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THE THERMODYNAMIC CHARACTERISTICS OF THE SYSTEMS WITH NONLINEAR PAIRWISE INTERACTIONS

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Abstract

In this report I summarize the results in the application of Direct Lyapunov method to the generalized replicator systems. This complex systems to define the properties of an system composed of agents that are coupled via pairwise interactions. It is shown that there are exist two types of Lyapunov functions: fitness-like and entropy-like. As example it'll be establish that practically all known entropy measures may be obtain from entropy-like Lyapunov function for replicator systems.

Key words

Nonlinear systems, complexity, stability, Lyapunov functions.

1 The generalized replicator systems

Replicator dynamics is an evolutionary strategy well established in different disciplines of biological sciences. It describes the evolution of self-reproducing entities called replicators in various independent models of, e.g., genetics, ecology, prebiotic evolution, and sociobiology. Besides this, replicator selection has been applied to problem solving in combinatorial optimization and to learning in neural networks and also in fluid mechanics, game and laser theory. So, the replicator systems arising in an extraordinary variety of modeling situations. In this report we'll consider the class of generalized replicator equations with nonlinear pairwise interactions. There have been some recent evidence that suggests that dynamics of arbitrary networks can be reduced to pairwise interactions [Schneidman et al, 2006] and [Bialek, Ranganathan, 2007]

Generalized replicator equations determine the evolution of probability distributions $p(t)=\left(p_1(t),...,p_n(t)\right)\in\sigma$ and has the next form [Pykh, 2003]:

$$\begin{aligned}\dot{p}_i = h(p)\, f_i(p_i)\Bigg(\sum_{j=1}^{n} w_{ij} f_j(p_j) - \\ -\theta^{-1}(p)\sum_{j,k=1}^{n} w_{jk} f_j(p_j) f_k(p_k)\Bigg)\; i=1,...,n\end{aligned}\tag{1}$$

Here, f_i are nonlinear response functions satisfying the conditions $f_i(0)=0$, $\partial f_i/\partial p_i>0$ $p_i>0$, and $\partial f_i/\partial p_i \geqslant 0$ for $p_i=0$; $W=(w_{ij})$ is the matrix of interactions; the function $h:\sigma\to(0,\infty]$ is determined by the particular problem under consideration; $\theta(p)=\langle \mathbf{e}, f(p)\rangle$, where $\langle\cdot,\cdot\rangle$ is the inner product; and $f(p)=\left(f_1(p_1),...,f_n(p_n)\right)$.

Obviously, since $\langle \dot{p}(t),\mathbf{e}\rangle\equiv 0$ and $f_i(0)=0$, the simplex σ and each of its faces are invariant sets of system (1).

System (1) has a very wide range of applications, from mathematical genetics and ecology to neural networks [Bomze, 1997] and [Hofbauer,Sigmund, 2003]. Recently, it was shown [Helbing, 1996] that system (1) can be obtained from Boltzmann-like equations. Thus, there are grounds for believing that system (1) determines the evolution of probability distributions for a fairly wide variety of processes.

To state the main theorem, we need some preliminary results. First, it is convenient to pass to the matrix form of representation. In this form, system (1) becomes

$$\dot{p}=h(p)\mathcal{D}(f)\left(Wf-\mathbf{e}\theta^{-1}(p)\langle f,Wf\rangle\right)\tag{2}$$

where $\mathcal{D}(f)=diag(f_1,...,f_n)$.

If the matrix W is nondegenerate, then system (2) has at most one isolated equilibrium point in $\mathrm{Int}\sigma$, which we call nontrivial.

Statement [Pykh, 2003]. *System (2) has a unique nontrivial equilibrium point* $\hat{p} \in \mathrm{Int}\sigma$ *if and only if the vector* $W^{-1}\mathbf{e}$ *is either strictly positive or strictly negative.* □

Theorem 1 [Pykh, 2003]. *If the matrix W is symmetric, then the function*

$$E(p) = \langle f(p), Wf(p) \rangle \theta^{-2}(p) \qquad (3)$$

is a Lyapunov energy function for system (2). □

Corollary [Pykh, 2003]. *If system (2) has a nontrivial equilibrium point* $\hat{p} \in \mathrm{Int}\sigma$, *then it is totally stable in* $\mathrm{Int}\sigma$ *if and only if the matrix W has* $(n-1)$ *negative characteristic numbers.* □

Theorem 2 [Pykh, 2003]. *If* $W = W^T$ *and system (2) has a nontrivial equilibrium point* $\hat{p} \in \mathrm{Int}\sigma$ *which is totally stable in* $\mathrm{Int}\sigma$, *then the entropy-like function:*

$$H(p) = \sum_{i=1}^{n} \int_{\hat{p}_i}^{p_i} \frac{\hat{f}_i dx}{f_i(x)} \qquad (4)$$

is a Lyapunov energy function for system (2), and

$$\dot{H} = h(p)\hat{\theta}\theta\left(E(\hat{p}) - E(p)\right) \geq 0. \quad □ \qquad (5)$$

Now, we can state the main result without restriction $W = W^T$:

Theorem 3. [Pykh, 2005] *If system (2) has a nontrivial equilibrium point* $\hat{p} \in Int\sigma$ *and the matrix* $(W^T + W)$ *has* $(n-1)$ *negative characteristic value, then the function*

$$H(p) = \sum_{i=1}^{n} \int_{\hat{p}_i}^{p_i} \frac{\hat{f}_i dx}{f_i(x)}$$

is a Lyapunov energy function for sytem (2). □

Corollary [Pykh, 2005]. *If the conditions of Theorem 3 are fulfilled and* W *is such that* $W^T W^{-1}$ *is a stochastic matrix, i.e.,*

$$W^T W^{-1}\mathbf{e} = \mathbf{e},$$

then the entropy production is defined by the formula

$$\dot{H}(p) = h(p)\theta\hat{\theta}\left(E(\hat{p}) - E(p)\right) \geqslant 0 \quad □ \qquad (5)$$

Based on this theorem we can recive a set of response function for existing entropy measures and constract new entropy measures for any response functions. Short summary of this approach listed below in table 1.

Table 1.
Different entropy measures

Response function	Entropy	Name
Logariphmic $f_i(p_i) = (1-\ln p_i)^{-1}$	$H(p) = \sum_{i=1}^{n} p_i \ln p_i$	Bolzmann Entropy
Power-law $f_i(p_i) = p_i^{1-q}$; $q \neq 1$	$H(p) = \frac{\left(\sum p_i^q - 1\right)}{1-q}$	Tsallis Entropy
Logistic $f_i(p_i) = \frac{1}{b + ce^{-\alpha p_i}}$ $b>0,\ c>0,$ $\alpha>0$	$H(p) = \sum_{i=1}^{n} \ln\left(1-e^{-\alpha p_i}\right)$	Logistic entropy (new)

2 Thermodynamic characteristics

We have received expression (3) for replicator's systems energy, expression (4) for systems entropy and expression (5) for entropy production. On the analogy of termodynamics laws, we can receive expression for systems temperature. Indeed if we redraft (5) as follows:

$$\frac{dH}{dE} = h(p)\hat{\theta}\theta(p),$$

then according to Clausius definition the systems temperature T is equal:

$$T = \left(h(\hat{p})\hat{\theta}\theta(p)\right)^{-1}$$

Note that in this case the temperature depends from systems steady-state. Now let us consider the exergy of the system. Exergy is a measurement of how far a certain system deviates from a state of equilibrium with its environment. Exergy for a system in an environment usually is written as:

$$Ex = T\left(\hat{H} - H\right)$$

So we have a lot of different expression for exergy dependance from entropy i.e. from response function. If we put:

$$f_i(p_i) = \frac{1}{1 - \ln \frac{p_i}{\alpha_i}}$$

where vector $\alpha = (\alpha_1, ..., \alpha_n) \in \sigma$ and interaction matrix is stochastic i.e. $W\mathbf{e} = \mathbf{e}$, then $\alpha_i = \hat{p}_i$. In this case we receive the next expression for exergy:

$$Ex = h(p)\hat{\theta}\theta(p)\left(\sum_{i=1}^{n} p_i \ln\frac{p_i}{\hat{p}_i} + (p_i - \hat{p}_i)\right)$$

It is easy to see that this expression almost coincide with formula proposed by Mejer and Jorgensen in 1979. Note, that in like manner we can receive all termodynamic potentials such as Helmholtz or Gibbs free energy, which is also Lyapunov, functions.

3 Conclusion

It is seen from the examples given above that many (and practically all) known entropy characteristics my be obtain from entropy-like Lyapunov function. We also emphasize that there exists a relation between the derivative of the function $H(p)$, which can be interpreted as a generalized entropy, and the function $E(p)$, which is often considered as an analog of the energy or fitness. This relationship for entropy production was establihed by Pykh [Pykh, 2004] and [Pykh, 2005] for different interactions matrix and has the next form:

$$\dot{H} = h(p)\hat{\theta}\theta\left(E(\hat{p}) - E(p)\right) \geqslant 0$$

We mention also that all results stated above were obtained by formally analyzing systems of generalized replicator equations, which arise in very diverse fields of natural sciences and, therefore, can serve as a basis for finding analogies between these domains of natural sciences. Also note that it was Ilya Prigogine who the first pointed out [Prigogine, 1977] the imoprtance of the relationship between Lyapunov functions and entropy.

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