D-robust control by uncertain objects

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Today the methods of the analysis of robust stability and robust stabilization of linear objects are the basis of the theory of robust control. Whereat not only one given linear system is investigated but also the stability of the entire family of systems fitting in initial (nominal) system with the ambiguity is examined. The tasks of control, as a rule, add up to the tasks of stabilization or optimal control without the given expiration time of the transient process. This allows to use frequency methods developed in the doctrines of automatic regulation [1]. The using of these methods for the synthesis of control actions for uncertain systems in the given interval of control is impossible.

Using of minimax approach lets to receive of necessary and sufficient condition of the existence of d-robust control for one class of not linear not stationary systems.

Main hypothesis and preliminaries

Let not stationary controlled dynamic object is described by the system of the ordinary differential equations of type

$$\frac{d}{dt}x(t) = f(x,u,\alpha,t), \quad x \in R^n,\ t \in [t_0,T]. \qquad (1)$$

The initial state of object (1) belongs to limited multitude X_0,

$$x(t_0) \in X_0. \qquad (2)$$

The conditions on the right end are also set:

$$g(x(T)) = 0, \qquad (3)$$

where $g(x(T))$ is a scalar function.

In equation (1) $\alpha(t) \in \Omega$ - are the parameters of object, Ω - is the closed limited multitude in Euclidean space R^P. It is assumed that control $u(t) \in U$ almost everywhere, U - is the closed limited multitude in Euclidean space R^r.

The functional estimating the effectiveness of the object control (1) is set:

$$J = J(x,u). \qquad (4)$$

Let $\alpha(t) = \alpha(t_0,T) \in \Omega$ – the possible trajectory of the change of the object (1) parameters. Then the decisions of differential equation (1) belong to the some differential inclusion

$$\frac{d}{dt}x(t) \subset f(x,u,\alpha,t),\ x(t_0) = x_0. \qquad (5)$$

In the known trajectories of the change of parameters for each object from multitude (5) $u^0(t) \in U$ can be synthesized in which condition (3) is fulfilled and the functional (4) accepts minimum. However, the optimal control for any known trajectory of the object parameters can turn out to be far from optimal in another trajectory of parameters. Moreover the control not always can provide the stability of system 'object-regulator" in the trajectories of parameters, different from that which was used in the synthesis of the optimal control.

In ignorance about significances which accept the parameters of object $\alpha(t_0,T) \in \Omega$ in the control interval that task is considered to be successfully solved if we can find the control $u^*(t) \in U$ transferring the system from $x(t_0) \in X_0$ to $x^*(T)$, in which purpose of control (3) will be fulfilled with given accuracy,

$$\left|g(x^*(T))\right| \le d. \qquad (6)$$

Here d is the fixed non-negative permanent, $x^*(T)$ - is the state accepted by object at the moment of the end of the control period in the concrete significances of parameters $\alpha(t_0,T) \in \Omega$ and fitting control $u^*(t_0,T)$.

Determining [3]

The system 'object-regulator" will be named the robust controlled system with the given indicator of robustness if we can find the control $u^*(t_0,T) \in U$ for object

$$\frac{d}{dt}x(t) = f(x,u,\alpha,t),\ x \in R^n,\ t \in [t_0,T],$$

which in any possible trajectories of parameters $\alpha(t_0,T)$ belonging to the given multitude of trajectories Ω ($\alpha(t_0,T) \in \Omega$) transfers object from the initial state $x(t_0)$

belonging to the given multitude of initial conditions X_0 ($x(t_0) \in X_0$) to the state $x^*(T)$, in which the purpose of control $g(x^*(T))$ is achieved with given accuracy ($|g(x^*(T))| \le d$).

For some tasks of d-robust control another condition can be added: the functional estimating the quality of control, must accepts significance, not exceeding given $J(x^*,u^*) \le J^{\max}(x,u)$, $u(t_0,T) \in U$.

The control $u^*(t_0,T) \in U$ will be named d-robust control.

The special type conditions (6) $\|\pi x^*(T)\| \le d$, where $\pi-$ is the projection operator from R^n on R^k.

Thus, robust control $u^*(t) \in U$ of object (1) is determined by correlations:

$$u^*(t_0,T) = \\ = \arg \sup_{x_0 \in X_0} \sup_{\alpha \in \Omega} \inf_{u \in U} |g(x^*(T))| \le d, \quad (7)$$

$$J(x^*,u^*) = \\ = \min_{u \in U} \sup_{\alpha \in A} J(x,u) \le J^{\max}(x,u). \quad (8)$$

Let's consider the value $\overline{J} = \sup_{\alpha \in A} J(x,u)\,_{|g(x(T))| \le d}$

to be the guaranteed significance of the criterion of quality in d-robust control (7).

The value

$$J^0 = \inf_{u \in U} \sup_{\alpha \in A} J(x,u)_{|g(x(T))| \le d}$$

will be the optimal guaranteed significance of the criterion of quality in robust control. It is evident that limitations on the control actions, in which task of robust control will be fulfilled, are dependent on initial state $x(t_0) \in X_0$, from the parameters significances α $(t_0,T) \in \Omega$ and from the period of control $T-t_0$. Thus, bossed multitude comprising robust control U is determined by the expression

$$U = \\ = \{x_0 \in X_0, \alpha(t_0,T) \in \Omega, t \in [t_0,T], u^*(t_0,T): \quad (9) \\ \frac{d}{dt}x(t) == f(x^*,u^*,\alpha,t), |g(x^*(T))| \le d\}.$$

If the robust control will be $u^*(t_0,T) \in U$ where U corresponds to the expression (9), the task of robust control will be successfully accomplished for the given period of control $T-t_0$ in any initial condition $x_0 \in X_0$ and in any trajectory of parameters α $(t_0,T) \in \Omega$. With this end in view is necessary in order

$$\sup_{x_0 \in X_0} \inf_{u \in U} |g(x^*(T))| \le d, \ \alpha(t_0,T) \in \Omega \ .(10)$$

Really since in any fixed $x_0 \in X_0$ and any fixed significances of the object parameters α $(t_0,T) \in \Omega$ must be fulfilled inequality

$$\inf_{u \in U} |g(x^*(T))| \le d, x(t_0) \in X_0, \ \alpha(t_0,T) \in \Omega$$

that the condition (10) follows. We shall note that the condition (10) will be sufficient in addition if for any $x_0 \in X_0$ and any significances of parameters $\alpha(t_0,T) \in \Omega$ $u^*(t_0,T) \in U$ is found such that $d \ge \inf_{u \in U} |g(x(T))|$.

In the d-robust control tasks actualized with using of minimax approach in the general case the point of minimax is not the saddle point, i. e. the permutation of operations *inf* and *sup* can lend non-coinciding results.

In given limitations on control actions and given initial condition $x(t_0) \in X_0$ the solving of the "dual" task can be interesting, i. e. definitions

$$\overline{\alpha}(t) = \arg \max_{a \in \Omega} \underline{J}(x,u), \quad \overline{\alpha}(t) \in \partial\Omega,$$

where $\underline{J}(x,u) = \inf_{u \in U} J(x,u)\,_{|g(x(T))| \le d}$.

Non-stationary system with non-linear sector element

Let non-linear non-stationary controlled object is described by vector differential equation

$$\frac{d}{dt}x(t) = f(x,t) + B(x,t)u(t), \\ x(t_0) \in X_0, \quad (11)$$

where $x \in R^n$ – is the vector of the entry condition of the object state, X_0 – is the multitude of the possible object initial conditions, $u \in R^r$ - is the vector of control actions. The functional of quality is set:

$$J(x,u)=\frac{1}{2}\Bigg[\;x^T(T)Fx(T)+ \\ +\int_0^T\left\{x^T(t)Qx(t)+u^T(t)Ru(t)\right\}dt\;\Bigg], \quad (12)$$

where $T-$ is the expiration time of the transient process is also set.
The task of control of object (11) consists in building $u(t)$ carrying to the minimum of the functional (12) to and the carrying out of condition

$$\left|\eta^T x(T)\right|\le d>0\,. \quad (13)$$

Let's suppose:
1) $f_i(x(t),t)$, $b_{ij}(x(t),t)$, $i=1,...,n,\ j=1,...,r$ – are the elements of matrixes f and B accordingly continuous relatively $x(t)$ and t;

2) $\dfrac{\partial f_i(x(t),t)}{\partial x_k(t)},\ \dfrac{\partial f_i(x(t),t)}{\partial t},\ \dfrac{\partial b_{ij}(x(t),t)}{\partial x_k(t)},\ \dfrac{\partial b_{ij}(x(t),t)}{\partial t}$ continuous as to $x(t)$ and t for $i,\,k=1,...,n,\ j=1,...,r$;

3) control is the linear function of the state of object (11), i. e.

$$u(t)=Kx(t)\,. \quad (14)$$

These assumptions [2] let present the initial equation of object in the neighborhoods of the point $x=0$ in the form of

$$\frac{d}{dt}x(t)=\left[A+\alpha(x,t)\right]x(t)+ \\ +\left[B_1+\beta(x,t)\right]Kx(t)+ \\ +\Im(x,a(x,t),\beta(x,t)), \quad (15)$$

$f(x,t)=\left[A+\alpha(x,t)\right]x(t)+\Im_f(x,t)$,
$B(x,t)Kx(t)=\left[B_1+\beta(x,t)\right]Kx(t)+\Im_B(x,t)$,
where $A+\alpha(x,t)=\partial f(x,t)/\partial x(t)_{npu\ x=0}$;
$\Im_f(x,t)$ - is the remaining member of the breakdown of function $f(x,t)$;

$B_1+\beta(x,t)=\sum_{i=1}^{n}\{\ \partial b_{ij}(x,t)/\partial x(t)\}^T{}_{npu\ x=0}$;

$\Im_B(x,t)$ - is the remaining member of the breakdown of function $B(x,t)Kx(t)$;
$\Im(x,a(x,t),\beta(x,t))=\Im_f(x,t)+\Im_B(x,t)-$ is the non-linear vector function, and

$\Im(x,a(x,t),\beta(x,t))=\Im(\sigma)=0$ in $x(0)=0$ and

$$\Im^T(\sigma)\{\Im(\sigma)-\sigma\}<0 \text{ in } x(t)\ne 0\,. \quad (16)$$

Let the initial state of object belongs to the area of closure the multitude of initial conditions $x_0^*\in\partial X_0$ in which conditions of the carrying out of the set task are "worst". Then, in the condition of the successful carrying out of the control task matrixes $\alpha(x,t)$, $\beta(x,t)$ and vector $\Im(\sigma)$ will have interval disposition of uncertainty.

Let Ω is the set of the possible trajectories $\alpha(x,t)$ and $\beta(x,t)$, i. e. $\alpha(x,t),\,\beta(x,t)\in\Omega$, and α^*, β^*- are the "worst" significances of matrixes lying on the border closed the multitudes of possible significances parameters disturbance, i. e. $\alpha^*,\beta^*\in\partial\Omega$ in which the set task of control of the object (15) can be fulfilled.

The synthesis of regulator (the matrix K search) will be carried out with the using of the linear model of object which is presented in the following way:

$$\frac{d}{dt}x_M(t)=[A+\alpha^*]x_M(t)+ \\ +\left[B_1+\beta^*\right]u^*(t),\ x_M(t_0)=x_0\,. \quad (17)$$

If to appoint matrix F in the first summand functional (12) in the form F=S where the positively definite matrix S is the solution of the Rikkati-Lurie equation:

$$S\left[A+\alpha^*\right]+\left[A+\alpha^*\right]^T S- \\ -S\left[B_1+\beta^*\right]R^{-1}\left[B_1+\beta^*\right]^T S+Q=0, \quad (18)$$

that the optimal control for model (17) with functional (12) in which instead of $x(t)$ we shall put $x_M(t)$, the control will be presented in the following way:

$$u^*(t)=-R^{-1}\left[B_1+\beta^*\right]Sx_M(t)\,. \quad (19)$$

We shall note that in this instance the matrix is $S(t)=const,\ t\in\left[0,T\right]$.
We use structure of control (19) for the building of the control of object (15):

$$u(t)=-R^{-1}\left[B_1+\beta^*\right]Sx(t)\,. \quad (20)$$

We shall find the necessary conditions of the existence of stabilizing control of type (20) for object (15).
The solution of the equation (15) with the control (20) is given by:

$$x(T)=\left\{\exp\left[A+\alpha^{*}-\left[B_1+\beta^{*}\right]R^{-1}\left[B_1+\beta^{*}\right]^{T}S\right]T\right\}\times$$
$$\times\{x^{*}(0)+$$
$$+\int_0^T\left\{\exp\left[-A-\alpha^{*}+\left[B_1+\beta^{*}\right]R^{-1}\left[B_1+\beta^{*}\right]^{T}S\right]\tau\right\}\Im(\sigma)\,d\tau\ \}$$

or

$$x(T)=\left[\exp(\mathcal{T}\!\!T\,T)\right]\{x^{*}(0)+$$
$$+\int_0^T\left[\exp(-\mathcal{T}\!\!T\,\tau)\ \right]\Im(\sigma)\,d\tau\},\qquad(21)$$

where

$$\mathcal{T}\!\!T=$$
$$=A+\alpha^{*}-\left[B_1+\beta^{*}\right]R^{-1}\left[B_1+\beta^{*}\right]^{T}S=const.$$

Let' s consider the norm of scalar product $\|\eta,x(0)\|$. If control (20) will stabilize the object (15) then in $T\to\infty$ must be fulfilled condition:

$$\left|\ \eta^{T}x^{*}(0)+\int_0^T\{\exp(-\mathcal{T}\!\!T\ \tau)\}\,\Im(\sigma)\,d\tau\ \right|\to 0$$

$$in\ \ T\to\infty$$

or

$$|\ \eta^{T}x^{*}(0)\,|-\int_0^T\{\exp(-\mathcal{T}\!\!T\,\tau)\ \}|\,\Im(\sigma\,)|\,d\tau\to 0$$

$$in\ T\to\infty.\qquad(22)$$

Because the first summand has final significance and the second summand in the control object stabilized must have a final significance in $T\to\infty$.The last condition is fulfilled in that case, if the integration element will be decreasing. The positively definite integration element will be demanded to monotonely decrease. This condition will be fulfilled, if the time derivative of the positively definite form

$$\eta^{T}\{\exp[-\mathcal{T}\!\!T\,t]\,|\,\Im(\sigma)\,|>0,$$
$$\sigma\neq 0,\ \ t\to\infty\qquad(23)$$

will be negative if

$$\eta^{T}\{\exp[-\mathcal{T}\!\!T\,t]\ \Im(\sigma)>0,\ \ \sigma\neq 0,\ \ t\to\infty,$$

i. e.

$$\frac{\partial}{\partial t}\left[\eta^{T}\{\exp[-\mathcal{T}\!\!T\,t]\ \}\Im(\sigma)\ \ \right]<0,\ \ \sigma\neq 0$$

and positive if

$$\eta^{T}\{\exp[-\mathcal{T}\!\!T\,t]\ \Im(\sigma)<0,\ \ \sigma\neq 0,\ \ t\to\infty$$

i. e.

$$\frac{\partial}{\partial t}\left[\eta^{T}\{\exp[-\mathcal{T}\!\!T\,t]\ \}\Im(\sigma)\ \ \right]>0,\ \ \sigma\neq 0\,.$$

In both cases the condition of the monotone decrease of the integration element (23) is given by

$$\eta^{T}\mathcal{T}\!\!T\,\{\exp\,[-\mathcal{T}\!\!T\,t]\,\}\Im(\sigma)>$$
$$\eta^{T}\{\exp\,[-\mathcal{T}\!\!T\,t]\}\left\{\frac{d\,\Im(\sigma)}{d\,t}\right\},\ \ \sigma\neq 0.\qquad(24)$$

From [4] the account

$$\frac{d}{dt}\{\exp[-\mathcal{T}\!\!T\,t]\}=-\mathcal{T}\!\!T\{\exp[-\mathcal{T}\!\!T\,t]\}=$$
$$=-\{\exp[-\mathcal{T}\!\!T\,t]\}\mathcal{T}\!\!T$$

The condition of non-linear function $\Im(\sigma)$ variation in time can be received from condition (24):

$$\left\|\frac{d\,\Im(x,t)}{d\,t}\right\|<\|\,\mathcal{T}\!\!T\,\|\ \|\Im(x,t)\|,\ x=x(0)\ .(25)$$

Because the fulfillment of condition (25) is provided the monotone decrease of the norm of integration element (22) the system "object (15) control (10)" in this case is asymptotically stable.

We shall note that the matrix

$$\mathcal{T}\!\!T=A+\alpha^{*}-\left[B_1+\beta^{*}\right]R^{-1}\left[B_1+\beta^{*}\right]^{T}S\,,$$

containing permanent parameters, is dependent on the row of parameters, essential of which for the carrying out of statement of problems the system "object (15) regulator (20)" stabilization is positively definite matrix S being deciding of equation (18).

Can be said that $S=S(Q,R)$.

Appointing accordingly matrixes Q and R can be received the solution of the equation (18) such that will be fulfilled conditions (24), (25).

Another type of the condition (24) can be received with the introduction of the Lyapunov’s function

$$V=x^{T}(t)Sx(t)\,,\qquad(26)$$

where the positively definite matrix S is the solution of the equation (18).

The derivative Lyapunova’s function (26) is given by:

$$\frac{d}{dt}V =$$
$$-x^T(t)\left[Q + S\left[B_1+\beta^*\right]R^{-1}\left[B_1+\beta^*\right]^T S\right]x(t) +$$
$$+x^T(t)SB\left[B_1+\beta^*\right]\Im(\sigma) +$$
$$\Im^T(\sigma)\left[B_1+\beta^*\right]^T Sx(t) < 0 \qquad (27)$$
in $x(t) \neq 0$.

Appointing accordingly matrixes Q and R can be received by the solution of the equation (18) the such matrixes that the condition (16) will be fulfilled, i. e.
$$\Im^T(x, a(x,t), \beta(x,t)))\times$$
$$\times\left\{ \Im(x, a(x,t), \beta(x,t))) - - \Im(\left[B_1+\beta^*\right]^T Sx(t))\right\} < 0, \qquad (28)$$
$$x(t)\neq 0,\ t\in[0,T].$$

Then the inequality (27) can be rewritten in the form
$$\frac{d}{dt}V =$$
$$-x^T(t)\left[Q + SBR^{-1}B^T S\right]x(t) + \qquad (29)$$
$$+2\Im^T(\left[B_1+\beta^*\right]^T Sx(t))\,\Im(\left[B_1+\beta^*\right]^T Sx(t)) < 0$$
in $x(t)\neq 0$.

In the concrete expressions of the analytical non-linear elements we can determine the spectrum of the system state $X(t)$ and the possible entry conditions X_0 in which in the task with the interval parametric ambiguity stabilizing control will exist.

It is evident that the border of the multitude of entry condition in which the stabilizing control of non-linear system (15) with control (20) exists, will be determined following correlation
$$\|x^*(0)\| =$$
$$= \int_0^\infty \| \{\exp\left[-A-\alpha^* + \left[B_1+\beta^*\right]R^{-1}\left[B_1+\beta^*\right]^T S\right]\tau\}\times$$
$$\times\Im(x, a(x,t), \beta(x,t))\ \| d\tau. \qquad (30)$$

The condition (25) determines the "sector" which the characteristics of the non-linear part of stabilized system with given control (20), with the known multitude of the parameters of non-stationary matrixes $\alpha(x,t),\ \beta(x,t)\in\Omega$ and by the multitude of entry condition $\{x(0)\} = X_0$ determined by the condition (30) must belong to.

Consider the question about the existence of the control of type (20) in the motion of non-linear non-stationary system in the given interval of time from any initial state belonging to the given multitude, to the given area.

Write down the condition of the set task of the d-robust control of the object (11):
$$\left\|\eta^T\left[\exp(\mathcal{T}T)\right]\right\|\times$$
$$\times\left\{ \ \|x^*(0)\| - \left\|\int_0^T\left[\exp(-\mathcal{T}\tau)\right]\Im(\sigma)\,d\tau\right\| \ \right\}\le d$$
or
$$\left\|\eta^T\left[\ \exp(\mathcal{T}T)\right]\right\|^{-1} d - \|x^*(0)\| \ge \qquad (31)$$
$$\left\|\int_0^T\left[\exp(-\mathcal{T}\tau)\right]\Im(x, a(x,t), \beta(x,t))\,d\tau\right\|.$$

If the condition (31) is not fulfilled it means that for the object
$$\frac{d}{dt}x(t) = \left[A+\alpha(x,t)\right]x(t) +$$
$$+\left[B_1+\beta(x,t)\right]Kx(t) + \Im(x, a(x,t), \beta(x,t))$$
with the initial condition $x(0)\in X_0$ and the given period of control [0, T] in the general case the control $u(t) = -R^{-1}\left[B_1+\beta^*\right]Sx(t)$ with permanent positively definite matrix S determined by Rikkati-Lurie solution
$$S\left[A+\alpha^*\right] + \left[A+\alpha^*\right]^T S -$$
$$-S\left[B_1+\beta^*\right]R^{-1}\left[B_1+\beta^*\right]^T S + Q = 0,$$
which can provide given robust indicator d, doesn't exist.

The carrying out of condition (25) provides to interim process asymptotic property producing fitting demands to the behavior of non-linear function entering system. Thus the carrying out of this condition is the necessary condition of the existence of the d-robust control. The condition (31) is the subsidiary condition providing the sufficient condition of the existence of the d-robust control. The carrying out of both conditions guarantees the carrying out of the task of the d-robust control of non- stationary object.

If to demand exponential decreasing of the integration element $\eta^T[\exp\mathcal{T}(T-\tau)]\,|\,\Im(\sigma)|$ the non-linear part of system "object (15) regulator (20)" must reply condition

$$\| \Im(x, a(x,t), \beta(x,t)) \| \le \le \| \eta^T [\exp \mathcal{T}(T-t)] \|, \; t \in [0,T]. \quad (32)$$

From the conditions of exponential decreasing of the integration element (32) and the executions of d-robust condition demands produced to matrix

$$\mathcal{T} = A + \alpha^* - \left[B_1 + \beta^*\right] R^{-1} \left[B_1 + \beta^*\right]^T S,$$

in carrying out which task of d-robust control will be fulfilled successful can be formulated.

The received result will be formulated in the theorem.

Theorem

In the task of control of the non-linear not stationary object of type

$$\frac{d}{dt} x(t) = \left[A + \alpha(x,t)\right] x(t) + + \left[B_1 + \beta(x,t)\right] u(t) + \Im(x,t),$$

where $u(t) = -R^{-1}\left[B_1 + \beta^*\right] S x(t)$ and the matrix S is the solution of the Rikkati-Lurie equation

$$S\left[A + \alpha^*\right] + \left[A + \alpha^*\right]^T S - - S\left[B_1 + \beta^*\right] R^{-1} \left[B_1 + \beta^*\right]^T S + Q = 0,$$

with the given interval of control, with the given interval of the parametric ambiguity $\alpha(x,t),\ \beta(x,t) \in \Omega$ and with the given area of the possible initial states X_0 of the condition

$$\left\| \frac{d\, \Im(x,t)}{d\,t} \right\| < \| \mathcal{T} \| \, \|\Im(x,t)\|, \; x \ne 0$$

and

$$\| x^*(0) \| \le \| \eta^T [\exp(\mathcal{T} T)] \|^{-1} d - - \left\| \int_0^T [\exp(-\mathcal{T} \tau)] \, \Im(x,t) \, d\tau \right\|$$

are accordingly the necessary and the sufficient conditions of the existence of d-robust control.

Statement

The necessary and the sufficient condition of the existence of the d-robust control for the row of tasks can be provided by fitting assigning of matrixes Q and R in the Rikkati-Lurie equation determining by its solution the matrix of the augmentations of the regulator (20).

This not difficultly to see that the condition (25) can be rewritten in form:

$$\left\| \frac{d\, \Im(\sigma)}{d\,t} \right\| < \left\| A + \alpha^* - - \left[B_1 + \beta^*\right] R^{-1} \left[B_1 + \beta^*\right]^T S(Q,R) \right\| \, \|\Im(\sigma)\|, \; \sigma \ne 0.$$

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