CONTROL PROBLEM OF A LINEAR NON-STATIONARY SYSTEM SUBJECT TO AN INTEGRAL CONSTRAINT

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Abstract

This paper investigates a nonlocal control problem of a linear non-stationary dynamic system described by an ordinary differential equation subject to an integral constraint imposed on the components of the state vector over a certain subinterval of the system's operating time. Conditions for complete controllability of the dynamic system under the imposed integral constraints are formulated. Conditions ensuring the existence of a solution to the control problem with such a nonlocal constraint are obtained. A constructive approach to solving the control problem is proposed, and the corresponding solution is constructed. The continuity and non-uniqueness of the control function are demonstrated. As an illustration, a control problem for the oscillatory motion of a material point with classical boundary conditions and an integral condition is solved.

Key words

dynamic system, integral condition, control, nonlocal control problem, complete controllability.

1 Introduction

Problems with nonlocal conditions, possessing significant theoretical and applied importance, constitute one of the actively developing directions in the theory of differential equations. The investigation of many applied control problems for dynamical processes leads to the necessity of considering control problems in which, along with classical boundary conditions (initial and terminal), nonlocal conditions are prescribed. It should be noted that control problems subject to nonlocal conditions arise, on the one hand, as mathematical models of real dynamical processes, and, on the other hand, from the impossibility of a correct formulation of problems with purely local conditions.

Nonlocal problems in the form of integral conditions arise in the mathematical modeling of phenomena of various natures, in particular when the boundary of the domain in which the process takes place is inaccessible to direct measurements [Lions, 1988]. In paper [Bazant and Jirsek, 2002] variety of nonlocal models of integral type are considered, and their physical justification, advantages, and numerical applications are discussed. Such conditions arise, for example, in the mathematical modeling of various processes in physics, biology, chemistry, ecology, and in many other phenomenons. In particular, problems with integral conditions appear in the modeling of heat propagation processes [Alekseeva and Yurchuk, 1998; Ionkin, 1977] in a thin heated rod; particle diffusion processes in turbulent plasma [Samarskii, 1980]; moisture transfer processes in porous media [Nakhushev, 1995]; biological problems related to the description of population dynamics [Nakhushev, 2012]; and certain technological processes [Muravei and Filinovskii, 1993]. In general, the presence of nonlocal conditions in applied problems is associated with the need to construct adequate mathematical models and to study the corresponding problems that meet the demands of modern natural science. Control problems for dynamic systems subject to nonlocal conditions have been studied both for ordinary differential equations, in particular [Barseghyan, 2025; Barseghyan and Barseghyan and Galyaev, 2026; Abdullaev, 2012; Baiburin, 2018; Amelin et al., 2022; Aleksandrov, 2023], and for partial differential equations [Pulkina, 2012; Avalishvili et al., 2011; Ilin and Moiseev, 2000]. A characteristic feature of such problems, alongside classical conditions, is the presence of constraints, for instance, relating the values of state variables at different intermediate points or over subintervals through integral conditions. Nonlocal control and optimal control problems for linear nonstationary dynamic system described by an ordinary differential equation with nonseparated multipoint conditions and a single integral condition imposed on the components of the state vector over a certain part of the system's time interval were studied in [Barseghyan, 2025]. It should be noted that the methods used to study such problems largely depend on the type of nonlocal condition. For example, in [Abdullaev, 2012] a numerical method for solving the system of linear differential equations with nonseparated multipoint and integral conditions is proposed. For a linear system of first-order differential equations subject to multipoint and integral conditions, necessary and sufficient conditions for unique solvability, the solvability criterion, and an explicit solution were established [Baiburin, 2018].

Significant applied control problems involving nonlocal conditions, specifically in the form of multipoint intermediate constraints, include systems described by partial differential equations, as considered in [Barseghyan and Solodusha, 2022; Barseghyan and Solodusha, 2020; Barseghyan et al., 2025], and systems described by ordinary differential equations, as examined in [Barseghyan et al., 2022; Nguyen et al., 2025].

It is worth noting that when an integral condition is imposed on specific subintervals of a dynamic system's operating period, control and optimal control problems with such nonlocal conditions are of considerable theoretical interest and may play an important role in accurately modeling corresponding applied problems. Nevertheless, these problems remain largely unexplored.

This paper addresses the control problem for a linear nonstationary dynamic system governed by an ordinary differential equation, subject to classical boundary conditions and an integral constraint on the components of the state vector over a specified subinterval of the system's operating period. A constructive approach to solving the control problem is proposed. Conditions ensuring complete controllability of a dynamic system under an imposed integral constraint are formulated. The criteria for the existence of a solution to the control problem with such a constraint are derived, and the corresponding solution is explicitly constructed. As an illustrative example, the control problem of a material point undergoing oscillatory motion under the influence of an external force, with classical boundary conditions and an integral constraint, is solved. Explicit expressions of the control function and phase trajectories are presented, accompanied by graphical representations of the phase coordinates.

2 Formulation of the Problem

Let us consider a controlled process whose dynamic is described by linear non-stationary differential equation

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + f(t), \quad t \in [t_0, T], \quad (1)$$

where $x(t) \in \mathbb{R}^n$, $A(t) - (n \times n)$, $B(t) - (n \times r)$ are matrices of appropriate dimensions, $t_0 \le t \le T$ (t_0 and T are given), $u(t) - (r \times 1)$, $(r \le n)$ is the control vector whose components are bounded functions that are continuous almost everywhere on the interval $[t_0, T]$, $f(t) - (n \times 1)$ is the vector of external disturbances.

Let the initial and terminal conditions of the system

$$x(t_0) = x_0, \quad x(T) = x_T \quad (2)$$

and fixed intermediate time instants $0 \le t_0 < t_1 < t_2 < T$ be known. We assume that the state vector $x(t)$ satisfies the following integral condition:

$$\int_{t_1}^{t_2} F(t)x(t)dt = \alpha, \quad (3)$$

where $F(t) - (q \times n)$ is a matrix of compatible dimension with elements that are bounded and almost everywhere continuous on the interval $[t_1, t_2]$, $\alpha - (q \times 1)$ is a column vector of compatible dimensions that are known real constants $(q \le n)$. In the introduction, the practical significance of condition (3) is provided. Here we just note that such constraints can characterize the limitation of resources and accumulation of memory in various control processes in problems of mechanics, energy, aviation, ets.

In the general case, it may be considered that over the time interval $[t_1, t_2]$, condition (3) may be assumed not on all components of the state vector $x(t)$, but only on a subset of them. In this case, the corresponding elements of the matrix $F(t)$ are taken to be zero.

We formulate the following problem.

Problem. It is required to find conditions under which there exist a control action $u(t), \ t \in [t_0, T]$ and a programmed trajectory $x = x(t)$ that transfer the system (1) from the initial state $x(t_0)$ to the terminal $x(T)$, while ensuring that the integral condition (3) is satisfied, and to construct them.

3 Solution to the Problem

We express the general solution of system (1) originating from the initial state:

$$x(t) = X[t, t_0]x(t_0) + \int_{t_0}^{t} X[t, \tau]B(\tau)u(\tau)d\tau + \\ + \int_{t_0}^{t} X[t, \tau]f(\tau)d\tau. \quad (4)$$

By substituting the state trajectory (4) into the integral condition (3), we obtain

$$\int_{t_1}^{t_2} F(t)x(t)dt = \int_{t_1}^{t_2} F(t)X[t, t_0]x_0 dt + \\ + \int_{t_1}^{t_2} \left(\int_{t_0}^{t} F(t)X[t, \tau]B(\tau)u(\tau)d\tau \right) dt + \\ + \int_{t_1}^{t_2} \left(\int_{t_0}^{t} F(t)X[t, \tau]f(\tau)d\tau \right) dt = \alpha,$$

or

$$\int_{t_1}^{t_2}\left(\int_{t_0}^{t} F(t)X[t,\tau]B(\tau)u(\tau)d\tau\right)dt = \alpha - \\ -\int_{t_1}^{t_2} F(t)X[t,t_0]x_0 dt - \\ -\int_{t_1}^{t_2}\left(\int_{t_0}^{t} F(t)X[t,\tau]f(\tau)d\tau\right)dt. \tag{5}$$

Let us formulate the following lemma.

Lemma. Let the matrix function $F(\xi)$ be almost everywhere continuous and bounded on the interval $[a,b]$, matrix functions $\Phi[\xi,\tau]$, $B(\xi)$ and vector $u(\xi)$ be almost everywhere continuous and bounded on the interval $[t_0,T]$, matrix $\Phi[\xi,\tau]$ possesses the composition property $\Phi[\xi,\tau] = \Phi[\xi,t_0]\Phi[t_0,\tau]$. Then, for any a and b, $0 \le t_0 < a < b < T$, the following equality holds:

$$\int_{a}^{b}\left(\int_{t_0}^{\xi} F(\xi)\Phi[\xi,\tau]B(\tau)u(\tau)d\tau\right)d\xi = \\ = \int_{t_0}^{a} P(a)\Phi[t_0,\tau]B(\tau)u(\tau)d\tau + \\ + \int_{a}^{b} P(\tau)\Phi[t_0,\tau]B(\tau)u(\tau)d\tau, \tag{6}$$

where

$$P(\tau) = \int_{\tau}^{b} F(\xi)\Phi[\xi,t_0]d\xi, \\ P(a) = \int_{a}^{b} F(\xi)\Phi[\xi,t_0]d\xi, \tag{7}$$

The proof of the lemma is based on Fubini's theorem (on the interchange of the order of integration) [Zorich, 2004], when changing the order of integration of

$$\int_{a}^{b}\left(\int_{t_0}^{\xi} F(\xi)\Phi[\xi,\tau]B(\tau)u(\tau)d\tau\right)d\xi$$

the value of an integral does not change.

Applying the lemma for double integral to the left-hand side of expressions (5), and noting that all conditions of the lemma are satisfied, we obtain

$$\int_{t_1}^{t_2}\left(\int_{t_0}^{t} F(t)X[t,\tau]B(\tau)u(\tau)d\tau\right)dt = \\ = \int_{t_0}^{t_1} P(t_1)X[t_0,\tau]B(\tau)u(\tau)d\tau + \\ + \int_{t_1}^{t_2} P(\tau)X[t_0,\tau]B(\tau)u(\tau)d\tau, \tag{8}$$

where

$$P(t_1) = \int_{t_1}^{t_2} F(\tau)X[\tau,t_0]d\tau, \\ P(t) = \int_{t}^{t_2} F(\tau)X[\tau,t_0]d\tau.$$

Substituting (8) into (5) we obtain the following integral relation:

$$\int_{t_0}^{t_1} P(t_1)X[t_0,\tau]B(\tau)u(\tau)d\tau + \\ + \int_{t_1}^{t_2} P(\tau)X[t_0,\tau]B(\tau)u(\tau)d\tau = \alpha - \\ -P(t_1)x_0 - \int_{t_1}^{t_2}\left(\int_{t_0}^{t} F(t)X[t,\tau]f(\tau)d\tau\right)dt. \tag{9}$$

We introduce the following notations:

$$H_1(t) = \begin{cases} P(t_1)X[t_0,t]B(t), & t \in [t_0,t_1], \\ 0, & t \in (t_1,T], \end{cases} \\ H_2(t) = \begin{cases} 0, & t \in [t_0,t_1), \\ P(t)X[t_0,t]B(t), & t \in [t_1,t_2], \\ 0, & t \in (t_2,T]. \end{cases} \tag{10}$$

In accordance with notation (10), relation (9) can be expressed as follows:

$$\int_{t_0}^{T} H_1(t)u(t)dt + \int_{t_0}^{T} H_2(t)u(t)dt = \alpha - \\ -P(t_1)x_0 - \int_{t_1}^{t_2}\left(\int_{t_0}^{t} F(t)X[t,\tau]f(\tau)d\tau\right)dt$$

or

$$\int_{t_0}^{T} \bar{H}(t)u(t)dt = \alpha - P(t_1)x_0 - \\ -\int_{t_1}^{t_2}\left(\int_{t_0}^{t} F(t)X[t,\tau]f(\tau)d\tau\right)dt, \tag{11}$$

where

$$\bar{H}(t) = \begin{cases} P(t_1)X[t_0,t]B(t), & t \in [t_0,t_1], \\ P(t)X[t_0,t]B(t), & t \in [t_1,t_2], \\ 0, & t \in [t_2,T]. \end{cases} \tag{12}$$

From formula (4), for $t = T$, we have:

$$x(T) = X[T,t_0]x(t_0) + \int_{t_0}^{T} X[T,\tau]B(\tau)u(\tau)d\tau + \\ + \int_{t_0}^{T} X[T,\tau]f(\tau)d\tau. \tag{13}$$

By moving all known terms in expression (13) to the left-hand side, we obtain:

$$\int_{t_0}^{T} X[T,\tau]B(\tau)u(\tau)d\tau = x_T - \\ -X[T,t_0]x(t_0) - \int_{t_0}^{T} X[T,\tau]f(\tau)d\tau. \tag{14}$$

We now define the following block matrices $H(t)$ and C as:

$$H(t) = \begin{pmatrix} \bar{H}(t) \\ X[T,t]B(t) \end{pmatrix},$$

$$C = \begin{pmatrix} \alpha - P(t_1)x_0 - \\ -\displaystyle\int_{t_1}^{t_2} \left(\int_{t_0}^{t} F(t)X[t,\tau]f(\tau)d\tau \right) dt \\ x_T - X[T,t_0]x_0 - \\ -\displaystyle\int_{t_0}^{T} X[T,\tau]f(\tau)d\tau \end{pmatrix}, \quad (15)$$

where matrix $H(t)$ and vector C have dimensions $((q+n) \times r)$ and $((q+n) \times 1)$.

Hence, using the notation introduced in (15), the integral conditions (11) and (14) are expressed as follows:

$$\int_{t_0}^{T} H(t)u(t)dt = C. \quad (16)$$

In a control problem, a principal issue is its solvability, which reduces to the analysis of the system's controllability. It follows from the integral relation (16) that system (1) with the integral condition (3) is completely controllable if, for any vector C (15) in the space $\mathbb{R}^{q+n}$, a control $u(t),\ t \in [t_0, T]$ can be found that satisfies condition (16).

Based on [Krasovskii, 1968; Zubov, 1975], a similar result concerning the complete controllability of system (1) under the integral condition (3) can be stated as the following theorem.

Theorem 1. For system (1) subject to an integral condition (3) to be completely controllable on the interval $[t_0, T]$, it is necessary and sufficient that the column vectors of the matrix $H(t)$ (16) be linearly independent on this interval.

The control action can be constructed in the form

$$u(t) = H^T(t)Q^{-1}C + v(t), \quad (17)$$

where

$$Q = \int_{t_0}^{T} H(t)H^T(t)dt$$

is a matrix of dimension $((q+n) \times (q+n))$, and a vector $v(t)$ of dimension $(r \times 1)$ satisfy the condition:

$$\int_{t_0}^{T} H(t)v(t)dt = 0. \quad (18)$$

Here and henceforth, the superscript "T" denotes the transpose operation.

Thus, the conditions required for the existence of a solution to the problem can be formulated as the following theorem, similarly to the theorem established in [Zubov, 1975].

Theorem 2. For the existence of a programmed control $u(t),\ t \in [t_0, T]$ and the corresponding solution of system (1) satisfying conditions (2) and (3), it is necessary and sufficient that the matrix Q be non-singular (i.e. $\det Q \neq 0$) or that the ranks of the matrix Q and the augmented matrix $\{Q, C\}$ coincide.

Note that, according to Theorem 2, in case of $\det Q \neq 0$, the control function $u(t),\ t \in [t_0, T]$, that solves the problem (1) - (3) is given by (17).

Taking into account the notations (10) and (12), the control action is represented in the following form:

$$u(t) = \begin{cases} u_1(t) + v(t), & t \in [t_0, t_1], \\ u_2(t) + v(t), & t \in [t_1, t_2], \\ u_3(t) + v(t), & t \in [t_2, T], \end{cases} \quad (19)$$

where

$$u_1(t) = B^T(t)((P(t_1)X[t_0,t])^T, X^T[T,t])Q^{-1}C,$$
$$u_2(t) = B^T(t)((P(t)X[t_0,t])^T, X^T[T,t])Q^{-1}C,$$
$$u_3(t) = B^T(t)(0, X^T[T,t])Q^{-1}C.$$

It is worth noting that the constructed control action $u(t),\ t \in [t_0, T]$ is not unique (due to the arbitrariness of vector $v(t)$) and a continuous function. Direct verification shows that $u_1(t_1) = u_2(t_1)$. Similarly, $u_2(t_2) = u_3(t_2)$. For instance, it can be showed that $u_1(t_1) = u_2(t_1)$.

For $t \in [t_0, t_1]$

$$u_1(t) = B^T(t)((P(t_1)X[t_0,t])^T, X^T[T,t])Q^{-1}C + v(t),$$

therefore

$$u_1(t_1) = B^T(t_1)((P(t_1)X[t_0,t_1])^T, X^T[T,t_1])Q^{-1}C + \\ + v(t_1),$$

for $t \in [t_1, t_2]$

$$u_2(t) = B^T(t)((P(t)X[t_0,t])^T, X^T[T,t])Q^{-1}C + v(t),$$

therefore

$$u_2(t_1) = B^T(t_1)((P(t_1)X[t_0,t_1])^T, X^T[T,t_1])Q^{-1}C + \\ + v(t_1),$$

Thus, it is obvious that the values of control function at the moment of joining the time intervals are the same. Continuity at point $t = t_2$ can be demonstrated in a similar manner (note that the value of $P(t)$ at the point t_2 is zero).

With the control function (19), the phase trajectories (system realizations) are constructed for each time interval of the system's functioning when $v(t) = 0$:

for $t \in [t_0, t_1]$

$$x(t) = X[t,t_0]x(t_0) + \int_{t_0}^{t} X[t,\tau]B(\tau)u_1(\tau)d\tau + \\ + \int_{t_0}^{t} X[t,\tau]f(\tau)d\tau,$$

for $t \in [t_1, t_2]$

$$x(t) = X[t,t_0]x(t_0) + \int_{t_0}^{t_1} X[t,\tau]B(\tau)u_1(\tau)d\tau + \\ + \int_{t_1}^{t} X[t,\tau]B(\tau)u_2(\tau)d\tau + \int_{t_0}^{t} X[t,\tau]f(\tau)d\tau,$$

for $t \in [t_2, T]$

$$x(t) = X[t,t_0]x(t_0) + \int_{t_0}^{t_1} X[t,\tau]B(\tau)u_1(\tau)d\tau + \\ + \int_{t_1}^{t_2} X[t,\tau]B(\tau)u_2(\tau)d\tau + \int_{t_2}^{t} X[t,\tau]B(\tau)u_3(\tau)d\tau + \\ + \int_{t_0}^{t} X[t,\tau]f(\tau)d\tau.$$

Note that the motions $x(t)$ are continuous functions on the time interval $[t_0, T]$.

4 Particular case

For system (1) with the initial and terminal conditions (2) we examine the case when $t_1 = t_0$, $t_2 = T$. At this point, the nonlocal integral condition will be as follows:

$$\int_{t_0}^{T} F(t)x(t)dt = \alpha. \tag{20}$$

Then the integral relation (5) can be written in the form:

$$\int_{t_0}^{T} \left(\int_{t_0}^{t} F(t)X[t,\tau]B(\tau)u(\tau)d\tau \right) dt = \alpha - \\ - \int_{t_0}^{T} F(t)X[t,t_0]x_0 dt - \\ - \int_{t_0}^{T} \left(\int_{t_0}^{t} F(t)X[t,\tau]f(\tau)d\tau \right) dt. \tag{21}$$

Since the conditions of the lemma are fulfilled, the integral relation (21) takes the following form:

$$\int_{t_0}^{t_0} P(t_0)X[t_0,\tau]B(\tau)u(\tau)d\tau + \\ + \int_{t_0}^{T} P(\tau)X[t_0,\tau]B(\tau)u(\tau)d\tau = \alpha - \\ - P(t_0)x_0 - \int_{t_0}^{T} \left(\int_{t_0}^{t} F(t)X[t,\tau]f(\tau)d\tau \right) dt,$$

or

$$\int_{t_0}^{T} P(\tau)X[t_0,\tau]B(\tau)u(\tau)d\tau = \alpha - P(t_0)x_0 - \\ - \int_{t_0}^{T} \left(\int_{t_0}^{t} F(t)X[t,\tau]f(\tau)d\tau \right) dt, \tag{22}$$

where

$$P(t) = \int_{t}^{t_2} F(\tau)X[\tau,t_0]d\tau.$$

Block matrix $H(t)$ and block vector C, according to (15), obtain the following forms:

$$H(t) = \begin{pmatrix} P(t)X[t_0,t]B(t) \\ X[T,t]B(t) \end{pmatrix},$$

$$C = \begin{pmatrix} \alpha - P(t_0)x_0 - \\ - \int_{t_0}^{T} \left(\int_{t_0}^{t} F(t)X[t,\tau]f(\tau)d\tau \right) dt \\ x_T - X[T,t_0]x_0 - \\ - \int_{t_0}^{T} X[T,\tau]f(\tau)d\tau \end{pmatrix}. \tag{23}$$

Consequently, the control action $u(t),\ t \in [t_0, T]$ is expressed in the following form, when $v(t) = 0$:

$$u(t) = B^T(t)((P(t)X[t_0,t])^T, X^T[T,t])Q^{-1}C. \tag{24}$$

Thus, the considered problem, subject to the integral condition (20), represents a particular case of the problem with condition (3).

5 Example

Let us consider the oscillatory motion of a material point in the presence of an external force. In this case, for small oscillations, the differential equation of the material point's motion can be written in the following form:

$$\ddot{x} + k^2 x = u, \tag{25}$$

where x is the state of the point, u is the control action.

Let us consider the case $k^2 = 1$ and introduce the notations $x_1 = x,\ x_2 = \dot{x}$. In this form, equation (25) can be represented as a system of first-order differential equations:

$$\begin{cases} \dot{x}_1(t) = x_2(t), \\ \dot{x}_2(t) = -x_1(t) + u(t), \end{cases} \quad t \in [t_0, T]. \tag{26}$$

Let

$$t_0 = 0,\ t_1 = \frac{\pi}{2},\ t_2 = 2\pi, T = 6\pi, \\ F = (0,1),\ \alpha = 32,$$

and an integral constraint be given

$$\int_{\frac{\pi}{2}}^{2\pi} x_2(t)dt = 32. \quad (27)$$

In this case, the integral condition (27) constrains the mean velocity of the material point's oscillatory motion over the interval $\left[\frac{\pi}{2}, 2\pi\right]$ [Blekhman, 2023].

The initial and terminal states of the system (26) are also given:

$$\begin{aligned} x(t_0) = x(0) &= \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 20 \\ 3 \end{pmatrix}, \\ x(T) = x(6\pi) &= \begin{pmatrix} x_1(6\pi) \\ x_2(6\pi) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned} \quad (28)$$

It is required to find a control action $u(t),\ t \in [0, 6\pi]$ under the influence of which the oscillatory motion of a material point transfers from the initial state $x(0)$ to the terminal state $x(6\pi)$, while satisfying the integral constraint (27).

To apply the obtained result, we verify that the matrix $Q = \int_0^{6\pi} H(t)H^T(t)dt$ is non-singular.

The fundamental matrix corresponding to the homogeneous part of the system (26) is given by:

$$X[t, t_0] = \begin{pmatrix} \cos(t - t_0) & \sin(t - t_0) \\ -\sin(t - t_0) & \cos(t - t_0) \end{pmatrix}. \quad (29)$$

Function $H(t)$ has the form,

$$H(t) = \begin{cases} H^{(1)}(t), & t \in [0, \frac{\pi}{2}], \\ H^{(2)}(t), & t \in [\frac{\pi}{2}, 2\pi], \\ H^{(3)}(t), & t \in [2\pi, 6\pi], \end{cases} \quad (30)$$

where

$$\begin{aligned} H^{(1)}(t) &= (-\cos t - \sin t \ \ -\sin t \ \ \cos t); \\ H^{(2)}(t) &= (\sin t \ \ -\sin t \ \ \cos t); \\ H^{(3)}(t) &= (0 \ \ -\sin t \ \ \cos t). \end{aligned}$$

Thus, the matrix Q has the form

$$Q = \begin{pmatrix} \frac{5\pi}{4} + 1 & \pi + \frac{1}{2} & -\frac{\pi}{4} \\ \pi + \frac{1}{2} & 3\pi & 0 \\ -\frac{\pi}{4} & 0 & 3\pi \end{pmatrix},$$

$$\det Q = \frac{129\pi^3}{16} + 6\pi^2 - \frac{3\pi}{4} \approx 306.85.$$

Then the control function, according to (19), has the following form, where $v(t) = 0$:

$$u(t) = \begin{cases} u_1(t), & t \in [0, \frac{\pi}{2}], \\ u_2(t), & t \in [\frac{\pi}{2}, 2\pi], \\ u_3(t), & t \in [2\pi, 6\pi], \end{cases}$$

where

$$\begin{aligned} u_1(t) = b[&4(739\pi^2 + 134\pi - 3)\cos t + \\ &+2\pi(598\pi - 417)\sin t], \end{aligned}$$

$$\begin{aligned} u_2(t) = b[&2(64\pi^2 - 28\pi + 6)\cos t + \\ &+2\pi(417 - 598\pi)\sin t], \end{aligned}$$

$$\begin{aligned} u_3(t) = b[&2(64\pi^2 - 28\pi + 6)\cos t + \\ &+2\pi(657 + 944\pi)\sin t], \end{aligned}$$

$$b = [3\pi(43\pi^2 + 32\pi - 4)]^{-1}.$$

Hence, the laws of motion $x_1(t)$ and $x_2(t)$ for each time interval of the system's functioning can be represented as follows:
for $t \in [0, \frac{\pi}{2}]$,

$$\begin{aligned} x_1(t) &= 20\cos t + 3\sin t + b[(598\pi^2 - 417\pi)t\cos t + \\ &+(6t - (1478\pi + 268)\pi t - 417\pi + 598\pi^2)\sin t], \\ x_2(t) &= 3\cos t - 20\sin t + b[(6 - (498\pi - 417)\pi t + \\ &+268\pi + 1478\pi^2)\sin t - 2(739\pi^2 + 134\pi - 3)t\cos t], \end{aligned}$$

for $t \in [\frac{\pi}{2}, 2\pi]$,

$$\begin{aligned} x_1(t) &= b[(2580\pi^3 + 3462\pi^2 + (598\pi - 417)\pi t)\cos t + \\ &+((64\pi^2 - 28\pi + 6)t - 384\pi^3 + 381\pi - 430\pi^2)\sin t], \\ x_2(t) &= b[2(64\pi^2 - 28\pi + 6)(6\pi t)\cos t + ((598\pi - \\ &-417)\pi t + 2580\pi^3 + 3398\pi^2 + 28\pi - 6)\sin t], \end{aligned}$$

for $t \in [2\pi, 6\pi]$,

$$\begin{aligned} x_1(t) &= b[\pi(657 + 944\pi)(6\pi - t)\cos t + ((64\pi^2 - 28\pi + \\ &+6)t - 384\pi^3 + 1112\pi^2 + 621\pi)\sin t], \\ x_2(t) &= b[((944\pi + 657)\pi t - 5664\pi^3 + 3878\pi^2 - 28\pi + \\ &+6)\sin t - 2(32\pi^2 - 14\pi + 3)(6\pi - t)\cos t]. \end{aligned}$$

From the above-derived laws of motion, one can verify that conditions (27) and (28) are satisfied. Indeed, by substituting $x_2(t)$ into (27) and integrating this function, we confirm that condition (27) is fulfilled.

Now, for $t \in [2\pi, 6\pi]$, by calculating $x_1(6\pi)$ and $x_2(6\pi)$, one can confirm that condition (28) is likewise fulfilled.

In this way, employing the constructive approach outlined above, explicit formulas for the programmed control and phase motion are obtained for the problem of controlling the oscillatory motion of a material point under an external force, with prescribed initial and terminal states and an integral constraint.

Graphical representations of functions $x_1(t)$ and $x_2(t)$ are the following figures:

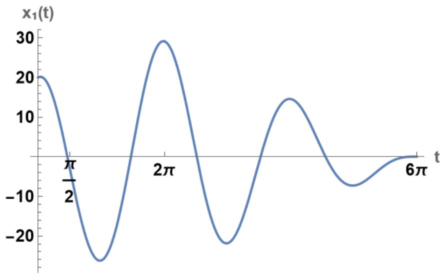

Figure 1. The plot of function $x_1 = x_1(t)$

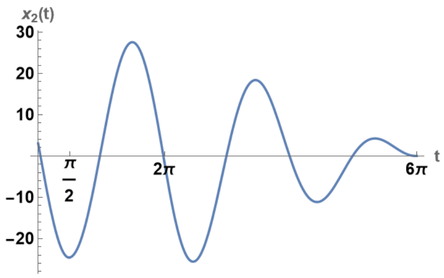

Figure 2. The plot of function $x_2 = x_2(t)$

6 Conclusion

A constructive approach for solving control problems of linear time-varying dynamic systems subject to an integral constraint is derived. Conditions for the complete controllability of a system under such nonlocal constraints are formulated. The existence conditions for a control solution under such nonlocal constraints are derived, and the solution is explicitly constructed. The control problem for the oscillatory motion of a material point, subject to classical boundary conditions and an integral constraint, is solved.

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