# Inertial particle's motion in geophysical fluid flows (1)

Themistoklis Sapsis and George Haller*
*Department of Mechanical Engineering*
*Massachusetts Institute of Technology*
*77 Massachusetts Ave., Cambridge, MA 02139, USA*

We derive a general reduced-order equation for the asymptotic motion of finite-size particles in unsteady fluid flows in a rotating frame. Our *inertial equation* is a small perturbation of passive fluid advection on a globally attracting slow manifold. Use of the inertial equation enables us to extract Lagrangian coherent structures for inertial particles motion in geophysical fluid flows. We illustrate these results on inertial particle motion in the three dimensional unsteady flow field of a hurricane. The dataset used is a simulation of the hurricane Isabel.

Keywords: Inertial particles, Slow manifolds, Nonautonomous systems.

## I. INTRODUCTION

Finite-size or inertial particle dynamics in fluid flows can differ markedly from infinitesimal particle dynamics: both clustering and dispersion are well documented phenomena in inertial particle motion, while they are absent in the incompressible motion of infinitesimal particles. As we show, these peculiar asymptotic features are governed by a lower-dimensional inertial equation which we determine explicitly.

Let $\mathbf{u}(\mathbf{x}, t)$ denote the velocity field of a two- or three-dimensional fluid flow of density $\rho_f$ observed in a coordinate frame that rotates with angular velocity $\boldsymbol{\Omega}$, with $\mathbf{x}$ referring to spatial locations and $t$ denoting time. The fluid fills a compact (possibly time-varying) spatial region $\mathcal{D}$ with boundary $\partial\mathcal{D}$; we assume that $\mathcal{D}$ is a uniformly bounded smooth manifold for all times. We also assume $\mathbf{u}(\mathbf{x}, t)$ to be $r$ times continuously differentiable in its arguments for some integer $r \geq 1$. We denote the material derivative of $\mathbf{u}$ by

$$\frac{D\mathbf{u}}{Dt} = \mathbf{u}_t + (\boldsymbol{\nabla}\mathbf{u})\,\mathbf{u},$$

where $\boldsymbol{\nabla}$ denotes the gradient operator with respect $\mathbf{x}$.

Let $\mathbf{x}(t)$ denote the path of a finite-size particle of density $\rho_p$ immersed in the fluid, observed in the rotating frame. If the particle is spherical, its velocity $\mathbf{v}(t) = \dot{\mathbf{x}}(t)$ satisfies the equation of motion (cf. Maxey and Riley [Max87] and Babiano et al. [Bab00])

$$\begin{aligned}
\rho_p \dot{\mathbf{v}} \;=\; & \rho_f \frac{D\mathbf{u}}{Dt} - 2\rho_p \boldsymbol{\Omega} \times \mathbf{v} \\
& + \left(\rho_p - \rho_f\right) \mathbf{g} \\
& - \frac{9\nu\rho_f}{2a^2}\left(\mathbf{v} - \mathbf{u} - \frac{a^2}{6}\Delta\mathbf{u}\right) \\
& - \frac{\rho_f}{2}\left[\dot{\mathbf{v}} + 2\boldsymbol{\Omega}\times\mathbf{v} - \frac{D}{Dt}\left(\mathbf{u} + \frac{a^2}{10}\Delta\mathbf{u}\right)\right] \\
& - \frac{9\rho_f}{2a}\sqrt{\frac{\nu}{\pi}}\int_0^t \frac{1}{\sqrt{t-s}}\left[\dot{\mathbf{v}}(s) + 2\boldsymbol{\Omega}\times\mathbf{v} \right. \\
& \left. - \frac{d}{ds}\left(\mathbf{u} + \frac{a^2}{6}\Delta\mathbf{u}\right)_{\mathbf{x}=\mathbf{x}(s)}\right] ds.
\end{aligned} \tag{1}$$

Here $\rho_p$ and $\rho_f$ denote the particle and fluid densities, respectively, $a$ is the radius of the particle, $\mathbf{g}$ is the constant vector of gravity, $2\boldsymbol{\Omega} \times \mathbf{v}$ is the Corriolis acceleration, and $\nu$ is the kinematic viscosity of the fluid. The individual force terms listed in separate lines on the right-hand side of (1) have the following physical meaning: (1) force exerted on the particle by the undisturbed flow (2) buoyancy force (3) Stokes drag (4) added mass term resulting from part of the fluid moving with the particle (5) Basset–Boussinesq memory term. The terms involving $a^2\Delta\mathbf{u}$ are usually referred to as the Fauxén corrections.

For simplicity, we assume that the particle is very small ($a \ll 1$), in which case the Fauxén corrections are negligible. We note that the coefficient of the Basset–Boussinesq memory term is equal to the coefficient of the Stokes drag term times $a/\sqrt{\pi\nu}$. Therefore, assuming that $a/\sqrt{\nu}$ is also very small, we neglect the last term in (1), following common practice in the related literature (Michaelides [Mic97]). We finally rescale space, time, and velocity by a characteristic length scale $L$, characteristic time scale $T = L/U$ and characteristic velocity $U$, respectively, to obtain the simplified equations of motion

$$\dot{\mathbf{v}} - \frac{3R}{2}\frac{D\mathbf{u}}{Dt} = -\mu\left(\mathbf{v} - \mathbf{u}\right) - 2\boldsymbol{\Omega}\times\mathbf{v} + \left(1 - \frac{3R}{2}\right)\mathbf{g}, \tag{2}$$

*Corresponding author. Email: ghaller@mit.edu

[1] This work summarizes the results presented at Haller and Sapsis [HalSap07] and also includes a new application to geophysical fluid flows.

with

$$R = \frac{2\rho_f}{\rho_f + 2\rho_p}, \qquad \mu = \frac{R}{St}, \qquad St = \frac{2}{9}\left(\frac{a}{L}\right)^2 \mathrm{Re},$$

and with $t$, $\mathbf{v}$, $\mathbf{u}$ and $\mathbf{g}$ now denoting nondimensional variables. Variants of equation (2) have been studied by Babiano, Cartwright, Piro and Provenzale [Bab00], Benczik, Toroczkai and Tél [Ben02], and Vilela, de Moura and Grebogi [Vil06].

In equation (2), $St$ denotes the particle Stokes number and $\mathrm{Re} = UL/\nu$ is the Reynolds number. The density ratio $R$ distinguishes neutrally buoyant particles ($R = 2/3$) from aerosols ($0 < R < 2/3$) and bubbles ($2/3 < R < 2$). In the limit of infinitely heavy particles ($R = 0$), equations (2) become the Maxey–Riley equations derived originally in [Max87]. The $3R/2$ coefficient represents the added mass effect: an inertial particle brings into motion a certain amount of fluid that is proportional to half of its mass. For neutrally buoyant particles, the equation of motion is simply $\frac{D}{Dt}(\mathbf{v} - \mathbf{u}) = -\mu(\mathbf{v} - \mathbf{u})$, i.e., the relative acceleration of the particle is equal to the Stokes drag acting on the particle.

Rubin, Jones and Maxey [Rub95] studied (2) with $R = 0$ in the special case when $\mathbf{u}$ describes a two-dimensional cellular steady flow model. They used a geometric singular perturbation approach developed by Fenichel [Fen79] to understand particle settling in the flow. The same technique was employed by Burns et al. [Burn99] in the study of particle focusing in the wake of a two-dimensional bluff body flow, which is steady in a frame co-moving with the von Kármán vortex street. Recently, Mograbi and Bar-Ziv [Mog06] discussed this approach for general steady velocity fields and made observations about possible asymptotic behaviors in two dimensions.

Here we consider finite-size particle motion in general unsteady velocity fields, extending Fenichel's geometric approach from time-independent to time-dependent vector fields. Such an extension has apparently not been considered before in dynamical systems theory, thus the present work should be of interest in other applications of singular perturbation theory where the governing equations are non-autonomous. We construct an attracting slow manifold that governs the asymptotic behavior of particles in system (2). We also obtain an explicit dissipative equation, the *inertial equation*, that describes the flow on the slow manifold. This equation has half the dimension of the Maxey-Riley equation; this fact simplifies both the qualitative analysis of inertial dynamics and the numerical tracking of finite-size particles.

## II. SINGULAR PERTURBATION FORMULATION

The derivation of the equation of motion (2) is only correct under the assumption $\mu \gg 1$, which motivates us to introduce the small parameter

$$\epsilon = \frac{1}{\mu} \ll 1,$$

and rewrite (2) as a first-order system of differential equations:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{v}, \\ \epsilon\dot{\mathbf{v}} &= \mathbf{u}(\mathbf{x}, t) - \mathbf{v} + \epsilon\left[\frac{3R}{2}\frac{D\mathbf{u}(\mathbf{x}, t)}{Dt} - 2\mathbf{\Omega}\times\mathbf{v}\right. \\ &\quad \left. + \left(1 - \frac{3R}{2}\right)\mathbf{g}\right]. \end{aligned} \tag{3}$$

This formulation shows that $\mathbf{x}$ is a slow variable changing at $\mathcal{O}(1)$ speeds, while the fast variable $\mathbf{v}$ varies at speeds of $\mathcal{O}(1/\epsilon)$.

To transform the above singular perturbation problem to a regular perturbation problem, we select an arbitrary initial time $t_0$ and introduce the fast time $\tau$ by letting

$$\epsilon\tau = t - t_0.$$

This type of rescaling is standard in singular perturbation theory with $t_0 = 0$. The new feature here is the introduction of a nonzero present time $t_0$ about which we introduce the new fast time $\tau$. This trick enables us to extend existing singular perturbation techniques to unsteady flows.

Denoting differentiation with respect to $\tau$ by prime, we rewrite (3) as

$$\begin{aligned} \mathbf{x}' &= \epsilon\mathbf{v}, \\ \phi' &= \epsilon, \\ \mathbf{v}' &= \mathbf{u}(\mathbf{x}, \phi) - \mathbf{v} + \epsilon\frac{3R}{2}\frac{D\mathbf{u}(\mathbf{x}, \phi)}{Dt} \\ &\quad -2\epsilon\mathbf{\Omega}\times\mathbf{v} + \epsilon\left(1 - \frac{3R}{2}\right)\mathbf{g}, \end{aligned} \tag{4}$$

where $\phi \equiv t_0 + \epsilon\tau$ is a dummy variable that renders the above system of differential equations autonomous in the variables $(\mathbf{x}, \phi, \mathbf{v}) \in \mathcal{D} \times \mathbb{R} \times \mathbb{R}^n$; here $n$ is the dimension of the domain of definition $\mathcal{D}$ of the fluid flow ($n = 2$ for planar flows, and $n = 3$ for three-dimensional flows).

## III. SLOW MANIFOLD AND INERTIAL EQUATION

The $\epsilon = 0$ limit of system (4),

$$\begin{aligned} \mathbf{x}' &= \mathbf{0}, \\ \phi' &= 0, \\ \mathbf{v}' &= \mathbf{u}(\mathbf{x}, \phi) - \mathbf{v}, \end{aligned} \tag{5}$$

has an $n+1$–parameter family of fixed points satisfying $\mathbf{v} = \mathbf{u}(\mathbf{x}, \phi)$. More formally, for any time $T > 0$, the compact invariant set

$$M_0 = \{(\mathbf{x}, \phi, \mathbf{v}) \;:\; \mathbf{v} = \mathbf{u}(\mathbf{x}, \phi), \quad \mathbf{x} \in \mathcal{D}, \ \phi \in [t_0 - T, t_0 + T]\}$$

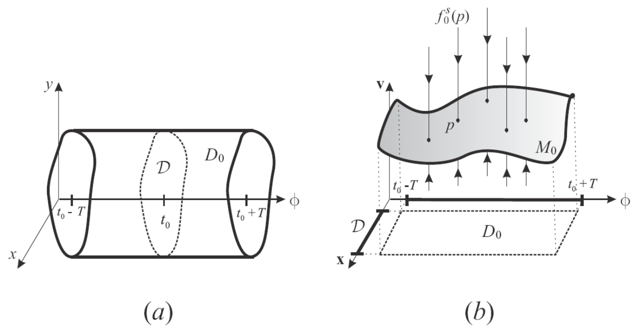


FIG. 1: (a) The geometry of the domain $D_0$ (b) The attracting set of fixed points $M_0$; each point $p$ in $M_0$ has a $n$-dimensional stable manifold $f_0^s(p)$ (unperturbed stable fiber at $p$) satisfying $(\mathbf{x}, \phi) = const.$

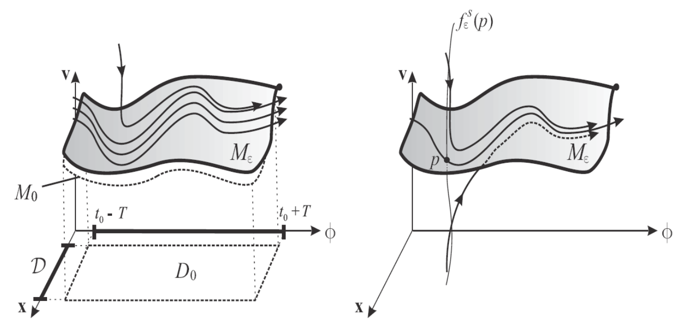


FIG. 2: (a) The geometry of the slow manifold $M_\epsilon$ (b) A trajectory intersecting a stable fiber $f_\epsilon^s(p)$ converges to the trajectory through the fiber base point $p$.

is completely filled with fixed points of (5). Note that $M_0$ is a graph over the compact domain

$$D_0 = \{(\mathbf{x}, \phi) \ : \ \mathbf{x} \in \mathcal{D}, \ \phi \in [t_0 - T, t_0 + T]\} \, ;$$

we show the geometry of $D_0$ and $M_0$ in Fig. 1.

Inspecting the Jacobian

$$\frac{d}{d\mathbf{v}} \left[\mathbf{u}(\mathbf{x}, \phi) - \mathbf{v}\right]_{M_0} = -\mathbf{I}_{n\times n},$$

we find that $M_0$ attracts nearby trajectories at a uniform exponential rate of $\exp(-\tau)$ (i.e., $\exp(-t/\epsilon)$ in terms of the original unscaled time). In fact, $M_0$ attracts all the solutions of (5) that satisfy $(\mathbf{x}(0), \phi(0)) \in \mathcal{D}\times [t_0 - T, t_0 + T]$; this can be verified using the last equation of (5), which is explicitly solvable for any constant value of $\mathbf{x}$ and $\phi$. Consequently, $M_0$ is a compact normally hyperbolic invariant set that has an open domain of attraction. Note that $M_0$ is not a manifold because its boundary

$$\partial M_0 = \partial\mathcal{D}\times [t_0 - T, t_0 + T] \bigcup \mathcal{D}\times \{t_0 - T\} \bigcup \mathcal{D}\times \{t_0 + T\}$$

has corners; $M_0 - \partial M_0$, however, is an $n+1$-dimensional normally hyperbolic invariant manifold.

By the results of Fenichel [Fen79] for autonomous systems, any compact normally hyperbolic set of fixed points on (5) gives rise to a nearby locally invariant manifold for system (4). (Local invariance means that trajectories can only leave the manifold through its boundary.) In our context, Fenichel's results guarantee the existence of $\epsilon_0\,(t_0, T) > 0$, such that for all $\epsilon \in [0, \epsilon_0)$, system (4) admits an attracting locally invariant manifold $M_\epsilon$ that is $\mathcal{O}(\epsilon)$ $C^r$-close to $M_0$ (See Fig. 2). The manifold $M_\epsilon$ can be written in the form of a Taylor expansion

$$\begin{aligned} M_\epsilon = \big\{(\mathbf{x}, \phi, \mathbf{v}) \ : \ \mathbf{v} = \mathbf{u}(\mathbf{x}, \phi) + \epsilon\mathbf{u}^1(\mathbf{x}, \phi) + \ldots \\ +\epsilon^r\mathbf{u}^r(\mathbf{x}, \phi) + \mathcal{O}(\epsilon^{r+1}), \quad (\mathbf{x}, \ \phi) \in D_0\big\} \, ; \end{aligned} \tag{6}$$

the functions $\mathbf{u}^k(\mathbf{x}, \phi)$ are as smooth as the right-hand side of (3). $M_\epsilon$ is a *slow manifold,* because (4) restricted to $M_\epsilon$ is a slowly varying system of the form

$$\begin{aligned} \mathbf{x}' &= \epsilon\mathbf{v}|_{M_\epsilon} \\ &= \epsilon\left[\mathbf{u}(\mathbf{x}, \phi) + \epsilon\mathbf{u}^1(\mathbf{x}, \phi) + \ldots + \epsilon^r\mathbf{u}^r(\mathbf{x}, \phi) + \mathcal{O}(\epsilon^{r+1})\right]. \end{aligned} \tag{7}$$

We find the functions $\mathbf{u}^k(\mathbf{x}, \phi)$ using the invariance of $M_\epsilon$, which allows us to differentiate the equation defining $M_\epsilon$ in (6) with respect to $\tau$. Specifically, differentiating

$$\mathbf{v} = \mathbf{u}(\mathbf{x}, \phi) + \sum_{k=1}^{r} \epsilon^k\mathbf{u}^k(\mathbf{x}, \phi) + \mathcal{O}(\epsilon^{r+1})$$

with respect to $\tau$ gives

$$\mathbf{v}' = \mathbf{u}_\mathbf{x}\mathbf{x}' + \mathbf{u}_\phi\phi' + \sum_{k=1}^{r} \epsilon^k \left[\mathbf{u}_\mathbf{x}^k\mathbf{x}' + \mathbf{u}_\phi^k\phi'\right] + \mathcal{O}(\epsilon^{r+1}), \tag{8}$$

on $M_\epsilon$, while restricting the $\mathbf{v}$ equations in (3) to $M_\epsilon$ gives

$$\begin{aligned} \mathbf{v}' &= \left[\mathbf{u} - \mathbf{v} + \epsilon\frac{3R}{2}\frac{D\mathbf{u}}{Dt} + \epsilon\left(1 - \frac{3R}{2}\right)\mathbf{g}\right]_{M_\epsilon} \\ &= -\sum_{k=1}^{r} \epsilon^k\mathbf{u}^k(\mathbf{x}, \phi) + \epsilon\frac{3R}{2}\frac{D\mathbf{u}}{Dt} - 2\epsilon\mathbf{\Omega}\times\mathbf{u}(\mathbf{x}, \phi) \\ &\quad -2\epsilon\mathbf{\Omega}\times\left(\sum_{k=1}^{r}\epsilon^k\mathbf{u}^k(\mathbf{x}, \phi)\right) + \epsilon\left(1 - \frac{3R}{2}\right)\mathbf{g}. \end{aligned} \tag{9}$$

Comparing terms containing equal powers of $\epsilon$ in (8) and (9), then passing back to the original time $t$, we obtain the following result.

**Theorem 1** *For small $\epsilon > 0$, the equation of particle motion (7) on the slow manifold $M_\epsilon$ can be rewritten as*

$$\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, t) + \epsilon\mathbf{u}^1(\mathbf{x}, t) + \ldots + \epsilon^r\mathbf{u}^r(\mathbf{x}, t) + \mathcal{O}(\epsilon^{r+1}), \tag{10}$$

*where $r$ is an arbitrary but finite integer, and the func-*

*tions* $\mathbf{u}^i(\mathbf{x},t)$ *are given by*

$$\begin{aligned}\mathbf{u}^1 &= \left(\frac{3R}{2}-1\right)\left[\frac{D\mathbf{u}}{Dt}-\mathbf{g}\right]-2\mathbf{\Omega}\times\mathbf{u}(\mathbf{x},\phi),\\ \mathbf{u}^k &= -\left[\frac{D\mathbf{u}^{k-1}}{Dt}+(\nabla\mathbf{u})\,\mathbf{u}^{k-1}\right. \\ &\quad \left. +\sum_{i=1}^{k-2}\left(\nabla\mathbf{u}^l\right)\mathbf{u}^{k-l-1}+2\mathbf{\Omega}\times\mathbf{u}^{k-1}(\mathbf{x},\phi)\right]\end{aligned} \tag{11}$$

*for* $k\geq 2$.

We shall refer to (10) with the $\mathbf{u}^i(\mathbf{x},t)$ defined in (11) as the *inertial equation* associated with the velocity field $\mathbf{u}(\mathbf{x},t)$, because (10) gives the general asymptotic form of inertial particle motion induced by $\mathbf{u}(\mathbf{x},t)$. A leading-order approximation to the inertial equations is given by

$$\dot{\mathbf{x}}=\mathbf{u}(\mathbf{x},t)+\epsilon\left(\frac{3R}{2}-1\right)\left[\frac{D\mathbf{u}}{Dt}-\mathbf{g}\right]-2\epsilon\mathbf{\Omega}\times\mathbf{u}(\mathbf{x},\phi); \tag{12}$$

this is the lowest-order truncation of (10) that has nonzero divergence, and hence is capable of capturing clustering or dispersion arising from finite-size effects.

The above argument renders the slow manifold $M_\epsilon$ over the fixed time interval $[t_0-T,t_0+T]$. Since the choice of $t_0$ and $T$ was arbitrary, we can extend the existence result of $M_\epsilon$ to an arbitrary long finite time interval.

Slow manifolds are typically not unique, but obey the same asymptotic expansion (11). Consequently, any two slow manifolds and the corresponding inertial equations are $\mathcal{O}(\epsilon^r)$ close to each other. Specifically, if $r=\infty$, then the difference between any two slow manifolds is exponentially small in $\epsilon$. The case of neutrally buoyant particles ($R=2/3$) turns out to be special: the slow manifold is the unique invariant surface

$$M_\epsilon=\{(\mathbf{x},\phi,\mathbf{v})\ :\ \mathbf{v}=\mathbf{u}(\mathbf{x},\phi),\quad (\mathbf{x},\ \phi)\in D_0\},$$

on which the dynamics coincides with those of infinitesimally small particles. This invariant surface survives for arbitrary $\epsilon>0$, as noticed by Babiano et al. [Bab00], but may lose its stability for larger values of $\epsilon$ (cf. Sapsis and Haller [SapHal07]).

## IV. CONVERGENCE TO THE SLOW MANIFOLD

The results of Fenichel [Fen79] guarantee exponential convergence of solutions of (4) to the slow manifold $M_\epsilon$. Translated to the original variables, exponential convergence with a uniform exponent to the slow manifold is only guaranteed over the compact time interval $[t_0-T,t_0+T]$.

Over finite time intervals, exponentially dominated convergence is not necessarily monotone. For instance, if the velocity field suddenly changes, say, at speeds comparable to $\mathcal{O}(1/\epsilon)$, then converged solutions may suddenly find themselves again at an increased distance from the slow manifold before they start converging again (cf. Fig. 3). Again, this is the consequence of the lack of compactness in time, which results in a lack of uniform exponential convergence to the slow manifold over infinite times. Where do solutions converging to the slow manifold tend asymptotically? Observe that for $\epsilon=0$, each solution converging to $M_0$ is confined to an $n$-dimensional plane

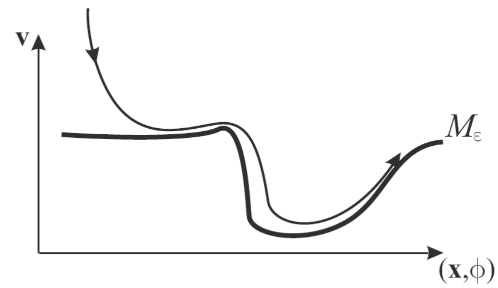


FIG. 3: Sudden changes in the velocity field delay convergence to the slow manifold.

$$f_0^s(p)=\left\{\left(\mathbf{x}_p,\phi_p,\mathbf{v}\right)\ :\ \ p=\left(\ \mathbf{x}_p,\phi_p,\mathbf{u}(\mathbf{x}_p,\phi_p)\right)\in M_0\right\}.$$

Fenichel refers to $f_0^s(p)$ as the *stable fiber* associated with the point $p$: each trajectory in $f_0^s(p)$ converges to the base point of the fiber, $p$. More generally, a stable fiber has the property that each solution intersecting the fiber converges exponentially in time to the solution passing through the base point of the fiber. The collection of all fibers intersecting $M_0$ is called the stable foliation of $M_0$, or simply the stable manifold of $M_0$.

Fenichel [Fen79] showed that the stable foliation of $M_0$ smoothly persists for small enough $\epsilon>0$. Specifically, associated with each point $p\in M_\epsilon$, there is an $n$-dimensional manifold $f_\epsilon^s(p)$ such that any solution of (4) intersecting $f_\epsilon^s(p)$ will converge at an exponential rate to the solution that runs through the point $p$ on $M_\epsilon$. The persisting stable fibers $f_\epsilon^s(p)$ are $C^r$ smooth in $\epsilon$, hence they are $\mathcal{O}(\epsilon)$ $C^r$-close to the invariant planes $f_0^s(p)$, as indicated in Fig. 2b.

## V. APPLICATION: INERTIAL PARTICLES IN THE UNSTEADY FLOW FIELD OF A HURRICANCE

### A. Asymptotics of finite-size particle motion

A general particle motion $(\mathbf{x}(t),\mathbf{v}(t))$ is attracted to a specific solution within the slow manifold $M_\epsilon$. This specific solution runs through the base points of stable fibers intersected by $(\mathbf{x}(t),\mathbf{v}(t))$. As a result, the forward-time asymptotic behaviors seen on the slow manifold are the only possible asymptotic behaviors for general inertial particle motion.

FIG. 4: NASA satellite photo taken at 11:50 am on September 18th 2003.

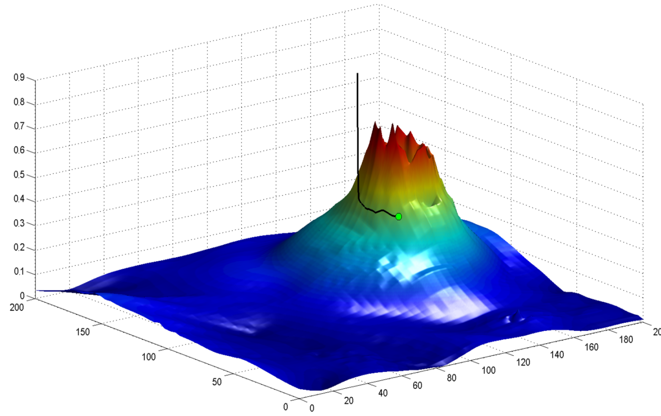


FIG. 5: Convergence of an inertial particle (bubble) on the slow manifold shown in the $(x, y, |\mathbf{v}|)$ space for $t = 16$. The particle is released and advected under the full Maxey-Riley equation.

Rapid changes in the velocity field $\mathbf{u}(\mathbf{x}, t)$ in time will lead to rapid changes in the slow manifold, as seen from the definition of $M_\epsilon$ in (6). In that case, particles that have already converged to the slow manifold may find themselves further away from the slow manifold (whose location has rapidly changed). Particles will converge exponentially to the new location of the slow manifold, but may again find themselves temporarily at a large distance from the manifold if a further rapid change occurs in the velocity field.

### B. The dataset

The dataset used in this paper is obtained from a weather simulation produced by the US National Center for Atmospheric Research (NCAR). It shows the Isabel hurricane, a large tropical depression that made landfall on the East Coast of the US on September 18th 2003 (cf. Fig. 4). The simulation covers a period of 48 time steps (hours). Each time step contains the instantaneous velocity field with a grid resolution 500x500x100 covering an atmospheric volume with coordinates running from 83W to 62W (Longitude), 23.7N to 41.7N (Latitude) and 0.035km to 19.835km (height). These coordinates corresponds to a square area with side length of approximatelly 2000km. To nondimensionalize the data we choose a characteristic lengthscale $L = 10km$, a characterstic velocity $U = 10m/\sec$ and a characterstic timescale $T = \frac{L}{U} = 1000\sec$. The earth rotates with an angular velocity $\Omega$ which we will take, for our analysis to be constant with time ($7.29 \times 10^{-5}\sec^{-1}$) although including a time variation for $\Omega$ may be nessecary for dynamics on very long geological time scales.

### C. Slow-manifold in the flow

Here we show that the inertial equation (10) indeed gives the correct asymptotic motion of finite-size particles in this application. For particles, we choose bubbles with $R = 1.55$ and $\epsilon = 0.1$. We solve the full six-dimensional Maxey-Riley equation (3) on the time interval $[10, 16]$ using a 4th-order Runge-Kutta algorithm with absolute integration tolerance $10^{-6}$. The initial velocity of the particle was taken much larger in absolute value than the velocity corresponding to the same initial location on the slow manifold. In the same figure, we also show the projection of the six-dimensional solution of (3) onto the $(x, y, |\mathbf{v}(\cdot, \cdot, z_p(t))|)$ space, where $z_p(t)$ is the instantaneous $z-$coordinate of the particle. We show the slow manifold $M_\epsilon$ (blue surface); we use color to indicate the instantaneous leading-order geometry of the slow manifold (6) computed for $t = 16$ at the instantaneous vertical particle position $z_p(t)$.

Specifically, colors ranging from dark blue to dark red indicate increasing values of $|\mathbf{v}| = |\mathbf{u}(\cdot, \cdot, z_p(t), T)|$, which is a measure of the "height" of the slow manifold at leading order in the $(\mathbf{x}, \mathbf{v})$ coordinate space. Note the rapid convergence of the particle trajectory to the invariant slow manifold.

### D. Extraction of Lagrangian coherent structures

Coherent structures in the Lagrangian (particle-based) frame can be defined as distingushed sets of fluid particles. These Lagrangian Coherent Structures (LCS) have a decisive impact on fluid mixing by their special stability properties (cf. Haller and Yuan [Hal00]). For the detection of LCS we used an extended Lyapunov-exponent-based LCS detection scheme applied previously in two-

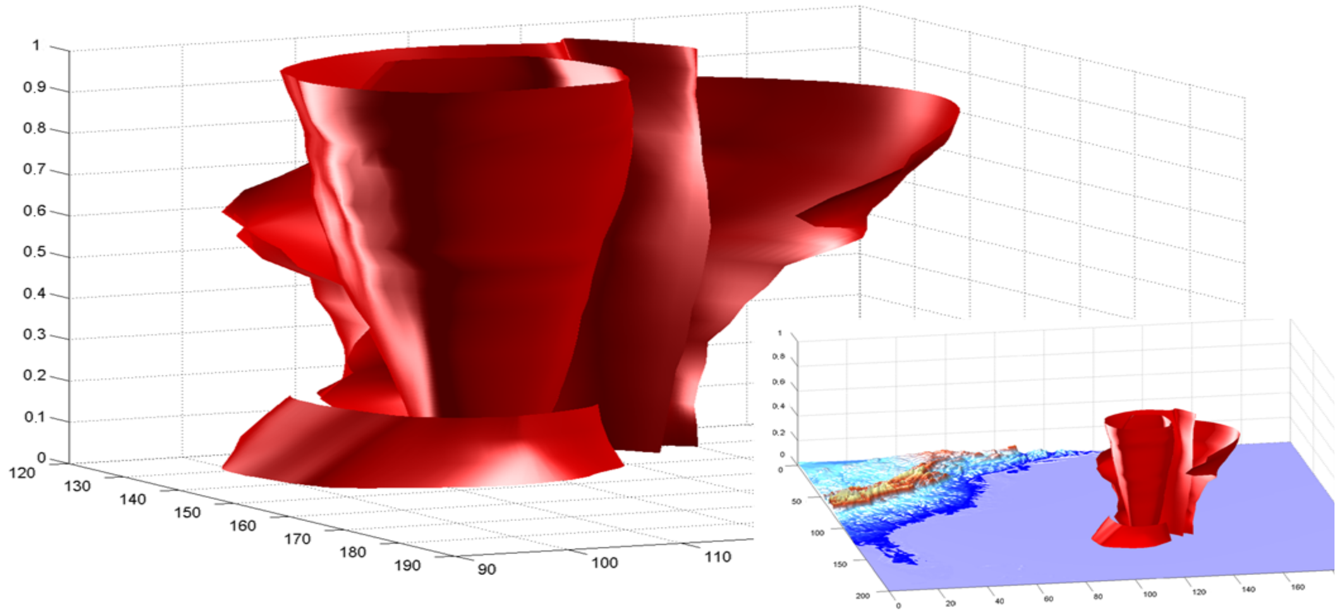


FIG. 6: Attracting manifolds at $t_0 = 16$ extracted from the inertial equation as ridges of the backward-time DLE fields ($t = 13$).

dimensional turbulence (Mathur et al. [Mat07]).

Specifically, by solving numerically the inertial equation (12) for a grid of initial conditions $\mathbf{x}_0$ at $t_0$, we determine the particle trajectories $\mathbf{x}(t, \mathbf{x}_0)$ .By numerical differentiation, we compute the largest singular-value field $\lambda_{\max}(t, t_0, \mathbf{x}_0)$ of the deformation-gradient tensor field $[\partial \mathbf{x}(t, t_0, \mathbf{x}_0) / \partial \mathbf{x}_0]^T [\partial \mathbf{x}(t, t_0, \mathbf{x}_0) / \partial \mathbf{x}_0]$ . We then use the local maximizing surfaces of the direct Lyapunov exponent (DLE) field $\sigma_{t_0}^t(\mathbf{x}_0) = [\ln \lambda_{\max}(t, t_0, \mathbf{x}_0)] / (2(t - t_0))$ plotted over initial positions $\mathbf{x}_0$ to visualize the LCS. In Fig. 6 we show the attracting LCS as local maximizing surfaces of $\sigma_{t_0}^t(\mathbf{x}_0)$ for $t \ll t_0$.

## VI. CONCLUSIONS

In this paper, we have described a way to reduce the Maxey–Riley equation in a rotating frame to a slow manifold that captures the asymptotics of inertial particle dynamics. The slow manifold arises in a singular perturbation approach that is valid for small particle Stokes numbers. We treat general unsteady flows, as opposed to earlier applications of singular perturbation theory in this context that were restricted to concrete steady flows.

Our main result is an explicit inertial equation for motions on the slow manifold suitable for the description of particles' motion in oceanography and meteorology. For small enough Stokes numbers, particles approach trajectories of this inertial equation exponentially fast. We have illustrated the use of the inertial equation on a three dimensional unsteady velocity field that describes the motion of the hurricane Isabel by extracting the attracting Lagrangian coherent structures for the motion of inertial particlers.

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