \eta < 1$ there exist $C_\eta > 0$ and $n_{C_\eta} \ge 0$ such that for any $C > C_\eta$ and any $n > n_\eta$

$$\eta < \left(1 - \frac{\pi^2}{24C^2}\right)^2 < \frac{W_n(\pi/(Cn))}{W_n(0)} < 1.$$

For instance, there exists $n_C > 0$ such that for $n > n_C$, $W_n(\pi/(Cn))/W_n(0) > .949$ for $C \ge 4$

On the other hand, given $0 < \delta < \pi$, we have

$$\frac{W_n(\lambda)}{W_n(0)} \sim \mathscr{O}\left(\frac{1}{n^2}\right),$$

as $n \to \infty$, where the $\mathscr{O}$ is uniform with respect to λ provided $\mathrm{dist}(\lambda, 0) > \delta$.

2.2 Strong local maxima

For computing stage n, compute $a_n^{(w)}(\lambda)$ for the values of λ taken on the uniform (equally spaced) grid $\Pi_n = \{\lambda_j^{(n)} = 2\pi j/(Cn) : j = 0,\ldots,Cn-1\}$ where C is a fixed positive integer. The constant C determines the density of the sampling net and is chosen (qualitatively) so that several values of $W_n(\lambda_j^{(n)})$ near to $W_n(0)$ are present in the sample. We will assign $\lambda_j^{(n)}$ to Λ_n if $a_n^{(w)}(\lambda_j^{(n)})$ is *locally maximum* and also *strong*. More precisely

Definition 2.1 *A frequency index j^* is said to produce a* strong local maximum *with parameters $K_1, K_2, K_3 \in (1,\infty)$ if*

$$\left|a_n^{(w)}(\lambda_{j^*}^{(n)})\right| \geq \left|a_n^{(w)}(\lambda_j^{(n)})\right| \quad \textit{for} \quad |j^*-j| \leq K_1,$$

and

$$\left|a_n^{(w)}(\lambda_{j^*}^{(n)})\right| \geq K_3 \left|a_n^{(w)}(\lambda_j^{(n)})\right| \quad \textit{for} \quad K_1 < |j^*-j| \leq K_2.$$

In this paper the point $\lambda_{j^*}^{(n)} \in \mathbb{T}$ is called a strong local maximum.

At a strong local maximum, the amplitude $\left|a_n^{(w)}(\lambda_{j^*}^{(n)})\right|$ is at least K_3 times larger than its neighbors except for those nearby ($|j^*-j| \leq K_1$). Moreover, when $\lambda_{j^*}^{(n)}$ is a strong local maximum then the $\lambda_{j^*+k}^{(n)}$s for $k = \pm(K_1+1),\ldots,\pm K_2$, cannot be ones. Further if $\lambda_{j^*}^{(n)}$ and $\lambda_{j^*+k}^{(n)}$ are strong local maxima for some $k = \pm 1,\ldots,\pm K_1$ then $\left|a_N^{(w)}(\lambda_{j^*}^{(N)})\right| = \left|a_N^{(w)}(\lambda_{j^*+k}^{(N)})\right|$.

The next proposition states that for any frequency λ_ν, the algorithm of strong local maxima produces a sequence which converges to λ_ν. This shows only that strong local maxima satisfy the first requirement for $\lim_n \Lambda_n = \Lambda$.

Proposition 2.2 *Let $\{x_k\}$ be an almost periodic sequence and let λ_ν be one of its frequencies. Then there exists a sequence of strong local maxima $\{\lambda_{\nu_n}^{(n)}\}$ determined by the algorithm of strong local maxima using the Bartlett kernel with $C \geq 2, K_1 = 3C, K_2 = 4C, K_3 = 10$, which converges to λ_ν as $n \to \infty$, and verifies*

$$\operatorname{dist}(\lambda_{\nu_n}^{(n)}, \lambda_\nu) \leq \frac{\pi}{Cn} \quad \textit{and} \quad \left|a_n^{(w)}(\lambda_{\nu_n}^{(n)})\right| > .8\,|a_\nu|,$$

for any $n > n_0$. The integer n_0 can be chosen large enough for there is no strong local maximum $\lambda_j^{(n)} \in \Lambda_n$ such that

$$\frac{\pi}{Cn} < \operatorname{dist}(\lambda_j^{(n)}, \lambda_\nu) \leq \frac{8\pi}{n}.$$

Hence we obtain that, for $n > n_0$, n_0 being sufficiently large, any $\lambda_\nu \in \Lambda$ admits at most two strong local maxima $\lambda_{\nu_n}^{(n)}$ such that $\operatorname{dist}(\lambda_{\nu_n}^{(n)}, \lambda_\nu) \leq 2K_2\pi/(Cn) = 8\pi/n$. If there are two, they are the two nearest points of the grid from each side of λ_ν, and they have the same modulus $\left|a_n^{(w)}(\lambda_{\nu_n}^{(n)})\right|$.

3. Λ IS FINITE

Whenever Λ is finite, the sequence $\{x_k\}$ is a trigonometric polynomial. As described above, we compute $a_n^{(w)}(\lambda)$ for the values $\lambda = \lambda_j^{(n)} = j\,2\pi/(Cn)$, $j = 0, 1, \ldots, Cn-1$, where C is a positive integer. Then using the properties of the discrete Fourier transform of the Bartlett kernel, we state that convergent sequences taken from the Λ_ns converge only to elements of Λ when Λ is finite.

Proposition 3.1 *If $\{x_k\}$ is an almost periodic sequence with Λ finite, and if Λ_n is determined by the algorithm of strong local maxima using the Bartlett kernel with $C \geq 2, K_1 = 3C, K_2 = 4C, K_3 = 10$, then every convergent sequence taken from the sets $\{\Lambda_n\}$ converges to an element of Λ.*

From Propositions 2.2 and 3.1, we deduce that $\lim_n \Lambda_n = \Lambda$, for Λ finite.

Results of simulation. In order to help our understanding

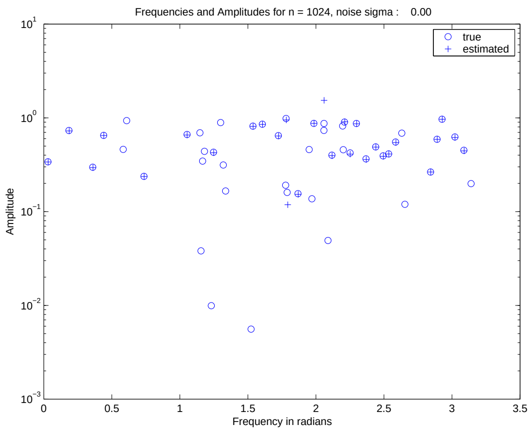

Figure 1: $\circ$: card $\Lambda = 50$ randomly chosen amplitudes and frequencies without additive noise. $+$: frequencies and amplitudes determined by strong local maxima with $n = 1024, C = 4,\ K_1 = 16, K_2 = 13, K_3 = 100$.

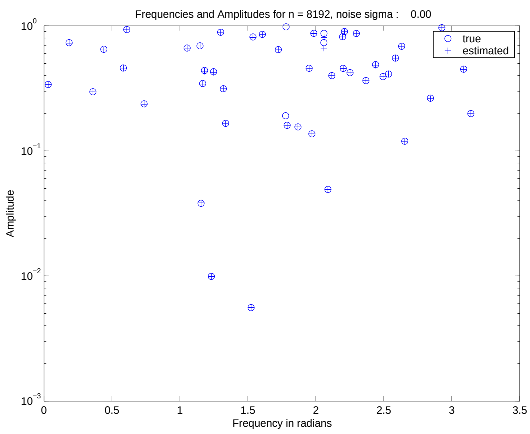

Figure 2: $\circ$: card $\Lambda = 50$ randomly chosen amplitudes and frequencies without additive noise. $+$: frequencies and amplitudes determined by strong local maxima with $n = 8192, C = 4,\ K_1 = 16, K_2 = 13, K_3 = 100$.

of the situation, the algorithm of strong local maxima was implemented in MATLAB code. For programming convenience we also took the sequence $\{x_k\}$ to be real. Then we set $\operatorname{card}\Lambda = 50$ and choose the frequencies from a uniform distribution on $[0,\pi]$ and the amplitudes from a uniform distribution on $[0,1]$.

In figures 1 and 2 where $n = 1024$ and 8142, there are 27 and 48 frequencies identified. However, even at $n = 8142$, there are a few frequencies that are not yet resolved. We need to take a greater value of n to separate them.

4. Λ IS COUNTABLE

Let $\{x_k\}$ be an almost periodic sequence. Then we know that its set of frequencies Λ is at most countable, say $\Lambda = \{\lambda_1,\ldots,\lambda_\nu,\ldots\}$ and it admits a Fourier series representation $x_k \sim \sum_\nu a_\nu \exp(i\lambda_\nu k)$, with $\sum_\nu |a_\nu|^2 < \infty$. Moreover there exists a sequence of trigonometric polynomials $\{\sigma_N\}$ such that $\lim_N \sigma_N(k) = x_k$ uniformly with respect to $k\in\mathbb{Z}$. The trigonometric polymonials can be chosen such that

$$\sigma_N(x) = \sum_{\nu=1}^{n(N)} r_{N,\nu}\, a_\nu \exp(i\lambda_\nu x),$$

and with the notation $r_{N,\nu} = 0$ for $\nu > n(N)$, we have $0 \le r_{N,\nu} \le 1$ for any ν, and $\lim_N \uparrow r_{N,\nu} = 1$ (see e.g. [3]). Then $\lim_N r_{N,\nu}\, a_\nu = a_\nu$, and for any N the set Λ_N of frequencies of σ_N is finite, $\Lambda_N \triangleq \{\lambda_\nu \in \Lambda : r_{N,\nu}\, a_\nu \neq 0\} \subset \Lambda$. Hence for any N we write

$$a_n^{(w)}(\lambda) = a_{N,n}^{(w)}(\lambda) + R_{N,n}^{(w)}(\lambda),$$

where

$$a_{N,n}^{(w)}(\lambda) \triangleq \frac{1}{n}\sum_{k=0}^{2n} \sigma_N(k)\, w_{n-k}^{(n)} \exp(-i\lambda k)$$

$$R_{N,n}^{(w)}(\lambda) \triangleq \frac{1}{n}\sum_{k=0}^{2n} \big(x_k - \sigma_N(k)\big)\, w_{n-k}^{(n)} \exp(-i\lambda k).$$

Then we apply the same arguments as in Section 3 for the trigonometric polynomial sequence $\{\sigma_N(\cdot)\}$. Thus the claim of Proposition 2.2 remains valid for any almost periodic sequence $\{x_k\}$.

However, consider the almost periodic sequence $\{x_k\}$ defined by $x_k = \sum_\nu a_\nu \exp(i\lambda_\nu k)$ with $\Lambda = \{\lambda_\nu = 1 - 1/\nu : \nu \in \mathbb{N}^*\}$ and $a_\nu = 1/\nu^2$. Then $1 \notin \Lambda$ and the set of strong local maxima converges to $\overline{\Lambda} = \Lambda \cup \{1\}$. So Proposition 3.1 is not generally valid. Neverthless, a localization property is still proved using thresholds on $\big|a_n^{(w)}\big(\lambda_j^{(n)}\big)\big|$.

Lemma 4.1 *Assume that for some fixed $a > 0$, the closed interval $B(\lambda,r)$ contains no frequency λ_ν such that $|a_\nu| > a$: $B(\lambda,r)\cap\Lambda^{(a)} = \emptyset$ where $\Lambda^{(a)} \triangleq \{\lambda_\nu \in \Lambda : |a_\nu| > a\}$. Then for any $\varepsilon > 0$ there exists an integer n_ε such that for any $n > n_\varepsilon$, the closed interval $B(\lambda,r)$ contains no strong local maximum $\lambda_j^{(n)} \in \Lambda_n$ with $\big|a_n^{(w)}\big(\lambda_j^{(n)}\big)\big| > a+\varepsilon$: $B(\lambda,r)\cap\Lambda_n^{(a+\varepsilon)} = \emptyset$ where $\Lambda_n^{(a+\varepsilon)} \triangleq \{\lambda_j^{(n)} \in \Lambda_n : \big|a_n^{(w)}\big(\lambda_j^{(n)}\big)\big| > a+\varepsilon\}$.*

Hence we easily deduce that for any $a > 0$, every convergent sequence of strong local maxima from the sets $\{\Lambda_n^{(a)}\}$ converges to an element of Λ.

Finally, whenever $\{\lambda_n\}$ is a sequence of strong local maxima which converges to some $\lambda_0 \in \mathbb{T}$, we cannot say whether the limit λ_0 belongs to Λ. In order to eliminate the points of $\mathbb{T}\setminus\Lambda$ we study the behavior of $a_n^{(w)}(\lambda_n)$ as $n\to\infty$. We state the following characterization of the frequency set.

Theorem 4.2 *Let $\{x_k\}$ be an almost periodic sequence with frequency set Λ. An element $\lambda_0 \in \mathbb{T}$ is a frequency, that is $\lambda_0 \in \Lambda$, if and only if there is a sequence of strong local maxima $\{\lambda_n\}$ which converges to λ_0 and such that*

$$\limsup_{n\to\infty} \big|a_n^{(w)}(\lambda_n)\big| = a_0 > 0.$$

Furthermore if $\lambda_0 = \lambda_\nu \in \Lambda$ then we have $0 < a_0 \le |a_\nu|$.

Then we deduce

Corollary 4.3 *If a sequence of strong local maxima $\{\lambda_n\}$ converges to some $\lambda_0 \notin \Lambda$ then*

$$\lim_{n\to\infty} a_n^{(w)}(\lambda_n) = 0.$$

Remark that thanks to Proposition 2.2, for any frequency $\lambda_\nu \in \Lambda$, there exists a sequence of strong local maxima $\{\lambda_{\nu_n}^{(n)}\}$ which converges to λ_0 and such that $\big|a_n^{(w)}\big(\lambda_{\nu_n}^{(n)}\big)\big| \ge .8|a_\nu|$ for sufficiently large n. Thus for this sequence

$$.8\,|a_\nu| \le \liminf_{n\to\infty}\Big|a_n^{(w)}\big(\lambda_{\nu_n}^{(n)}\big)\Big| \le \limsup_{n\to\infty}\Big|a_n^{(w)}\big(\lambda_{\nu_n}^{(n)}\big)\Big| \le |a_\nu|.$$

5. THE RANDOM CASE

Working now in a stochastic context, we first consider the fundamental case of a non-random signal with an additive noise. Then we fit our analysis to the case of an almost periodically correlated stochastic sequence.

5.1 Almost periodic sequence observed with noise

Here we prove that the strong local maxima algorithm using the Bartlett kernel also produces the desired result when we only observe

$$y_k = x_k + z_k$$

where $\{x_k\}$ is a non-random almost periodic sequence and $\{z_k\}$ is a sequence of zero-mean random variables defined on a probability space $(\Omega,\mathscr{F},\mathrm{P})$.

In a classical way we assume that the random variables $\{z_k\}$ satisfy some asymptotic independence. For simplicity of exposition we assume here that $\{z_k\}$ fulfills the ρ-mixing property that is $\lim_{k\to\infty}\rho(k) = 0$ where $\rho(k) := \sup\{|\mathrm{corr}[f,g]|\}$ where the supremum is taken over all $f \in \mathscr{F}_{-\infty}^{s}\cap L^2(\mathrm{P})$, $g \in \mathscr{F}_{k+s}^{\infty}\cap L^2(\mathrm{P})$ and $s\in\mathbb{Z}$, with $\mathscr{F}_s^t$ being the σ-field generated by $\{z_k : s \le k \le t\}$. For example, any stationary Gaussian sequence is ρ-mixing, as well as any M-dependent sequence.

At each stage of the strong local maxima algorithm we compute

$$\begin{aligned} a_n^{(w*)}(\lambda,\omega) &= \frac{1}{2n}\sum_{k=0}^{n} \big(x_k + z_k(\omega)\big) w_{n-k}^{(n)} \exp(-i\lambda k) \\ &= a_n^{(w)}(\lambda) + b_n^{(w)}(\lambda,\omega) \end{aligned}$$

for any $\lambda \in \mathbb{T}$ and any $\omega \in \Omega$.

In order to show that the strong local maxima algorithm applied to the observation $\{y_k\}$ gives the desired sequence of frequency sets, we note that the first term of the right hand side of the previous equality is no random, and can be studied as previoulsy. The second one is negligible and we first show the convergence in quadratic mean and almost sure of $b_n^{(w)}(\lambda)$ to 0 uniformly with respect to λ in $\mathbb{T}$ as $n \to \infty$.

Lemma 5.1 *Assume that the sequence $\{z_k\}$ is ρ-mixing and $\sup_k \mathrm{E}\left[|z_k|^4\right] < \infty$.*
i) Then

$$\lim_{n\to\infty} \sup_{\lambda\in\mathbb{T}} \left| a_n^{(w*)}(\lambda,\omega) - a_n^{(w)}(\lambda) \right| = 0 \qquad q.m.$$

ii) Assume in addition that $\rho(n) = o(n^{-1/2})$ as $n \to \infty$ then

$$\lim_{n\to\infty} n^{\varepsilon} \sup_{\lambda} \left| a_n^{(w*)}(\lambda,\omega) - a_n^{(w)}(\lambda) \right| = 0 \quad q.m. \quad if \quad \varepsilon < 1/4,$$

and

$$\lim_{n\to\infty} n^{\varepsilon} \sup_{\lambda} \left| a_n^{(w*)}(\lambda,\omega) - a_n^{(w)}(\lambda) \right| = 0 \quad a.s. \quad if \quad \varepsilon < 1/8.$$

Then we readily verify that the results of the non-radom case can be easily translated in our noisy setting. Proposition 2.2 is adapted in the following way

Proposition 5.2 *Assume that the noise $\{z_k\}$ is ρ-mixing with $\rho(n) = o(n^{-1/2})$ as $n \to \infty$, and $\sup_k \mathrm{E}\left[|z_k|^4\right] < \infty$. For any frequency $\lambda_\nu \in \Lambda$ of the signal $\{x_k\}$, there exists a sequence of random variables $\left\{\widehat{\lambda}_{\nu_n}^{(n)}\right\}$ such that*

i) $\widehat{\lambda}_{\nu_n}^{(n)}$ takes its values in $\left\{2\pi j/(Cn) : j = 0, \ldots, Cn-1\right\}$,

ii) $\lim_{n\to\infty} \mathrm{P}\left[\widehat{\lambda}_{\nu_m}^{(m)} \quad \text{is a SLM for any } m \geq n\right] = 1$ where the strong local maxima (SLM) are determined by the algorithm of strong local maxima using the Bartlett kernel with $C \geq 2, K_1 = 3C, K_2 = 4C, K_3 = 10$,

iii) $\mathrm{dist}\left(\widehat{\lambda}_\nu^{(n)}, \lambda_\nu\right) \leq 2\pi/n + \pi/(Cn) \qquad$ everywhere,

iv) $\lim_{n\to\infty} \mathrm{P}\left[\left|a_n^{(w*)}\left(\widehat{\lambda}_\nu^{(n)}\right)\right| > .8\left|a_\nu\right|\right] = 1$.

As a converse result we state the following.

Proposition 5.3 *Assume that the noise $\{z_k\}$ is ρ-mixing, $\sup_k \mathrm{E}\left[|z_k|^4\right] < \infty$. Let $\{\lambda_n\}$ be a sequence of random variables such that*

i) λ_n takes its values in $\left\{2\pi j/(Cn) : j = 0, \ldots, Cn-1\right\}$,

ii) $\lim_{n\to\infty} \mathrm{P}\left[\lambda_m \quad \text{is a SLM for any } m \geq n\right] = 1$ where the strong local maxima (SLM) are determined by the algorithm of strong local maxima using the Bartlett kernel with $C \geq 2, K_1 = 3C, K_2 = 4C, K_3 = 10$.

iii) the sequence $\{\lambda_n\}$ converges a.s. to some random variable λ_0,

iv) $\mathrm{P}\left[\limsup_{n\to\infty} \left|a_n^{(w*)}(\lambda_n)\right| = a_0\right] = 1 \qquad$ for some random variable a_0 such that $a_0 > 0$ a.e.

Then $\mathrm{P}\left[\lambda_0 \in \Lambda\right] = 1$. Moreover $0 < a_0 \leq \left|a_\nu\right|$ a.e. on $\left\{\lambda_0 = \lambda_\nu\right\}$, $\lambda_\nu \in \Lambda$.

Results of simulation. We apply the strong local maxima algorithm to a signal which the sum of an non-random almost periodic sequence $\{x_k\}$ and a noise $\{z_k\}$. As in Section 3 the almost periodic sequence $\{x_k\}$ has a finite number of frequencies $\mathrm{card}\,\Lambda = 50$ which are chosen from a uniform distribution on $[0, \pi]$ and the amplitudes from a uniform distribution on $[0, 1]$. Here the additive noise $\{z_k\}$ is a sequence of independent, identically distributed random variables following the zero-mean Gaussian law with unit variance.

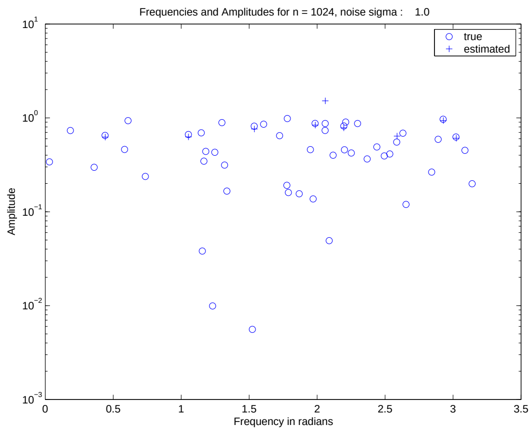

Figure 3: $\circ$: $\mathrm{card}\,\Lambda = 50$ randomly chosen amplitudes and frequencies with additive noise, $\sigma = 1$. $+$: frequencies and amplitudes determined by strong local maxima with $n = 1024, C = 4$, $K_1 = 16, K_2 = 13, K_3 = 100$.

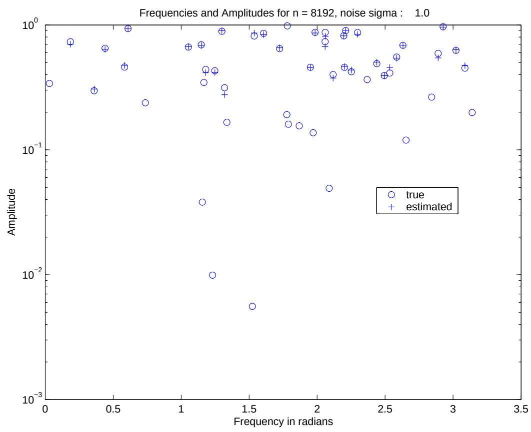

Figure 4: $\circ$: $\mathrm{card}\,\Lambda = 50$ randomly chosen amplitudes and frequencies with additive noise, $\sigma = 1$. $+$: frequencies and amplitudes determined by strong local maxima with $n = 8192, C = 4$, $K_1 = 16, K_2 = 13, K_3 = 100$.

In figures 3 and 4 where $n = 1024$ and 8142, there are 9 and 33 frequencies identified. It is not surprising that in the noisy context there are less identified frequencies than in the non-random case in Section 3. Note that the identified frequencies are among those which have the largest amplitudes.

5.2 APC process

Consider a real-valued zero-mean almost periodically correlated process $\{x_k\}$, that is a zero-mean second order process such that for any $\tau \in \mathbb{Z}$ the shifted covariance function

$k \mapsto \mathrm{E}\left[x_{k+\tau}\overline{x_k}\right]$ is almost periodic [7]. Then the spectral covariance of the process $\{x_k\}$ is defined by

$$a(\lambda,\tau) \triangleq \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \mathrm{E}\left[x_{k+\tau}\overline{x_k}\right]\exp(-ik\lambda)$$

for any $\tau \in \mathbb{Z}$ and any $\lambda \in \mathbb{T}$. It is well known that the set of frequencies $\Lambda \triangleq \left\{\lambda \in \mathbb{T} : a(\lambda,\tau) \neq 0 \text{ for some } \tau\right\}$ is at most countable. The problem here is the detection of the frequencies in Λ.

In order to solve our detection problem let $Y_{\tau,k} \triangleq x_{k+\tau}\overline{x_k}$, $X_{\tau,k} \triangleq \mathrm{E}\left[x_{k+\tau}\overline{x_k}\right]$ and $Z_{\tau,k} \triangleq x_{k+\tau}\overline{x_k} - \mathrm{E}\left[x_{k+\tau}\overline{x_k}\right]$ for any τ and any k. Then we have $Y_{\tau,k} = X_{\tau,k} + Z_{\tau,k}$ where $X_\tau = \{X_{\tau,k}\}$ is a non-random almost periodic sequence and $Z_\tau = \{Z_{\tau,k}\}$ is a random sequence for any τ. Whenever the process $\{x_k\}$ is ρ-mixing, then for any τ the process Z_τ is ρ_τ-mixing with $\rho_\tau(k) = \rho(k-|\tau|)$ for any $k \geq |\tau|$. Moreover $\sup_k \mathrm{E}\left[|Z_{\tau,k}|^4\right] \leq 16\sup_k \mathrm{E}\left[|x_k|^8\right]$.

Under the hypotheses $\sup_k \mathrm{E}\left[|x_k|^8\right] < \infty$, and $\rho(n) = o\left(n^{-1/2}\right)$ the results of Section 5.1 can be applied for the determination of the set $\Lambda_\tau = \left\{\lambda \in \mathbb{T} : a(\lambda,\tau) \neq 0\right\}$, for any $\tau \in \mathbb{Z}$.

6. CONCLUSION

In this paper we present an algorithm for the estimation of the frequencies of an almost periodically correlated sequences, which applies even when we do not know whether the number of frequencies is finite or infinite. This algorithm is based on the determination of the strong local maxima of a weighted version of the discrete Fourier transform of the sequence.

The Bartlett kernel with its good frequency localization properties, is used as the weighting sequence. The Parzen kernel, or higher order Bartlett kernels may be of interest for future studies.

When the set Λ is finite the frequencies can be isolated, and the algorithm works perfectly. When the set Λ is properly countable each frequency can no longer be isolated, and we can obtain a sequence of strong local maxima which converges to a point which is not a true frequency. The problem is resolved using weighted Fourier coefficients.

In additive noisy context, when the noise presents asymptotic independence (e.g. a mixing property), this method still applies. Of course it is less efficient. The detection of the hidden frequencies of an almost periodically correlated sequence can be done with this algorithm.

7. ACKNOWLEDGEMENTS

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