Delayed Feedback Control of Chaos in Economic Models Based On quasi-sliding mode

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ABSTRACT

In this paper the method of quasi-sliding mode control is extended via delayed feedback for stabilizing unstable fixed points of uncertain economic systems which exhibit chaotic behavior. This method, in contrast to quasi-sliding method proposed for such systems, doesn't require the position of unstable fixed points. The control scheme is presented and the performance of the proposed approach is examined by applying it to two chaotic economic models, the Behrens Feichtinger and the Cournot duopoly model with complements goods. Simulation results show the effectiveness and feasibility of the method for chaos control in uncertain chaotic models.

KEY WORDS

Delayed Feedback Control, Chaos Control, Quasi-sliding mode.

1.INTRODUCTION

So many phenomena take place in real world which can no longer be illustrated with the aids of simple models. Economical systems are one of the kinds. It is unanimously accepted that economy belongs to very complex systems which usually take the form of being either stochastic or deterministic. It is due to the fact that the linear models could no longer anticipate the ongoing events of the system; hence the economists started developing complex systems for macroeconomic and microeconomic theories.

Goodwin probably was the first economist to realize the importance of nonlinear mechanism for the economical systems [1] as one of his works; Goodwin enhanced and reduced the deficiencies of the Keynesian fiscal policies. Up to now, many researchers spending time on the matter, for example Majumdar, Mitra and Nishimura [2] for overview of nonlinear dynamic theory in economics, K. Ishiyama and Y. Saiki [3] for their quantitative analysis of Keynes-Goodwin macroeconomic growth model, M. Szydlowski, A. Krawiec [4] for investigating the stability problem of the Kaldor-Kalecki business cycle and so on.

The economical systems due to their complex dynamics usually evince chaotic manners. Researches on chaotic behavior and its effect on economical systems have attracted a great deal of research interests in recent years. For the first time, Strotz et al. [5] states that chaotic dynamics can exist in an economics model. Also, some researches show that even oligopolistic markets, which first proposed by Cournot in 1838, may exhibit chaotic behavior under certain condition[6, 7]. In another work [8], the authors show that . A simple Cournot dynamic model with nonlinear reaction functions may give rise to chaotic dynamic. It states that when the nonlinearity gets strong, the model shows chaotic fluctuations.

Chaotic behavior in an economic system is often an undesirable phenomenon which prevents the prediction in long time period, and may threaten the safety of investment. In [6], it has been shown that the performance of a chaotic economics can be improved by controlling chaos. Chaotic systems have infinitely many fixed points or periodic solutions in their chaotic attractors [9]. Stabilizing the periodic solutions of a chaotic market model may increase economic efficiency [10].

Since 1990, various control algorithms for chaos control by stabilizing the periodic orbits or fixed points of nonlinear dynamic systems have been proposed. Ott, Grebogi and Yorke [11] presented a perturbing method (OGY) for chaos control by linearizing the nonlinear map of the system. The OGY method was successfully applied for chaos control in some economic systems, e.g. see [12]. Also Pyragas [13] presented a method for chaos control by using a delayed feedback signal. Stabilizing the first order fixed point of the Cournot duopoly via delayed feedback control has been investigated in [14]. Other nonlinear control methods such as feedback linearization [15], continuous time sliding mode [16] and adaptive Lyapunov based control [17] have been greatly used for chaos suppression in numerous physical systems. Many models of economic systems are illustrated though nonlinear discrete maps [18]. If the governing equations of such systems are exactly known, then one can easily used the OGY or the Pyragas method to design stabilizing controllers for them. However due to uncertainties, determining an exact governing equation for a system is not possible, and the parameters of a system always have some uncertainties. So introducing a robust control strategy for chaos control in such systems is necessary. One of the famous nonlinear methods used as a robust control is the sliding mode. Sliding mode control initially developed for continuous time systems [19].

Due to some technical difficulties it cannot be used directly for discrete dynamical systems generated by nonlinear maps. In [20, 21] some concepts for adapting the sliding mode method to use in discrete time systems was presented. The modified method is called the quasi sliding mode control which is used to control chaos in [22, 23]. But in [22, 23], the desired input which is the fixed point should be provided. Computing the fixed point in system in uncertain models couldn't be calculated. In this paper it is shown that how combining the delayed feedback proposed by Pyragas [13] and the quasi-sliding mode method can be used for stabilizing the fixed points of chaotic systems without knowing the position of fixed point. Then the proposed method is applied for chaos elimination for two economic models. The main advantage of the new strategy is its ability to control chaos when the actual parameters of market are not available or have some uncertainties.

2. CHAOS CONTROL USING DELAYED FEEDBACK QUASI-SLIDING MODE CONTROL

In this section, first a Quasi-sliding mode control scheme presented in [23] is discussed. Then, by extending it the Delayed Feedback Quasi-Sliding Mode Control is introduced, which is later used for stabilization of unstable fixed points of chaotic systems.

2.1. QUASI-SLIDING MODE CONTROL

In this subsection, the concept of sliding mode control in discrete systems, known as the quasi-sliding mode, is presented based on the scheme introduced in [20].

Consider the system which can be viewed as number of discrete nonlinear single-input single-output (SISO) systems (For every output there is input):

$$x_{k+1} = f_k\left(x_k, \dots, x_{k-n}, u_{k-1}, \dots, u_{k-b}\right) + g_k\left(x_k, \dots, x_{k-c}, u_{k-1}, \dots, u_{k-d}\right)u_k \qquad (1)$$

Where x_{k+1} is the output, u_k is the control input, $f_k(.)$ and $g_k(.)$ are smooth nonlinear functions of past values of the input and output, and the constants *n*, *b*, *c*, and *d* are all positive integers.

For this system, it is assumed that the function $g_k(.)$ is bounded away from zero. Let us assume that a model for (1) exists, in the form of

$$\hat{x}_{k+1} = \hat{f}_k\left(x_k, \dots, x_{k-n}, u_{k-1}, \dots, u_{k-b}\right) + \hat{g}_k\left(x_k, \dots, x_{k-c}, u_{k-1}, \dots, u_{k-d}\right)u_k \qquad (2)$$

where $\hat{f}_k(.)$ and $\hat{g}_k(.)$ are estimated functions of $f_k(.)$ and $g_k(.)$, respectively. The switching function is defined as a linear combination of the past tracking errors as:

$$S_k = e_k + \alpha_1 e_{k-1} + \dots + \alpha_p e_{k-p} \qquad (3)$$

where *p* is an integer which can be chosen same as order of system and the coefficients $\alpha_1, \dots, \alpha_p$ in (3) are selected to make the switching function a stable linear combination of the past tracking errors. In other words, all of the roots of the polynomial $z^p + \alpha_1 z^{p-1} + \cdots + \alpha_p$ should lie inside the unit circle. This way, after a finite period of time, the tracking error will converge towards zero.

The desired input is x_k^d, so the tracking error can be defined as:

$$e_k = x_k - x_k^d \qquad (4)$$

In contrary to the continuous systems, “sliding mode” cannot be achieved for discrete systems and hence we have to try reaching a quasi-sliding mode. Some concepts regarding this matter can be found in [20]. A system is said to be in a quasi-sliding mode when the dynamics of S_k meets the following set of conditions:

I) Starting from any state, the S_k sequence moves toward the quasi-sliding surface, defined by $S_k=0$, and crosses it in a finite period of time.

II) Once the surface is crossed by the first time, the S_k value changes around the surface in a zigzag way.

III) The zigzag motion is stable and stays inside a fixed band.

Conditions I, II, and III can be mathematically expressed as:

$$S_k\left(S_{k+1} - S_k\right) < 0 \qquad (5)$$

and

$$if \ \operatorname{sgn}(S_{k+1}) = -\operatorname{sgn}(S_k) \Rightarrow \begin{cases} \operatorname{sgn}(S_{k+2}) = \operatorname{sgn}(S_k) \\ |S_k| < \xi \end{cases} \qquad (6)$$

where ξ is the fixed bound mentioned in the third condition.

Let us consider that, in closed loop, the switching function will exhibit the following behavior [20]:

$$S_{k+1} = \left(x_{k+1} - \hat{x}_{k+1}\right) - \varepsilon.\operatorname{sgn}\left(S_k\right), \ \varepsilon > 0 \qquad (7)$$

Starting from (7) and using (1), and (3) one can determine that the input signal must be:

$$u_k = \frac{-1}{\hat{g}_k}\left(\left(\hat{f}_k - x_{k+1}^d\right) + \alpha_1 e_k + \dots + \alpha_p e_{k-p+1} + \varepsilon \operatorname{sgn}\left(S_k\right)\right) \qquad (8)$$

We can now readily check if such behavior for the switching function can guarantee the conditions in (5) and (6). Inserting (7) right into (5) we get:

$$\begin{aligned} S_k\left(S_{k+1} - S_k\right) &= S_k\left(\left(x_{k+1} - \hat{x}_{k+1}\right) - \varepsilon.\operatorname{sgn}\left(S_k\right) - S_k\right) \\ &\le \left|x_{k+1} - \hat{x}_{k+1}\right|\left|S_k\right| - \varepsilon\left|S_k\right| \end{aligned} \qquad (9)$$

Hence, it is sufficient to take

$$\varepsilon = \eta H \qquad (10)$$

where η is an arbitrary constant larger than unity and H is an upper bound for the modeling error,

$$H > \left|x_{k+1} - \hat{x}_{k+1}\right| \quad , \quad \forall k \qquad (11)$$

The first condition in (6) is also easy to check. The expression for S_{k+2} can be directly derived from (7).

$$S_{k+2} = \left(x_{k+2} - \hat{x}_{k+2}\right) - \varepsilon.\operatorname{sgn}\left(S_{k+1}\right) \qquad (12)$$

Considering that S_k has just crossed the sliding surface, $\operatorname{sgn}(S_{k+1}) = -\operatorname{sgn}(S_k)$, thus:

$$S_{k+2} = \left(x_{k+2} - \hat{x}_{k+2}\right) + \varepsilon.\operatorname{sgn}\left(S_k\right) \qquad (13)$$

Taking into account the conditions in (10) and (11), the first condition in (6) is guaranteed. The second requirement in (6) needs the consideration of the closed loop system in which one gets the following representation for the closed-loop dynamics of the switching surface:

$$\begin{aligned} S_{k+1} = &\left(1 - \frac{g_k}{\hat{g}_k}\right)\alpha_1 S_k + \left(f_k - \frac{g_k}{\hat{g}_k}\hat{f}_k\right) \\ &+ \left(1 - \frac{g_k}{\hat{g}_k}\right)\vartheta - \left(1 - \frac{g_k}{\hat{g}_k}\right)x_{k+1}^d - \varepsilon\frac{g_k}{\hat{g}_k}\operatorname{sgn}\left(S_k\right) \end{aligned} \qquad (14)$$

where,

$$\begin{aligned} \vartheta = &(\alpha_2 - \alpha_1^2)e_{k-1} + \cdots \\ &+ (\alpha_p - \alpha_1\alpha_{p-1})e_{k-p+1} - \alpha_1\alpha_p e_{k-p} \end{aligned} \qquad (15)$$

The only recurrent terms in S_k are the first and the last terms on the right hand side. The latter is a bounded function of S_k, so considering that $\left|\alpha_1\right| < 1$, the switching function will remain stable only if :

$$\left|1 - \frac{g_k}{\hat{g}_k}\right| < 1 \qquad (16)$$

Remark: Although the stability of the proposed control technique has been proved theoretically, there are some technical problems such as strong chattering in implementation of control law due to use of sign function in Eq. (8). To overcome this problem one can use the saturation function instead of sign function:

$$sat\left(S/\phi\right)=\begin{cases} S/\phi & \left|S/\phi\right|<1 \\ sign\left(S/\phi\right) & otherwise \end{cases} \tag{17}$$

where ϕ is a positive small number. In this case, some steady state error is generated which implies that there exists some error between the stabilized trajectory and the actual one. By decreasing ϕ to zero, the mentioned error will converge to zero.

2.2. DELAYED FEEDBACK QUASI-SLIDING MODE CONTROL

In this subsection, the concept of delayed feedback used to stabilize unstable fixed points is merged with sliding mode control. In order to reach a stable first order fixed point in discrete systems, following conditions should be satisfied.

$$x_k = x_{k-1} \quad , \quad u_k = 0 \tag{18}$$

So, the error could be defined,

$$e_k = x_k - x_{k-1} \tag{19}$$

But if this error is used in the quasi-sliding mode controller, it makes the system to repeat the last steps but with non-zero control signals. In order to overcome this problem, we redefine the sliding surface as,

$$S_k = e_k + \alpha_1 e_{k-1} + \ldots + \alpha_p e_{k-p} + \hat{g}_{k-1}\left(\frac{1}{c_{k-1}} - 1\right)u_{k-1} \tag{20}$$

where c_k is one at first stages, which results in normal sliding surface, but c_k will change adaptively in time, so the effect of control signal also enters the sliding surface and the errors and control signals in the closed loop system simultaneously converges to zero.

Starting from (7) and using (1), and (20) one can determine that the input signal must be:

$$u_k = \frac{-c_k}{\hat{g}_k}\left(\left(\hat{f}_k - x_{k-1}\right) + \alpha_1 e_k + \ldots + \alpha_p e_{k-p+1} + \varepsilon\,\mathrm{sgn}\left(S_k\right)\right) \tag{21}$$

It can be shown similar to quasi sliding mode, this scheme is stable. An adaptive law which results in convergence to the unstable fixed point in chaotic system is found to be:

$$c_{k-1} = \begin{cases} c_{k-2} - \gamma u_{k-1} & if \quad \left|S_k\right| < \delta \\ 1 & if \quad \left|S_k\right| > \delta \end{cases} \quad , \; 0 < \gamma < 1 \; , c_0 = 1 \tag{22}$$

3. CHAOS CONTROL OF COURNOT MODEL WITH COMPLEMENTARY GOODS MODEL

The dynamical system model of Cournot Model with Complementary Goods can be obtained based on the model suggested in [8]. This model explains the ongoing dynamical behavior between two-market economy with complementary goods *x* and *y*. We are interested in the dynamic interactions between the two firms which can be described by

$$\begin{aligned} x_{k+1} &= (\alpha y_k - \alpha + 1)^2 \\ y_{k+1} &= (\beta x_k - 1)^2 \end{aligned} \tag{23}$$

where α, β are coefficients of system. To have economically meaningful system, the domain of parameters should be restricted [8]. Thus, when the parameters of the system is restricted to

$$A \equiv \left\{(\alpha,\beta) \mid 0 \le \alpha \le 2 \text{ and } 0 \le \beta \le 2\right\} \tag{24}$$

This parameter restriction prevents trajectory divergence in the system. Also it is known that, under certain conditions for the values of α and β, the above described system exhibits chaotic behavior. The aforementioned economic system in Eq. (23) exhibits chaotic behavior with the following parameters [8]:

$$\beta = 1.75, \alpha = 1.1 \tag{25}$$

Using mathematical and simulation, authors in [8] showed that for some values of *α* and *β*, the long-run average profit of the system is higher in chaotic region than its stationary points. For example, they show that when the value of *α* is fixed to 1.1, the long-run average profit of first firm is higher than the corresponding stationary profit when the value of *β* is larger than about 1.7, while the long-run average profit of second firm is less than its corresponding stationary point. This will continue until the value of *β* reaches 1.8, where the long-run average of firm 2 is become higher than its stationary point.

Form such example, one may see that the profit of the biggest firm (second firm) is less than its corresponding stationary profit in a bound of *β* = [1.7, 1.8], while the first firm is exceeding its stationary profit. Thus, for second firm to hold its position in the market, it should apply a control force to compensate such “Relative Loss”. Moreover, in a bound of *β* = [1.7, 1.8] both firms is in the chaos region and their long-run profit is not a “profitable” situation for both.

All this, will convince a manager to apply a private control strategy to this system in order to “re-maximize” the profit when needed. By considering some uncertainties in the model parameters, the method proposed is applied to the system.

In the previous section we derived the a nonlinear control techniques for suppressing chaotic dynamics in the system by adding a feedback control forcing signal. So, we chose to add control signal to the second state of the Eq. (23). The new governing equation including the form of control force, u, can be written as:

$$\begin{cases} x_{k+1} = (\alpha y_k - \alpha + 1)^2 \\ y_{k+1} = (\beta x_k - 1)^2 + u_k \end{cases} \tag{26}$$

By substituting the second relation of Eq. (26) in the first one, the SISO format of the equation is obtained as,

$$y_{k+1} = (\beta(\alpha y_{k-1} - \alpha + 1)^2 - 1)^2 + u_k \tag{27}$$

This input force can be assumed as the compensated policy of the second firm to control over the chaos in the market. The second firm inaccuracy of the other firms’ production prediction led to unstable patterns. This may alter the profit maximization process of the firm which is described by the nonlinear model. Thus, a feedback force should be added in order to stabilize the system whenever needed.

Comparing Eq. (26) with Eq. (1) one may obtain,

$$f_k = (\beta(\alpha y_{k-1} - \alpha + 1)^2 - 1)^2 \, , g_k = 1 \tag{28}$$

Assume that the exact value of α and β are unknown, their estimated value are denoted by $\hat{\alpha}$ and $\hat{\beta}$. One may set the following relations to show the bounds of uncertainties:

$$\left|\alpha-\hat{\alpha}\right|<\varepsilon_{\alpha} \quad , \quad \left|\beta-\hat{\beta}\right|<\varepsilon_{\beta} \tag{29}$$

where ε_{α} and ε_{β} are the upper limits of the uncertainties. Defining the tracking error as,

$$e_k = y_k - y_{k-1} \tag{30}$$

one may obtain the sliding surface as,

$$S_k = e_k + \alpha_1 e_{k-1} + \hat{g}_{k-1}\left(\frac{1}{c_{k-1}}-1\right)u_{k-1} \tag{31}$$

Considering Eq. (11) one may write,

$$\begin{aligned} H &> \left|f_{k+1}-\hat{f}_{k+1}\right| \\ &= \left|(\beta(\alpha y_{k-1}-\alpha+1)^2-1)^2-(\hat{\beta}(\hat{\alpha} y_{k-1}-\hat{\alpha}+1)^2-1)^2\right| \end{aligned} \tag{32}$$

one may easily see that,

$$\left|f_{k+1}-\hat{f}_{k+1}\right| \le \left| \begin{aligned} &((\hat{\beta}+\varepsilon_{\beta})((\hat{\alpha}+\varepsilon_{\alpha})(y_{k-1}-1)+1)^2-1)^2 \\ &-(\hat{\beta}(\hat{\alpha} y_{k-1}-\hat{\alpha}+1)^2-1)^2 \end{aligned} \right| \tag{33}$$

Thus, we may set,

$$H = \left| \begin{aligned} &((\hat{\beta}+\varepsilon_{\beta})((\hat{\alpha}+\varepsilon_{\alpha})(y_{k-1}-1)+1)^2-1)^2 \\ &-(\hat{\beta}(\hat{\alpha} y_{k-1}-\hat{\alpha}+1)^2-1)^2 \end{aligned} \right| \tag{34}$$

So, the control action, according to Eq. (21) can be obtained as,

$$u_k = -c_k\left(\left(\hat{f}_k - y_{k-1}\right)+\alpha_1 e_k + \eta H \ \mathrm{sat}\left(\frac{S_k}{\phi}\right)\right) \tag{35}$$

where ϕ is the boundary thickness and

$$c_{k-1} = \begin{cases} c_{k-2}-\gamma u_{k-1} & if \ \ |S_k|<\delta \\ 1 & if \ \ |S_k|>\delta \end{cases} \quad , \ 0<\gamma<1 \ , c_0 = 1 \tag{36}$$

As $g_k = 1$ the Eq. (16) representing the stability criterion for switching function, is already satisfied.

Regarding Eq. (29), the following uncertainties are considered in the simulation:

$$\varepsilon_{\alpha} = 0.02\hat{\alpha} \quad \varepsilon_{\beta} = 0.02\hat{\beta} \tag{37}$$

where $\hat{\alpha} = 1.11$ and $\hat{\beta} = 1.76$. Also α_1 is assumed to be 0.1 and $\delta = 0.01$, $\gamma = 0.4$, $\phi_0 = 0.05$.

Also, it is to be noted that the fixed point of the system can be found by solving the following Equation:

$$y^f = (\beta(\alpha y^f - \alpha+1)^2-1)^2 \tag{38}$$

For above parameters, one may find only a meaningful solution for such equation as:

$$y^f = 0.474324 \tag{39}$$

The simulation results for this uncertain system are shown in Figure 1. It is observed that, the fixed point of the system , y^f , is stabilized and the control action converge to zero.

It is to be noted that the amplitude of the control signal is sufficiently small and converges to zero in a small finite duration of time.

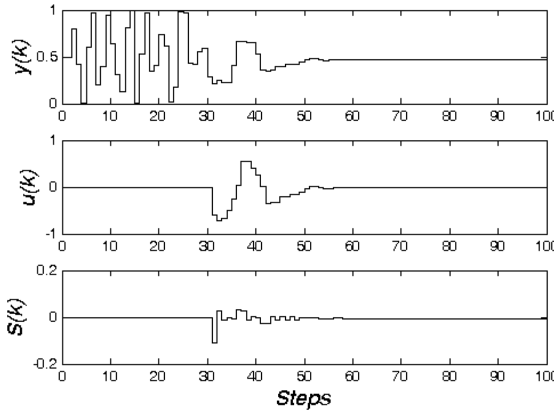

Figure 1- Time series of state y and control signal and distance from sliding surface in the Cournot model (controller starts at step 30)

4.CHAOS CONTROL OF BEHRENS-FEICHTINGER MODEL

The Behrens-Feichtinger model denotes a simple micro-economical model [24] of two firms *X* and *Y* competing on the same market of goods. The firms perform active investment strategies, i.e. their temporary investments depend on their relative position on the market. The strategies are asymmetric.

The firm *X* invests more when it has an advantage over the firm *Y* while the firm *Y* invests more if it is in a disadvantageous position to the firm *X*. The sales x_k and y_k of both firms are measured in discrete time periods $k = 1, 2, 3, \ldots$ and the system governing equations are modeled by [10]:

$$\begin{aligned} x_{k+1} &= (1-\alpha)x_k + \frac{a}{1+\exp\left[-c\left(x_k-y_k\right)\right]} \\ y_{k+1} &= (1-\beta)y_k + \frac{b}{1+\exp\left[-c\left(x_k-y_k\right)\right]} \end{aligned} \tag{40}$$

The constants α and β which $0<\alpha,\beta<1$ are the time rates at which the sales of both firms decay in absence of investments while the second terms in the right hand side of Eq.(40) describes the investment effect at the k-th time period on the sale quantities at (*k-1*)-th time period. Parameters *a* and *b* define the investment effectiveness of the firms and *c* is an "elasticity" measure of the investment strategies. The regular or irregular behavior of dynamic system described by Eq. (40) depends on the values of parameters α, β , a, b and c.

For $\alpha = 0.46$, $\beta = 0.7$, $a = 0.16$, $b = 0.9$, $c = 105$, system (40) shows chaotic. The first order fixed points of the system is $x_f^1 = 0.01182$, $y_f^1 = 0.04370$.

For chaos elimination from the system, the first order fixed point of the system is selected to be stabilized, and both firms achieve this objective by applying the control action to the parameters *a* and *b*, i.e. the investment effectiveness. In this case the controlled system is written as:

$$\begin{aligned} x_{k+1} &= (1-\alpha)x_k + \frac{a+u_k}{1+\exp\left[-c\left(x_k-y_k\right)\right]} \\ y_{k+1} &= (1-\beta)y_k + \frac{b+v_k}{1+\exp\left[-c\left(x_k-y_k\right)\right]} \end{aligned} \tag{41}$$

Comparing Eq. (41) and Eq. (1) yields

$$\begin{aligned} x_{k+1} &= f_{1,k}\left(x_k, y_k\right)+g_{1,k}\left(x_k, y_k\right)u_k \\ y_{k+1} &= f_{2,k}\left(x_k, y_k\right)+g_{2,k}\left(x_k, y_k\right)v_k \end{aligned} \tag{42}$$

where

$$f_{1,k}(x_k,y_k)=(1-\alpha)x_k+\frac{a}{1+\exp\left[-c\left(x_k-y_k\right)\right]}$$
$$f_{2,k}(x_k,y_k)=(1-\beta)y_k+\frac{b}{1+\exp\left[-c\left(x_k-y_k\right)\right]} \qquad (43)$$
$$g_{1,k}(x_k,y_k)=g_{2,k}(x_k,y_k)=\frac{1}{1+\exp\left[-c\left(x_k-y_k\right)\right]}$$

It is assumed that all of the state variables can be measured and fed back. Due to market uncertainties the system parameters are not known while their nominal or estimated values are assumed to be known which are denoted by $\hat{\alpha},\hat{\beta},\hat{a},\hat{b},\hat{c}$. The upper limits of the errors between the actual and nominal values of parameters are assumed to be specified and denoted by $\varepsilon_\alpha,\varepsilon_\beta$, $\varepsilon_a,\varepsilon_b,\varepsilon_c$, as in

$$\left|\alpha-\hat{\alpha}\right|<\varepsilon_\alpha \;,\; \left|\beta-\hat{\beta}\right|<\varepsilon_\beta \;,\; \left|c-\hat{c}\right|<\varepsilon_c$$
$$\left|a-\hat{a}\right|<\varepsilon_a \;,\; \left|b-\hat{b}\right|<\varepsilon_b \qquad (44)$$

Two sliding surfaces are considered for designing the control actions,

$$S_{1k}=e_{1,k}+\alpha_1 e_{1,k-1}+\hat{g}_{1,k-1}\left(\frac{1}{c_{1,k-1}}-1\right)u_{k-1}$$
$$S_{2k}=e_{2,k}+\alpha_1 e_{2,k-1}+\hat{g}_{2,k-1}\left(\frac{1}{c_{2,k-1}}-1\right)v_{k-1} \qquad (45)$$

where

$$e_{1,k}=x_k-x_{k-1}$$
$$e_{2,k}=y_k-y_{k-1} \qquad (46)$$

and α_1 is selected such that $\left|\alpha_1\right|<1$. Regarding Eq. (21) the control inputs, u_k and v_k are calculated as:

$$u_k=\frac{-c_{1,k}}{\hat{g}_{1,k}}\left(\hat{f}_{1,k}-x_k-\alpha_1 e_{1,k}+\varepsilon_1\mathrm{sat}\left(\frac{S_{1,k}}{\phi}\right)\right)$$
$$v_k=\frac{-c_{2,k}}{\hat{g}_{2,k}}\left(\hat{f}_{2,k}-y_k-\alpha_1 e_{2,k}+\varepsilon_2\mathrm{sat}\left(\frac{S_{2,k}}{\phi}\right)\right) \qquad (47)$$

where ϕ is the boundary thickness and regarding Eq. (22)

$$c_{1,k-1}=\begin{cases}c_{1,k-2}-\gamma_1 u_{k-1} & if\ \left|S_{1,k}\right|<\delta\\ 1 & if\ \left|S_{1,k}\right|>\delta\end{cases}$$
$$c_{2,k-1}=\begin{cases}c_{2,k-2}-\gamma_2 v_{k-1} & if\ \left|S_{2,k}\right|<\delta\\ 1 & if\ \left|S_{2,k}\right|>\delta\end{cases} \qquad (48)$$

and ε_j , $j=1,2$ are obtained according to Eq. (11) as $\varepsilon_j=\eta H_j$ where:

$$H_j>\left|f_{1,k}-\hat{f}_{1,k}\right|+\left|g_{1,k}-\hat{g}_{1,k}\right|U_j \quad ,\forall k,j=1,2 \qquad (49)$$

where U_j is the upper bound for control signals. H_j can be calculated with some cumbersome calculations like the ones performed in [23] .

Numerical simulation for $\alpha=0.46$, $\beta=0.7$, $a=0.16$, $b=0.9$, $c=105$, $\hat{\alpha}=0.45,\hat{\beta}=0.71,\hat{a}=0.17,\hat{b}=0.9$, $\hat{c}=105.5$, $\varepsilon_\alpha=0.01$, $\varepsilon_\beta=0.01$, $\varepsilon_a=0.01$, $\varepsilon_b=0.01$ and $\varepsilon_c=1$ is shown in Figure 2 and Figure 3. The control parameters are set as $\alpha_1=0.1$, $\gamma_1=\gamma_2=0.2$, $\delta=0.01$ and $U_i=0.2$. It is observed that the fixed point of the system is asymptotically stabilized, even although that no where abouts of the fixed point is provided for the controller.

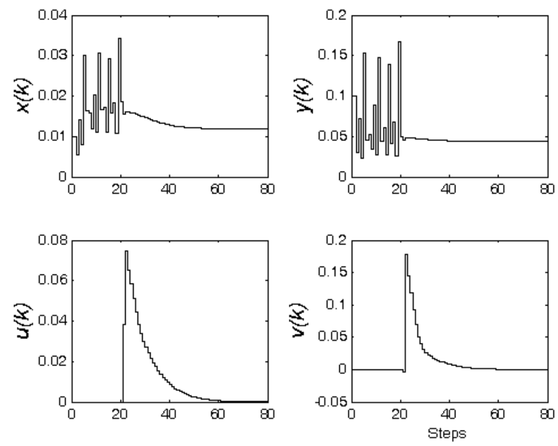

Figure 2-Time series state and control signal in the Behrens-Feichtinger model (controller starts at step 20)

It is also remarkable that the proposed Delayed Feedback Quasi Sliding Mode Controller is more sensitive to uncertainties, and might take a longer time to stabilize such systems regarding the normal quasi sliding mode [23], but as the fixed point of uncertain systems aren't exactly known, despite normal quasi sliding mode, this scheme can achieve the zero controlling signal that is desired in the control of chaos.

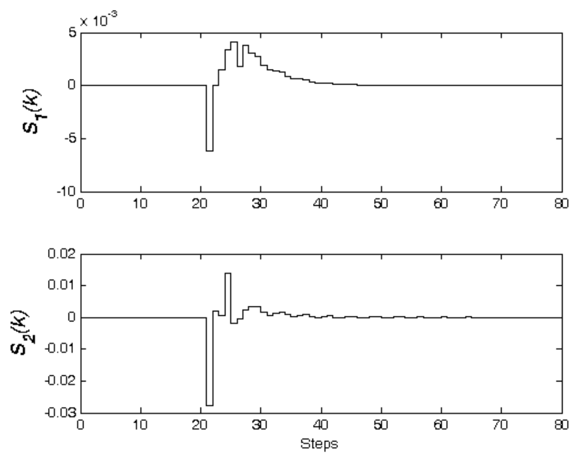

Figure 3-Time series of distance from variable sliding surface in the Behrens-Feichtinger model (controller starts at step 20)

CONCLUSION

In this paper the problem of chaos control in uncertain economic models is investigated. To this purpose the delayed feedback quasi-sliding mode controller is designed and utilized for stabilization of unstable fixed points of economic system. The performance is examined by applying it to two chaotic models, the Behrens-Feichtinger and the Cournot duopoly model with complements. Simulation results illustrate that the presented algorithm may be successfully applied to chaotic economic models to obtain regular and stable behaviors. It also mentioned that considering normal sliding mode controller, the suggested method is less resilient to uncertainties, but can reach the zero control signal and stabilize the real fixed point of system.

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