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# CONTINUOUS APPROXIMATION OF THE COMPLEX DYNAMICS OF A DISCONTINUOUS SYSTEM

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**Abstract**

A continuous approximation of a second-order, piecewise-linear system, modeled as a discontinuous system, is presented. The discontinuous system includes a signum function, approximated by a saturation-type function, whose complex dynamics is analyzed based on some recent results. A numerical comparison between the analytical solutions of both systems shows the accuracy of the approximation.

**Key words**

Discontinuous system, piecewise-linear systems, continuous approximation, strange invariant.

## 1 Introduction

Piecewise-smooth (PWS) systems have attired the attention of many researchers in the last years ([di Bernardo *et al*, 2008], [Filippov, 1988]). They describe with good accuracy important phenomena and practical systems, like friction, impacts, commutation, sliding motion, mechanical, electronic, and even biological systems ([Brogliato, 1999],[di Bernardo *et al*, 2008]). In consequence, they appear frequently in several mathematical fields, in control theory and engineering, and so on. They can exhibit particular behaviors like equilibria intervals, sliding motion, non-typical bifurcations (border collision, grazing, sky blue, sliding, etc.), and chaotic dynamics.

An important class of PWS systems can be described by

$$\dot{x} = F_i(x), \text{ if } x \in S_i, \; i = 1, \cdots, m$$

where $S_i$, $i = 1, 2, \cdots, m$ are open, disjoint sets in $\mathbb{R}^n$ such that $\cup_{i=1}^{m} \bar{S}_i = \mathbb{R}^n$, where $\bar{S}_i$ is the closure of $S_i$. The border between the adjacent sets $S_i$ and $S_j$ can be given by a function $H_{ij}$, that is,

$$\Sigma_{ij} := \bar{S}_i \cap \bar{S}_j = \left\{x : H_{ij}(x) = 0\right\}, \quad i \neq j = 1, \cdots, m.$$

In general, $F_i$ and $H_{ij}$ are smooth; in this paper we will suppose also that they are linear.

These systems have been analyzed with several tools, like the convex method of Filippov [Filippov, 1988]. Some conditions to have diverse kinds of typical bifurcations of discontinuous systems are given in [di Bernardo *et al*, 2008].

A different approach reported in many works is to use continuous functions to approximate discontinuous systems [Danca and Codreanu, 2001], applying well known analytical results of ordinary differential equations. For example, [Feckan, Awrejcewicz and Olejnik, 2005] use continuous approximations to calculate periodic orbits. However, discontinuous systems can have dynamical behavior not possible to reproduce by continuous systems, and the accuracy of the approximation is very often evaluated numerically.

In this paper we use a continuous approximation to analyze the existence of complex, chaotic-type orbits, in a class of second-order, piecewise-linear systems. We approximate the discontinuous term, given by a signum function, by a saturation function. From a comparison of the explicit solutions of both systems, it can be observed a good convergence of the approximated solution to the response of the discontinuous system. Moreover, by applying a result given in [Kukučka, 2007], it is possible to calculate the so-called nonsmooth Melnikov function, from which it is possible to predict a chaotic behavior of the approximate system. Because this system can be arbitrarily approximated to the discontinuous system, the ability to produce complex orbits of this last system can be concluded.

The paper is organized as follows. In section 2 we present the discontinuous system and its approximation. In Section 3 we analyze the conditions the approximate system must satisfy to have a strange invariant set. Explicit solutions of both systems are given in Section 4, and a numerical comparison of the corresponding dynamical behavior is shown in Section 5. Finally, in Section 6 some final comments are presented.

## 2 Discontinuous system

We consider a class of discontinuous, second-order systems described by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 - 2\xi x_2 + \alpha \mathrm{sign}\,(x_1) + u\,(t)\,, \end{aligned} \tag{1}$$

where $0 < \xi < 1$, $\alpha > 0$, $u\,(t) = r\sin(\omega t)$, and the discontinuous term is defined as

$$\mathrm{sign}\,(v) := \begin{cases} -1, \text{ if } v < 0; \\ 0, \quad \text{if } v = 0; \\ 1, \quad \text{if } v > 0. \end{cases} \tag{2}$$

If $x = (x_1, x_2)^T$, a compact notation is given by

$$\dot{x} = \begin{cases} Ax - b + q\,(t)\,, \text{ if } x_1 \in S_-; \\ Ax + b + q\,(t)\,, \text{ if } x_1 \in S_+; \end{cases} \tag{3}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -2\xi \end{pmatrix}, b = \begin{pmatrix} 0 \\ \alpha \end{pmatrix}, q\,(t) = \begin{pmatrix} 0 \\ r\sin(\omega t) \end{pmatrix},$$

$S^{-(+)} = \left\{x \in \mathbb{R}^2 | x_1 < (>) 0\right\}$. The border surface is the $x_2$-axis, denoted by $\Sigma = \left\{x \in \mathbb{R}^2 | x_1 = 0\right\}$.

We approximate the discontinuous term (2) by a saturation function that, for $n$ a positive, integer number (which will be, in general, a large number), is defined as

$$\mathrm{sat}_n\,(v) := \begin{cases} -1, \text{ if } nv < -1; \\ nv, \text{ if } |nv| \le 1; \\ +1, \text{ if } nv > 1. \end{cases} \tag{4}$$

The approximate system is then given by

$$\dot{x} = \begin{cases} Ax - b + q, \text{ if } x_1 \in R^-; \\ A_n x + q, \quad \text{ if } x_1 \in R^n; \\ Ax + b + q, \text{ if } x_1 \in R^+; \end{cases} \tag{5}$$

where

$$A_n = \begin{pmatrix} 0 & 1 \\ \alpha n - 1 & -2\xi \end{pmatrix}$$

and

$$\begin{aligned} R^- &= \left\{x \in \mathbb{R}^2 \mid nx_1 \le -1\right\}; \\ R^n &= \left\{x \in \mathbb{R}^2 \mid |nx_1| \le 1\right\}; \\ R^+ &= \left\{x \in \mathbb{R}^2 \mid \ nx_1 \ge 1\right\}. \end{aligned}$$

## 3 Chaotic dynamics of the approximate system

The Melnikov method is a well known technique to analyze the generation of homoclinic tangles of second-order dynamical systems perturbed by a periodic, small driving input. The nominal scenario assumes the existence of a saddle point giving place to a homoclinic orbit, and the existence of periodic orbits inside the region encircled by the homoclinic trajectory. This orbit persists under small enough perturbations, and after that it can be broken, giving place to a homoclinic bifurcation, producing eventually a strange invariant set. The Melnikov method can be used to predict this last scenario.

This method works well for differentiable systems. However, the nominal scenario may not be produced by nonsmooth systems, particularly the homoclinic orbit with an infinite evolution time. Furthermore, classical results require smoothness of the vector field, and the application of this method to systems like (1) or (5) is not adequate. Nevertheless, some recent results given in [Kukučka, 2007] can be applied to the approximate (nonsmooth) system described before(system (5)) , and the possible generation of chaotic orbits can be predicted for this kind of systems.

Let us assume that parameters $\xi$ and $r$ can be given by $\xi = \epsilon\gamma$ and $r = \epsilon R$. Then system (5) can be described by a perturbed system given by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -x_1 + \alpha \mathrm{sat}_n\,(x_1) + \epsilon\left[-\gamma x_2 + R\sin(\omega t)\right]. \end{aligned} \tag{6}$$

When $\epsilon = 0$, system (6) can be described as a Hamiltonian system with a Hamiltonian function given by

$$H(x_1, x_2) = \frac{x_2^2}{2} + V(x_1), \tag{7}$$

where

$$V(x_1) = \begin{cases} \frac{x_1^2}{2} + \alpha\left(x_1 + \frac{1}{2n}\right), \text{ if } x_1 < -\frac{1}{n}; \\ \quad -a\frac{x_1^2}{2}, \qquad \text{ if } |x_1| \le \frac{1}{n}; \\ \frac{x_1^2}{2} - \alpha\left(x_1 - \frac{1}{2n}\right), \text{ if } x_1 > \frac{1}{n}; \end{cases}$$

with $a = \alpha n - 1$.

This system has three equilibrium points, two centers at $(\pm\alpha, 0)$ and a saddle point placed at the origin, with two homoclinic orbits described by

$$\begin{pmatrix} u_0^1 \\ u_0^2 \end{pmatrix} = \begin{cases} \frac{1}{n}\begin{pmatrix} e^{\sqrt{a}(t+t_1)} \\ \sqrt{a}e^{\sqrt{a}(t+t_1)} \end{pmatrix}, & \text{if}\, t \le -t_1; \\ \begin{pmatrix} \alpha + \sqrt{\frac{a\alpha}{n}}\cos t \\ -\sqrt{\frac{a\alpha}{n}}\sin t \end{pmatrix}, & \text{if}\, |t| \le t_1; \\ \frac{1}{n}\begin{pmatrix} e^{-\sqrt{a}(t-t_1)} \\ -\sqrt{a}e^{-\sqrt{a}(t-t_1)} \end{pmatrix}, & \text{if}\, t \ge t_1; \end{cases} \quad (8)$$

where $t_1 = \arccos\left(-\sqrt{\frac{a}{\alpha n}}\right)$.

### 3.1 Melnikov function

The Melnikov function can be calculated with the help of the results given in [Kukučka, 2007], summarized in the Appendix. Given the symmetry of the vector field, we can analyze the right homoclinic orbit in the interval $[0, \infty)$; the other case is similar.

Let us describe system (5) in the following form (see the Appendix),

$$x' = \begin{cases} f_-(x) + \epsilon g_-(t, x), & \text{if } x \in S^-; \\ f_+(x) + \epsilon g_+(t, x), & \text{if } x \in S^+; \end{cases} \quad (9)$$

where $f_- = (x_2, ax_1)^T$, $f_+ = (x_2, -x_1 + \alpha)^T$, $g_- = g_+ = \left(0, -\gamma x_2 + R\sin(\omega t)\right)^T$. The border surface is given by $\sum = \left\{x \in \mathbb{R}^2 \,\middle|\, x_1 = 1/n, x_2 \in [0, \infty)\right\}$, which divides the sections $S^- = \left\{x \in \mathbb{R}^2 \,\middle|\, 0 \le x_1 < 1/n, x_2 \in [0, \infty)\right\}$ and $S^+ = \left\{x \in \mathbb{R}^2 \,\middle|\, x_1 > 1/n, x_2 \in [0, \infty)\right\}$.

The origin is a saddle point of $\dot{x} = f(x)$, and the homoclinic orbit crosses $\sum$ at the times $\tau_1 = -t_1$ and $\tau_2 = t_1$ in the points $u_0(\tau_1) = \left(1/n, \sqrt{a}/n\right)^T$ and $u_0(\tau_2) = \left(1/n, -\sqrt{a}/n\right)^T$. We have that the system is Hamiltonian, then $\mathrm{tr}(Df_-) = \mathrm{tr}(Df_+) = 0$, and the perturbation is periodic. Moreover, because the vector field is continuous, a direct application of Theorem 2 of the Appendix needs only the calculation of the next integral, equivalent to the smooth case,

$$M(\theta) = \int_{-\infty}^{\infty} f\left(u_0(t)\right) \wedge g\left(t + \theta, u_0(t)\right) dt. \quad (10)$$

A straightforward calculation of this integral leads to

$$M(\theta) = -\frac{\alpha\gamma\sqrt{a}}{n}\left(1 + \sqrt{a}t_1\right) + \\ + \frac{2\alpha R\sqrt{a}\sin\left(\omega t_1 + \arctan\left(\frac{\omega}{\sqrt{a}}\right)\right)}{\left(\omega^2 - 1\right)\sqrt{\omega^2 + a}}\cos(\omega\theta)$$

Note that, if $M(\theta) = 0$, then it has simple zeroes if, and only if,

$$\left|\frac{r}{\left(\omega^2 - 1\right)\sqrt{\omega^2 + a}}\right| > \left|\frac{\xi}{n}\left(1 + \sqrt{a}t_1\right)\right|. \quad (11)$$

## 4 System solutions

In this section we obtain the solutions of the discontinuous and the approximate systems, and show how this last solution can be arbitrarily close to that of the discontinuous system.

### 4.1 Discontinuous system

Because system (1) is piecewise linear, a solution $\varphi$ can be obtained from the general solution of nonautonomous, linear systems, yielding, for $x \in S^-$,

$$\varphi^-(x_0, t) = \begin{pmatrix} \phi_1 - \alpha + a\cos(\omega t) + b\sin(\omega t) \\ \phi_2 + b\omega\cos(\omega t) - a\omega\sin(\omega t) \end{pmatrix}, \quad (12)$$

where

$$\phi_1 = e^{-\xi\Delta t}\left(B_1\cos(\kappa\Delta t) + B_2\sin(\kappa\Delta t)\right),$$
$$\phi_2 = e^{-\xi\Delta t}\left(B_3\cos(\kappa\Delta t) + B_4\sin(\kappa\Delta t)\right),$$

$\Delta t = t - t_0$, $\kappa = \sqrt{1 - \xi^2}$, $a = -2r\xi\omega/d$, $b = r(1 - \omega^2)/d$, $d = (1 - \omega^2)^2 + (2\omega\xi)^2$,

$$B_1 = \alpha + x_{1_0} - a\cos(\omega t_0) - b\sin(\omega t_0)$$
$$\kappa B_2 = x_{2_0} + \xi(x_{1_0} + \alpha) - (a\xi + b\,\omega)\cos(\omega t_0) + (a\omega - b\xi)\sin(\omega t_0),$$

$$B_3 = x_{2_0} - b\omega\cos(\omega\, t_0) + a\omega\sin(\omega t_0),$$
$$\kappa B_4 = -x_{2_0} - x_{1_0} - \alpha + (a + b\omega\xi)\cos(\omega t_0) + (b - a\omega\xi)\sin(\omega t_0).$$

For $x \in S^+$ we have

$$\varphi^+(x_0, t) = \begin{pmatrix} \phi_3 + \alpha + a\cos(\omega t) + b\sin(\omega t) \\ \phi_4 + b\omega\cos(\omega t) - a\omega\sin(\omega t) \end{pmatrix}, \quad (13)$$

where

$$\phi_3 = e^{-\xi\Delta t}C_1\cos(\kappa\Delta t) + C_2\sin(\kappa\Delta t)$$
$$\phi_4 = e^{-\xi\Delta t}B_3\cos(\kappa\Delta t) + C_4\sin(\kappa\Delta t)$$

$$\begin{aligned} C_1 &= -\left(\alpha - x_{1_0} + a\cos(\omega t_0) + b\sin(\omega t_0)\right) \\ \kappa C_2 &= x_{2_0} + \xi\left(x_{1_0} - \alpha\right) - (a\xi + b\omega)\cos(\omega t_0) \\ &\quad + (a\omega - b\xi)\sin(\omega t_0), \\ \kappa C_4 &= -x_{2_0} - x_{1_0} + \alpha + (a + b\omega\xi)\cos(\omega t_0) \\ &\quad + (b - a\omega\xi)\sin(\omega t_0). \end{aligned}$$

Because the interaction of the orbits with the discontinuity surface is transversal, a complete solution can be obtained by concatenating (12)-(13). For example, if $x_0 \in S^+$ and $t_\Sigma$ denotes the switching time, then we have, for an interval $(t_0, t_0+\sigma)$, $\sigma > 0$, where there is only one commutation, that the solution is given by

$$\varphi(x_0,t) = \begin{cases} \varphi^+(x_0,t), \text{ for } t_0 \le t < t_\Sigma; \\ \varphi^-(\xi_0,t), \text{ for } t_\Sigma \le t < t_\Sigma + \sigma; \end{cases} \quad (14)$$

where $\xi_0 = \varphi^+(x_0, t_\Sigma)$.

### 4.2 Approximate solution

Similarly, the solution $\varphi_n$ of (5) can be obtained by concatenation of the orbits in the regions $R^-$, $R^n$, and $R^+$, denoted $\varphi_n^-$, $\varphi_n^n$, and $\varphi_n^+$, respectively.

For $x \in R^-$ it is straightforward to obtain $\varphi_n^- = \varphi^-$. Similarly, for $x \in R^+$ we have $\varphi_n^+ = \varphi^+$. Finally, for $x \in R^n$, we can obtain the expression

$$\varphi_n^n(x_0,t) = \begin{pmatrix} \phi_5 + a_1\cos(\omega t) + b_1\sin(\omega t) \\ \phi_6 + b_1\omega\cos(\omega t) - a_1\omega\sin(\omega t) \end{pmatrix}, \quad (15)$$

where

$$\begin{aligned} \phi_5 &= e^{-\xi\Delta t}\left(D_1\cosh(\kappa_1\Delta t) + D_2\sinh(\kappa_1\Delta t)\right) \\ \phi_6 &= e^{-\xi\Delta t}\left(D_3\cosh(\kappa_1\Delta t) + D_4\sinh(\kappa_1\Delta t)\right) \end{aligned}$$

$$\kappa_1 = \sqrt{\xi^2 + \alpha n - 1}, \quad a_1 = -2r\xi\omega/f,$$

$$b_1 = \frac{r\left(1 - \alpha n - \omega^2\right)}{f}, \quad f = \left(1 - \alpha n - \omega^2\right)^2 + (2\omega\xi)^2,$$

$$\begin{aligned} D_1 &= x_{1_0} - a_1\cos(\omega t_0) - b_1\sin(\omega t_0), \\ \kappa_1 D_2 &= x_{2_0} + \xi x_{1_0} - (a_1\xi + b_1\omega)\cos(\omega t_0) \\ &\quad + (a_1\omega - b_1\xi)\sin(\omega t_0), \\ D_3 &= x_{2_0} - b_1\omega\cos(\omega t_0) + a_1\omega\sin(\omega t_0), \\ \kappa_1 D_4 &= -x_{2_0}\xi + (\alpha n - 1)\, x_{1_0} + \omega\,(a\omega - b\xi)\cos(\omega t_0) \\ &\quad + \left((1 - \alpha n)\, b - a\omega\,\xi\right)\sin(\omega\, t_0). \end{aligned}$$

Similar to the discontinuous system (1), a complete solution can be obtained by concatenating $\varphi_n^- = \varphi^-$ (12), $\varphi_n^+ = \varphi^+$ (13), and $\varphi_n^n$ (15). For example, if $x_0 \in R^+$, and the orbit enters region $R^n$ at time $t_{n_1}$, then to region $R^-$ at time $t_{n_2}$ and stay there, then we have, for an interval $(t_0, t_0 + t_{n_1} + t_{n_2} + \sigma)$, $\sigma > 0$, that the solution is given by

$$\varphi_n(x_0,t) = \begin{cases} \varphi_n^+(x_0,t), \; t_0 \le t \le t_{n_1}; \\ \varphi_n^n(y_0,t), \; t_{n_1} \le t \le t_{n_2}; \\ \varphi_n^-(z_0,t), \; t_{n_2} \le t \le t_{n_2} + \sigma; \end{cases} \quad (16)$$

where $y_0 = \varphi_n^+(x_0, t_{n_1})$, $z_0 = \varphi_n^n(y_0, t_{n_2})$.

## 5 Numerical results

In this section we show some numerical results obtained from the explicit solutions presented in Section 4. These results were calculated with MatLab©. Parameter values are $\xi = 0.08$, $\alpha = 1$, $r = 1.1$, $\omega = \pi/10$. The simulation interval was from 0 to 140 sec, and the integration step $h = 0.001$ sec. Initial conditions were set to $x_0 = (0.1, 0.1)^T$. We obtained the difference between the position and velocity of the two system responses, the discontinuous and the approximate, for different values of the saturation slope $n \in \{2, 25, 500\}$. These results are shown in figures 1 and 2 for the position and the velocity, respectively.

Note that, for $n > 500$, the difference between both

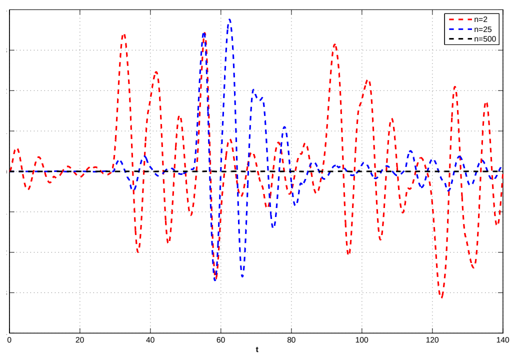


Figure 1. Position difference between the discontinuous and the continuous system, for $n = 2, 25, 500$.

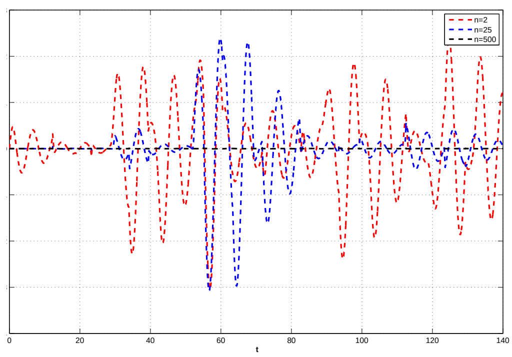


Figure 2. Velocity difference between the discontinuous and the continuous system, for $n = 2, 25, 500$.

systems is negligible. This difference can be arbitrarily small if $n$ is big enough. Moreover, condition (11) is satisfied for big values of $n$. This means that the dynamical behavior displayed by the approximate system can be arbitrarily close to the discontinuous system.

Figures 3-5 show some responses of the discontinuous system, for different parameter values satisfying contidition (11).

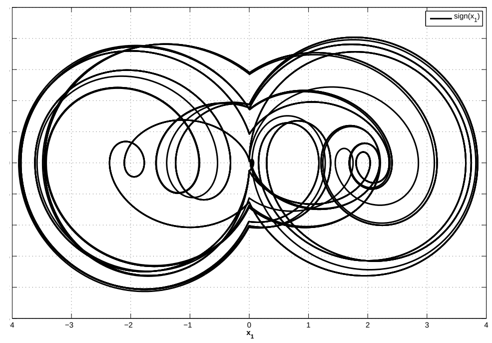


Figure 3. Response of the discontinuous system. $c_i = [0.001\,,\,0.01], \xi = 0.08, \alpha = 1, r = 1.1, \omega = 0.1\,\pi$.

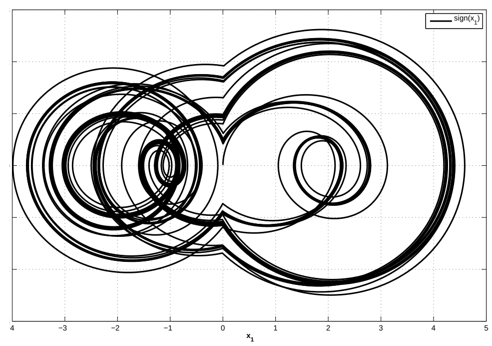


Figure 4. Response of the discontinuous system. $c_i = [0.001\,,\,0.01], \xi = 0.01, \alpha = 1, r = 1.1, \omega = 0.1\,\pi$.

## 6 Conclusions

The application of a recent result about the persistence of homoclinic orbits in a class of nonsmooth systems, and the possibility that a homoclinic bifurcation may occur, has permitted to analyze the complex dynamics exhibited by a piecewise linear system that can be seen as an approximation of a discontinuous system. From a comparison of the explicit solutions of both systems, it can be observed a good convergence of the approximated solution. From this fact, the ability of this kind of systems to produce chaotic-like orbits can be concluded. However, to prove that the discontinuous system had the usual properties defining a system as chaotic deserves a more profound analysis.

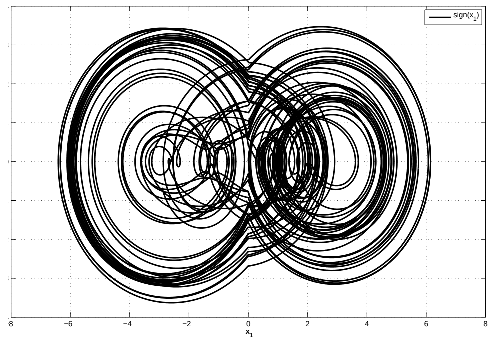


Figure 5. $c_i = [0.001\,,\,0.01], \xi = 0.01, \alpha = 1, r = 2, \omega = 0.1\,\pi$.

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## Appendix A Melnikov method for discontinuous systems

Let us consider the system

$$\dot{x} = f(x) + \epsilon g(t, x), \quad t \in \mathbb{R}, x \in \mathbb{R}^2 \tag{17}$$

where $f(g) : G \to \mathbb{R}^n$, $G := J \times G' \subset \mathbb{R} \times \mathbb{R}^n$, and $f$ and $g$ satisfy the following assumptions,

(i) $f$ and $g$ have the form

$$f(t, x)\big(g(t, x)\big) = \begin{cases} f_-(g_-) \text{ if } & x \in S_- \\ f_+(g_+) \text{ si } & x \in S_+ \end{cases} \tag{18}$$

where $G'$ is partitioned in two disjoint, open subsets $S_-$, $S_+$ by a surface $\Sigma$, such that $G' = S_- \cup \Sigma \cup S_+$. The surface $\Sigma$ is defined by a scalar function $H : G' \to \mathbb{R}$, $H \in C^r, r \geq 1$. The subsets $S_-$ y $S_+$, and the surface $\Sigma$ can be defined by

$$S_- = \left\{x \in G' | H(x) < 0\right\}$$
$$S_+ = \left\{x \in G' | H(x) > 0\right\} \quad (19)$$
$$\sum = \left\{x \in G' | H(x) = 0\right\}$$

(ii) The normal to the surface $\Sigma$ is given by

$$n(x) = \left[DH(x)\right]^T, \quad x \in \Sigma \quad (20)$$

and it is chosen in such a way that $n(x) \neq 0$ for each $x \in \Sigma$.

(iii) There exist functions $h_-(k_-) : J \times D_- \to \mathbb{R}^n$ and $h_+(k_-) : J \times D_+ \to \mathbb{R}^n$ with the properties

(a) $S_- \cup \Sigma \subset D_-$, $S_+ \cup \Sigma \subset D_+$, where $D_-$ and $D_+$ are domains in $\mathbb{R}^n$.

(b) $h_-(k_-) \in C^r$, $h_+(k_+) \in C^r$

(c)

$$\begin{aligned} h_-(k_-) &= f_-(g_-), \forall t \in J, \forall x \in S_- \\ h_+(k_+) &= f_+(g_+), \forall t \in J, \forall x \in S_+ \end{aligned} \quad (21)$$

Moreover, the following conditions hold,

1. The system $\dot{x} = f_-$ has an saddle point $x_0 \in S_-$.
2. $$tr(Df_-) = tr(Df_+) = 0 \quad (22)$$
   in their respective domains.
3. $g$ is a $T$-periodic function, that is, there exists $T > 0$ such that
   $$g(t+T, x) = g(t, x) \quad (23)$$
4. Let $u_0(t)$ be the homoclinic trajectory of the equilibrium $x_0$, which has at least one transversal intersection with $\Sigma$.

**Theorem: 1.** *The function* $d(\epsilon, \theta)$[1], $\epsilon \in (-\epsilon_4, \epsilon_4)$, $\theta \in \mathbb{R}$, *can be expressed as*

$$d(\epsilon, \theta) = \frac{\epsilon}{\left\| f_+(u_0(t)) \right\|} M(\theta) + O\left(\epsilon^2\right), \quad (24)$$

*where*

$$\begin{aligned} M(\theta) = &\int_{-\infty}^{\infty} f\left(u_0(t)\right) \wedge g\left(t+\theta, u_0(t)\right) dt \\ &+ \sum_{j=1}^{2i_0-1} \left[\Delta^u\left(j+1, \tau_j, \theta\right) - \Delta^u\left(j, \tau_j, \theta\right)\right] \\ &+ \sum_{j=2i_0}^{2k} \left[\Delta^s\left(j+1, \tau_j, \theta\right) - \Delta^s\left(j, \tau_j, \theta\right)\right] \end{aligned} \quad (25)$$

Finally, we have the next theorem.

**Theorem: 2.** *Let $\epsilon_0$ be small enough, and all the conditions (18)-(23) satisfied.*

*(i) If there exists a number $\theta_0 \in \mathbb{R}$ such that*

$$M(\theta_0) = 0, \quad DM(\theta_0) \neq 0, \quad (26)$$

*then there exists a mapping $C^{r-1}$ $\theta : (-\epsilon_0, \epsilon_0) \to \mathbb{R}$ such that*

$$\theta(0) = \theta_0 \ y \ d\left(\epsilon, \theta(\epsilon)\right) = 0, \ \forall \epsilon \in (-\epsilon_0, \epsilon_0). \quad (27)$$

*(ii) System (17) has a homoclinic solution $x(\epsilon, t)$, corresponding to the $T$-periodic solution $\varphi_{\epsilon,0}(0, \xi_\epsilon)$, for each $\epsilon \in (-\epsilon_0, \epsilon_0)$.*

[1] where $d$ is a length of the projection of the vector $u^u_{\epsilon,\theta}(0) - u^s_{\epsilon,\theta}(0)$ to line L.