CHAOS CONTROL BY TIME–DELAYED FEEDBACK WITH AN UNSTABLE CONTROL LOOP

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Abstract
We demonstrate by electronic circuit experiments the feasibility of an unstable control loop to stabilize torsion-free orbits by time-delayed feedback control. Our experiments show the importance of the coupling scheme for global control performance.

Key words
Control of chaos, experimental methods, numerical methods

1 Introduction

Control of complex and chaotic behaviour has been one of the most rapidly developing topics in applied nonlinear science for more than one decade (cf. [Schuster, 1999] and references therein). Contrary to traditional control schemes which have been developed by engineers and applied mathematicians for more than half a century the emphasis of non–invasive methods has lead to new concepts like time–delayed feedback techniques [Pyragas, 1997]. While such a concept is easy to implement in experiments to stabilise time periodic states, a deeper theoretical understanding has been gained only recently (cf. e.g. [Just et al., 1997]). So far the control performance has been evaluated on the basis of linear stability analysis but no systematic treatment of global properties, like the dependence of the control performance on perturbations or the size of basins of attraction is available in the literature. Although there exist meanwhile software packages for analysing global features of differential–difference equations [Engelborghs, Luzyanina and Rose, 2000] such tools are of limited use since time–delayed feedback control mainly targets at systems where a proper mathematical model is not available. Thus generic properties of the control system are of interest and such features are difficult to estimate from numerical simulations. We will illustrate the experimental relevance with electronic circuit experiments.

Time-delayed feedback methods are based on measuring a time signal $s(t)$. The control force is generated by a time-delayed difference signal:

$$F(t) = k\,(s(t) - s(t-\tau)) \qquad (1)$$

For proper choice of the control amplitude k stable periodic oscillations can be achieved. Such a scheme is non-invasive if the delay time τ is chosen to coincide with the period of the target state. Once the system trajectory has settled on the UPO the control force $F(t)$ vanishes by construction.

A large number of successful applications of TDFC have been reported in various fields of physics, engineering, chemistry and biology [Schuster, 1999]. While TDFC is a convenient technique for controlling chaos, a serious drawback called the 'odd number limitation' became evident [Nakajima, 1997] predicting that in driven systems UPOs with an odd number of real unstable Floquet multipliers can never be stabilized by conventional TDFC. In other words only UPOs with finite torsion can be stabilized [Just et al., 1997]. To overcome this limitation Pyragas introduced the counterintuitive idea of introducing an unstable time-delayed feedback controller (UTDFC) [Pyragas, 2001]. Such a controller has an additional unstable loop variable which artificially increases the number of the real Floquet multipliers to become even and, thus, avoids the odd number limitation.

2 Unstable van der Pol oscillator

A prominent paradigm showing such torsion-free unstable orbits is the unstable van der Pol oscillator which is described by the following equations of motion:

$$\dot{x}(t) = -y(t) + \varepsilon x(t) + x^3(t)/3, \tag{2a}$$
$$\dot{y}(t) = x(t) \quad . \tag{2b}$$

Here, ε is the bifurcation parameter of system, and the time scale is normalized to the inverse oscillator frequency. Equation (2) differs from that for the conventional van der Pol oscillator merely by the sign of the nonlinear coefficient. For $\varepsilon < 0$, this equation has two coexisting solutions, a stable fixed point at the origin $x = y = 0$, and an unstable limit cycle with period $\tau = 2\pi + \mathcal{O}(\varepsilon)$, amplitude $2\sqrt{-\varepsilon} + \mathcal{O}(\varepsilon)$, and a real positive Floquet exponent $\lambda = -\varepsilon + \mathcal{O}(\varepsilon^{3/2})$. The real positive Floquet exponent indicates that the limit cycle is unstable and shows no torsion.

3 Applying the concept of an unstable controller

We assume that x is a system variable accessible in experiment. To stabilize the unstable periodic orbit appearing for $\varepsilon < 0$ we consider the following control algorithm:

$$\dot{x}(t) = -y(t) + \varepsilon x(t) + x^3(t)/3 + w(t)f(x(t)) \tag{3a}$$
$$\dot{y}(t) = x(t) \tag{3b}$$
$$\dot{w}(t) = \lambda_c w(t) - k(x(t) - x(t-\tau))f(x(t)) \tag{3c}$$

The term $wf(x)$ in Eq. (3a) is the control signal perturbing the x-variable. The specific form of this coupling is given by the function $f(x)$ and will be specified later. Equation (3c) describes an unstable delayed feedback controller with $\lambda_c > 0$. Here $w(t)$ is the dynamical variable of the controller and k determines the feedback strength. Note that the control scheme does not change the solution of the free system corresponding to the unstable orbit of period τ, since for $x(t) = x(t-\tau)$ Eq. (3c) is satisfied by $w = 0$ and the control signal $w(t)f(x(t))$ in Eq. (3a) vanishes.

We just mention that in a recent paper [Pyragas, Pyragas and Benner, 2004] Eq. (3) has been considered as a paradigm of a subcritical Hopf bifurcation showing an unstable torsion-free limit cycle. The possibility to stabilize such an orbit was explored, both analytically and numerically, for the specific choice of a linear coupling function, $f(x) = x$, and successful control was achieved.

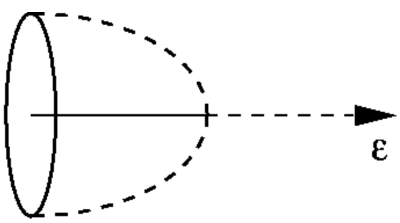

Figure 1. Bifurcation diagram of the unstable van der Pol oscillator. For $\varepsilon < 0$ the stable fixed point at the origin coexists with an UPO of amplitude $2\sqrt{-\varepsilon}$. For $\varepsilon > 0$ there is only an unstable fixed point at the origin.

4 Design of the experiment

In order to probe the concept of an unstable controller in experiment we designed an autonomous electronic circuit which is related to Eq. (3).

Fig. 2 shows the equivalent circuit scheme of the unstable van der Pol oscillator system with an unstable controller based on active elements. The upper part corresponds to the unstable van der Pol oscillator. The linear terms were implemented by operational amplifiers TL084 and the nonlinearities by AD633 multiplier ICs. The fundamental period T of the unstable van der Pol oscillator was set to 0.628 ms by trimming the time constants of the integrators, so that the resulting dynamics was easy to handle with our equipment. The lower part of Fig. 2 marked by the dashed frame shows the unstable control loop. It was designed from the same type of active components. The control amplitude k and the positive exponent λ_c as well as the bifurcation parameter ε of the unstable van der Pol oscillator were simply determined by the gain of the electronic amplifiers which were tuned by resistors.

For the time-delayed signal, we designed a digital delay system based on a combination of an analog-to-digital converter (ADC) of 8-bits resolution, a shift register ("FIFO") and a digital-to-analog converter (DAC). The accuracy of the delay time could be set to better than 1% of T by using a clock frequency of a few hundred kHz. The output signal of the delay system is smoothed by a low-pass filter with a bandwidth of about 10% of the clock frequency.

In our experiment the coupling function $f(x)$ of the

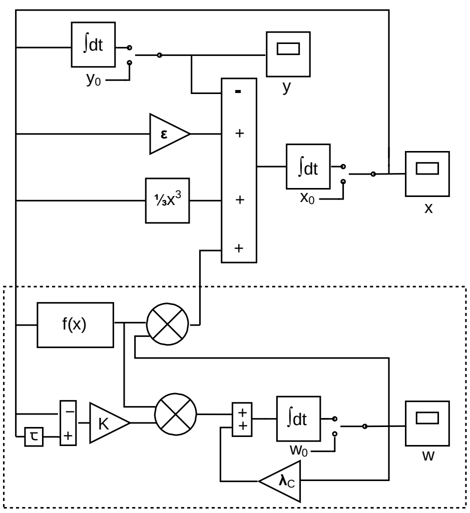

Figure 2. Block diagram of the van der Pol oscillator with unstable control loop (setup built from active components).

control algorithm Eq. (3) was chosen to be either a linear function of $x(t)$ or of sigmoidal type $f(x) = sign(x)$. In the linear case the coupling function was just given by the voltage representing the x-component. In the sigmoidal case the coupling function was experimentally realized by means of a comparator.

To ensure accurate initial conditions we used electronic switches parallel to each of the integrator outputs which generate the variables $x(t), y(t)$ and $w(t)$. These switches allowed to apply adjustable constant voltages x_0, y_0 and w_0, respectively, and a precise timing. Thus, when switching on the system at $t = 0$, the variables $x(t)$ and $y(t)$ started from a well-defined state. At about one cycle later the feedback loop generating the control signal was switched on, simultaneously with the controller variable $w(t)$. Such a time-lag was necessary to obtain an appropriate delayed signal reflecting the dynamics of the uncontrolled system close to the initial state. Note that the control generally failed when the feedback is switched on earlier than one cycle or later than three or four cycles. This is understandable since in the former case a proper delay signal has not yet developed while in the latter case the unstable system has already escaped too far away from the target state. The data were recorded by a digital oscilloscope with a sampling rate higher than 100 MHz and with file length larger than 2^{16} words.

This setup allowed to study in detail the basin of attraction of the controlled state. In the following subsection we describe control experiments for both linear and sigmoidal coupling.

5 Experimental results

5.1 Linear coupling

As pointed out in [Pyragas, Pyragas and Benner, 2004] for the most trivial choice of the coupling function, $f(x(t)) = const$ the system variable $x(t)$ and the unstable control loop variable $w(t)$ immediately decouple and the control fails. The next simplest coupling is of linear type. It was shown by both analytical and numerical investigations that such an unstable time-delayed feedback controller can stabilize the torsion-free UPO of the unstable van der Pol oscillator.

The first step was to confirm this result in experiment. We adjusted the parameters to $\varepsilon = -0.1$, $k = 0.3$, and $\lambda_c = 0.051$. Fig. 3(a) shows a time series of $x(t)$ and $w(t)$ for the initial conditions $(x_0, y_0, w_0) = (0.6, 0, 0)$, which is pretty close to the UPO. The feedback loop was switched on one period later in order to generate a proper delay signal $x(t-\tau)$. This switching defines time zero in Fig. 3. After a transient process of about 2500 time-steps a stable oscillation in x appears and the control variable $w(t)$ converges to zero. The UPO is stabilized, and the control force does not affect anymore the stabilized van der Pol oscillator. The situation changes dramatically for initial conditions far away from the UPO. In Fig. 3(b) we have chosen the initial conditions $(x_0, y_0, w_0) = (0.1, 0, 0)$. The $x(t)$-component immediately starts oscillating between the saturation limits of the active elements, and the control variable $w(t)$ diverges. This examples already indicates the importance of global properties in practical applications.

We probed systematically the dependence of control performance on the chosen initial conditions. We fixed w_0 at zero and varied both x_0 and y_0 from -1.0 V to 1.0 V in steps of 10 mV. The result is shown in Fig. 4. The black areas denote the basin of attraction for the stabilized periodic state. It consists of a narrow annulus surrounding the UPO which is indicated by the white line. For initial conditions outside the basin of attraction the time series shows a behavior as given in Fig. 3(b). Small basins of attraction make the UTDFC method unsuitable for practical applications.

5.2 Phase coupling

In order to extent the basins of attraction a different type of coupling was applied. A possible reason

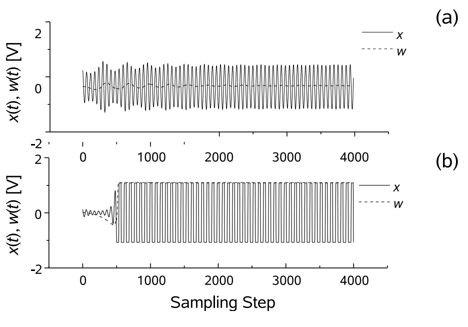

Figure 3. Linear coupling: Time series of the system variable $x(t)$ (line) and the control loop variable $w(t)$ (dashed) for fixed parameters $\varepsilon = -0.1$, $k = 0.3$ and $\lambda_c = 0.051$, but different initial conditions. (a) Initial conditions close to the UPO $(x_0, y_0, w_0) = (0.6, 0, 0)$: control successful. (b) Initial conditions close to the origin $(x_0, y_0, w_0) = (0.1, 0, 0)$: control fails.

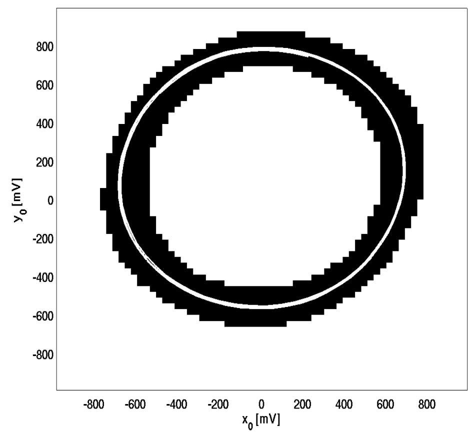

Figure 4. Experimental basins of attraction for linear coupling in black. Periodic orbit in white.

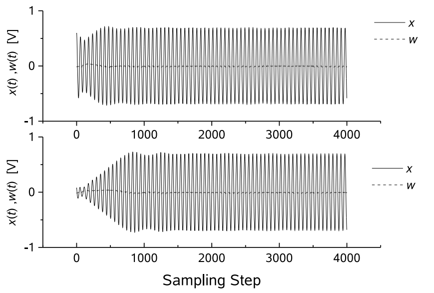

Figure 5. Phase coupling: Time series of the observed variable $x(t)$ (line) and the control loop variable $w(t)$ (dashed) for fixed parameters but different initial conditions. The parameters are $\varepsilon = -0.1, k = 0.10$ and $\lambda_c = 0.051$. (a) Initial conditions close to UPO $(x_0, y_0, w_0) = (0.6, 0, 0)$. (b) Initial conditions close to origin $(x_0, y_0, w_0) = (0.1, 0, 0)$

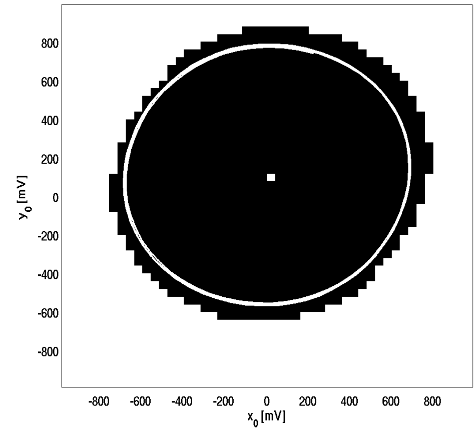

Figure 6. Experimental basins of attraction for phase coupling in black. Periodic orbit in white.

for failing control might have been the overshooting of the feedback signal when the control is switched on and high-amplitude transients occur. To suppress such undesired feedback oscillations the control signal has to be limited to a certain level. Obviously a sigmoidal control function could do this job much better. So we replaced the linear control function by a hyperbolic tangent, $f(x) = \tanh(\beta x)$ with $\beta \gg 1$, which was easily implemented in our electronic circuits by means of a operational amplifier acting as a comparator. Now the coupling function $f(x)$ just probes the sign of $x(t)$, which means that the coupling in eq.(3) appears through the phase of of the variable only, without any dependence on the amplitude.

We carry out the same experiment as in the case of linear coupling. The parameters were fixed to $\varepsilon = -0.1$, $k = 0.10$, and $\lambda_c = 0.051$. In Fig. 5 we have shown the time series for $x(t)$ and $w(t)$ for the same initial conditions as in Fig. 3, i.e. (a) close to the orbit $(x_0, y_0, w_0) = (0.6, 0, 0)$ and (b) far from the UPO $(x_0, y_0, w_0) = (0.1, 0, 0)$. The obvious difference to Fig. 3 is that for both conditions the UPO is stabilized and the controller variable vanishes. Comparing the time series of Fig. 5 it is evident that the transient time is much longer for initial condition far away from the UPO (b) than for initial conditions close to the UPO (a).

We found that phase coupling results in a dramatic increase of the basin of attraction , c.f. Fig. 6. Now, even close to the limiting bifurcation the basin of attraction includes the whole area inside the UPO (except for the very centre).

6 Conclusions

We have shown for the first time by experimental means that the concept of an unstable time-delayed feedback controller is able to overcome the odd number limitation. When applying a linear coupling $f(x) = x$ successful control was obtained only for a small range of initial conditions close to the target orbit. This experimental results are in good agreement with numerical results.

For many technical application it is difficult or even impossible to set initial conditions precisely. So the size of the basin of attraction of the target orbit is of similar importance as stability considerations. The phase coupling, which can technically be implemented in a rather simple way, is able to change the size of the basin of attraction dramatically. Apart from the origin the basin of attraction now covers the full area inside the orbit. With this type of coupling the unstable time-delayed feedback controller becomes suitable for various practical applications.

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