SHIL'NIKOV SADDLE-FOCUS HOMOCLINIC ORBITS IN SINGULARLY PERTURBED SYSTEMS IN DIMENSION HIGHER THAN THREE

Flaviano Battelli
Dipartimento di Scienze Matematiche
Marche Polytechnic University
60100 Ancona - Italy
battelli@dipmat.univpm.it

Kenneth J. Palmer
Department of Mathematics
National Taiwan University
Taipei 106 - Taiwan
palmer@math.ntu.edu.tw

Abstract
We consider a singularly perturbed system depending on two parameters with a normally hyperbolic centre manifold. We assume that the unperturbed system has a homoclinic orbit connecting a hyperbolic fixed point on the centre manifold. We give conditions concerning the persistence of this connecting orbit and apply the result to construct a class of singularly perturbed systems in R^{m+2} which possess Shilnikov saddle-focus homoclinic orbits.

Key words
Bifurcations, Chaos, Nonlinear systems

1 Introduction

In this talk we consider a singularly perturbed system like:

$$\begin{cases} \dot{x} = \varepsilon f(x, y, \lambda, \varepsilon) \\ \dot{y} = g(x, y, \lambda, \varepsilon) \end{cases} \tag{1}$$

where $x \in \mathbf{R}^2$, $y \in \mathbf{R}^m$, λ and ε are small real parameters and $f(x, y, \lambda, \varepsilon)$, $g(x, y, \lambda, \varepsilon)$ are C^r–functions in their arguments bounded with their derivatives, $r \geq 1$. We suppose that the following conditions hold:

(i) for any $x \in \mathbf{R}^2$, the equation

$$g(x, y, 0, 0) = 0$$

has a solution $y = v(x) \in C^r_b(\mathbf{R}^2)$ (i.e. $v(x)$ and its first r derivatives are bounded on $\mathbf{R}^2$),

(ii) there exists $\delta_0 > 0$ such that, for any $x \in \mathbf{R}^2$, the eigenvalues $\lambda(x)$ of $g_y(x, v(x), 0, 0)$ satisfy $|\text{Re}\lambda(x)| > \delta_0$

(iii) the equation on the centre manifold

$$\dot{x} = F(x) := f(x, v(x), 0, 0)$$

has an unstable focus ξ_0. We denote with $\mu \pm i\omega$ (with $\mu, \omega > 0$) the eigenvalues of the Jacoban matrix $F'(\xi_0)$.

(iv) the equation

$$\dot{y} = g(\xi_0, y, 0, 0)$$

has a solution $\gamma_0(t)$ satisfying $\gamma_0(t) \to v(\xi_0)$ as $|t| \to \infty$ (homoclinic orbit) and $\dot{\gamma}_0(t)$ is the unique bounded solution, up to a multiplicative constant, of the variational system $\dot{y} = g_y(\xi_0, \gamma_0(t), 0)y$.

(v) let $\psi(t)$ be the unique (up to a multiplicative factor) bounded solution of the adjoint system

$$\dot{y} + g^*_y(\xi_0, \gamma_0(t), 0, 0)y = 0.$$

Then the following generic condition holds:

$$\int_{-\infty}^{\infty} \psi^*(t) g_x(\xi_0, \gamma_0(t), 0, 0)y \neq 0.$$

Conditions (i) and (ii) imply the existence of a centre manifold $y = v(x, \lambda, \varepsilon)$ for the perturbed system together with their associated centre–stable and centre–unstable manifolds. Condition (iii) implies that the system on the perturbed centre manifold:

$$\dot{x} = F(x, \lambda, \varepsilon) := f(x, v(x, \lambda, \varepsilon), \lambda, \varepsilon) \tag{2}$$

has a hyperbolic fixed point $\xi_0(\lambda, \varepsilon)$ and

$$q(\lambda, \varepsilon) = (\xi_0(\lambda, \varepsilon), v(\xi_0(\lambda, \varepsilon), \lambda, \varepsilon)$$

is a hyperbolic fixed point of system (1). Condition (iv) is a kind of nondegenerateness condition which is automatically satisfied when (as we will assume in this paper) $g_y(x, v(x), 0)$ has a simple negative eigenvalue and all the other eigenvalues have positive real parts. Condition (v) implies that the *centre-stable* and the *centre-unstable* manifold of system (1) intersect *transversally* in a family of solutions which are homoclinic to the centre manifold $y = v(x, \lambda, \varepsilon)$. Here by centre-stable manifold we mean the submanifold of $\mathbf{R}^{m+2}$ consisting of the initial point we have to assign to (1) so that the distance of the corresponding solution to the perturbed centre manifold $y = v(x, \lambda, \varepsilon)$ tends to zero as $t \to \infty$. Centre-unstable manifold has a similar meaning.

Our purpose is *to give a general class of singularly perturbed systems in* $\mathbf{R}^{m+2}$ *which possess Shil'nikov saddle-focus homoclinic orbits*.

To reach this goal we proceed in two steps. Using a result of [Battelli and Palmer, to appear] we find $\lambda = \lambda(\varepsilon)$ such that system (1) with $\lambda = \lambda(\varepsilon)$ has an orbit $p(t, \varepsilon) = (x(t, \varepsilon), y(t, \varepsilon))$ which is homoclinic to the fixed point and, finally, we give a condition so that this homoclinic orbit satisfies the Shil'nikov-Deng conditions (see [Deng, 1993])

The theory of Shil'nikov saddle-focus homoclinic orbits is developed in [Shil'nikov, 1970; Deng, 1993]. Such orbits have been found in special systems (see, for example, [Deng, 1993; Deng and Hines, 2002; Feng and Wiggins; Hastings, 1982]) but not many general classes of systems with such orbits have been found, apart from that of Rodriguez [Rodriguez, 1986] where, however, only three-dimensional systems are studied. On the other hand, in higher dimensions, two extra conditions must be verified.

2 Homoclinic orbits to the fixed point

In the following theorem, we treat two cases: the first where the homoclinic orbit $\gamma_0(t)$ breaks as λ passes through $\lambda = 0$ and a second degenerate case where $\gamma_0(t)$ does not break as λ passes through $\lambda = 0$, so that there is a one-parameter family $y(t, \lambda)$ of homoclinic orbits of $\dot{y} = g(y, \xi_0(\lambda, 0), \lambda, 0)$.

Let α, σ be positive numbers such that $\alpha < \mu$ and $\sigma < \delta_0$. In [Battelli and Palmer to appear] the following theorem has been proved.

Theorem *Let f and g be C^r functions ($r \geq 2$), bounded together with their derivatives and satisfying conditions* (i)-(v). *Suppose also that either the condition*

(vi)

$$\int_{-\infty}^{\infty} \psi^*(t)[g_x(\xi_0, \gamma_0(t), 0, 0)\xi_0'(0) + g_\lambda(\xi_0, \gamma_0(t), 0, 0)]dt \neq 0$$

or the following two conditions

(vii) *the stable and unstable manifolds of the hyperbolic equilibrium $y = v(\xi_0, \lambda, 0)$ of*

$$\dot{y} = g(\xi_0(\lambda), y, \lambda, 0)$$

intersect near $\gamma_0(0)$ so that there is a solution $\gamma_0(t, \lambda) \to v^\pm(\xi_0, \lambda, 0)$ as $t \to \pm\infty$ with $\gamma(0, \lambda)$ depending continuously on λ and $y_0(0, 0) = y_0(0)$;

(viii) *if we denote with $\psi(t, \lambda)$ the unique (up to a multiplicative constant), bounded solution of the adjoint linear system*

$$\dot{y} + g_y^*(\xi_0(\lambda), \gamma_0(t, \lambda), \lambda, 0)y = 0,$$

then the Melnikov function

$$\mathcal{M}(\lambda) = -\int_{-\infty}^{\infty} \psi(t, \lambda)^* \Big\{ g_\varepsilon(\xi_0(\lambda), \gamma_0(t), \lambda), \lambda, 0) + g_x(\xi_0(\lambda), \gamma_0(t, \lambda), \lambda, 0) \cdot \left(\int_t^{\infty} f(\xi_0(\lambda), \gamma_0(\tau, \lambda), \lambda, 0)d\tau - \frac{\partial \xi_0}{\partial \varepsilon}(\lambda, 0) \right) \Big] \Big\} dt$$

has a simple zero at $\lambda = 0$

hold. Then there exists a C^{r-1}–function (C^{r-2} in the second case) $\lambda(\varepsilon)$ with $\lambda(0) = 0$ such that for ε sufficiently small and nonnegative, system (1) *with $\lambda = \lambda(\varepsilon)$ has a homoclinic solution $p(t, \varepsilon) = (x(t, \varepsilon), y(t, \varepsilon))$, that is,*

$$p(t, \varepsilon) \neq q^\pm(\lambda(\varepsilon), \varepsilon)$$

but $p(t, \varepsilon) \to q^\pm(\lambda(\varepsilon), \varepsilon)$ as $t \to \pm\infty$. Moreover $p(t, 0) = (\xi_0, \gamma_0(t))$, and

$$\begin{array}{l} \sup_{t \in \mathbf{R}_\pm} |x(t, \varepsilon) - \xi_0^\pm(\lambda(\varepsilon), \varepsilon)| e^{\varepsilon \alpha t} = O(\varepsilon), \\ \sup_{t \in \mathbf{R}} |y(t, \varepsilon) - \gamma_0(t)| = O(\varepsilon). \end{array} \tag{3}$$

Finally, $\dot{p}(t, \varepsilon)$ is not in the tangent space to the unstable fibre through $p(t, \varepsilon)$, provided that

$$\int_{-\infty}^{\infty} f(\xi_0, \gamma_0(t), 0, 0)dt \neq 0, \tag{4}$$

where, according to Theorem 3 *in [Battelli and Palmer, to appear], vectors in the tangent space to the unstable fibre at $p(t, \varepsilon)$ are the initial values of the solutions of the variational system along $p(t, \varepsilon)$ which approach zero as $t \to -\infty$ at an exponential rate greater than σ.*

3 Shil'nikov-Deng condition

Here we recall the definition of saddle-focus homoclinic orbit as given in [Deng and Hines, 2002]. Let $\dot{z} = F(z)$ be an autonomous system. The conditions for Shil'nikov chaos are:

(D1) q is an equilibrium such that the eigenvalues of $F'(q)$ having the smallest positive real part are $\mu \pm i\omega$ with $\omega > 0$ and

$$0 < \mu < -\mathrm{Re}(\lambda)$$

for all eigenvalues λ with negative real parts;

(D2) there is a homoclinic orbit $p(t)$ to q, that is, $p(t) \neq q$ and $p(t) \in \mathcal{W}^s \cap \mathcal{W}^u$ ($\mathcal{W}^s$, $\mathcal{W}^u$ denote the stable and unstable manifolds of q), such that

$$\dim T_{p(t)}\mathcal{W}^s \cap T_{p(t)}\mathcal{W}^u = 1.$$

(D3) as $t \to -\infty$, $p(t)$ is asymptotically tangent to the linear span of the eigenvectors of $\mu \pm i\omega$;

(D4) there is a submanifold $\mathcal{M}_0$ of $\mathcal{W}^u$ containing $p(0)$ with $\dim\mathcal{M}_0 = \dim\mathcal{W}^{uu}$ such that

$$\lim_{t\to\infty} T_{p(t)}\mathcal{M}_t = T_q\mathcal{W}^{uu},$$

where $\mathcal{M}_t = \phi(t, \mathcal{M}_0)$ and $\mathcal{W}^{uu}$ is the *strong unstable manifold* of the equilibrium q that is a locally invariant manifold containing q whose tangent space at q consists of the sum of the generalized eigenspaces of $F'(q)$ corresponding to the eigenvalues with real part greater than μ.

Conditions (D1), (D2) are the only conditions needed in $\mathbf{R}^3$, although in $\mathbf{R}^3$ the second part of (D2) is automatically satisfied. In higher dimensions, we have to add conditions (D3) and (D4). Note that solutions of (1) starting in $\mathcal{W}^{uu}$ approach q as $t \to -\infty$ at an exponential rate faster than μ.

If there is such a homoclinic orbit, Shil'nikov and Deng show the presence of chaotic dynamics near it.

Assuming the conditions of Theorem 1 hold, we take $z = (x, y)$, $F(z) = (\varepsilon f(x, y, \lambda(\varepsilon), \varepsilon), g(x, y, \lambda(\varepsilon), \varepsilon))$, $q = q(\lambda(\varepsilon), \varepsilon)$ and $p(t) = p(t, \varepsilon)$. The Jacobian matrix $F'(q)$ is

$$\begin{pmatrix} \varepsilon f_x(q(\lambda(\varepsilon),\varepsilon)) & \varepsilon f_y(q(\lambda(\varepsilon),\varepsilon)) \\ g_x(q(\lambda(\varepsilon),\varepsilon)) & g_y(q(\lambda(\varepsilon),\varepsilon)) \end{pmatrix}$$

and has the eigenvalues $\varepsilon(\mu \pm i\omega + O(\varepsilon))$ and $\pm\lambda + O(\varepsilon)$. Thus (D1) is satisfied, if $\varepsilon > 0$ is sufficiently small. Next, since $\dim \mathcal{W}^s$ equals the number of eigenvalues with negative real parts and we only have one such eigenvalue, we see that (D2) is satisfied. Next (D3) is equivalent to saying that $\dot{p}(t, \varepsilon)$ is not in the tangent space to the unstable fibre through $p(t, \varepsilon)$ and this follows from Theorem 1, provided condition (4) holds. Thus we only have to check that (D4) is satisfied. It turns out this is equivalent to the condition: $\dim(T_{p(0,\varepsilon)}\mathcal{W}^u \cap W^{cs}) = 2$ where $W^{cs} = T_{p(0,\varepsilon)}\mathcal{M}^{cs}$ is the tangent space to the centre stable manifold. We show that $T_{p(0,\varepsilon)}\mathcal{W}^u = T_{p(0,\varepsilon)}\mathcal{M}^{cu}$, the tangent space to the centre unstable manifold. For this, it suffices to show that $T_{p(0,\varepsilon)}\mathcal{W}^u \subset T_{p(0,\varepsilon)}\mathcal{M}^{cu}$ since both subspaces have the same dimension. However it follows from [Battelli and Palmer, to appear] that $T_{p(0,\varepsilon)}\mathcal{M}^{cu}$ consists of the initial values of solutions of the variational system which do not grow at too high an exponential rate as $t \to -\infty$. However, all the solutions beginning in $T_{p(0,\varepsilon)}\mathcal{W}^u$ tend to zero as $t \to -\infty$. So the inclusion follows. Then (D4) follows, since (vi) implies that $\mathcal{M}^{cs}$ and $\mathcal{M}^{cu}$ intersect transversely at $p(0, \varepsilon)$ (see [Battelli and Palmer, 2001]). So we obtain the following

Theorem. *Assume that conditions (i)–(iv), equation (4) and that either (v)-(vi) or (vi)–(viii) hold. Then system (1) has a Shil'nikov saddle focus homoclinic orbit with the induced chaotic behaviour.*

4 An example

We consider the following system in $\mathbf{R}^5$

$$\begin{cases} \dot{x} = \varepsilon f(x, y, \varepsilon) + \varepsilon^2 f_1(x, y, z, \lambda, \varepsilon) \\ \dot{y} = g(x, y, z) + \lambda g_1(x, y, z, \varepsilon) \\ \dot{z} = z h(z) + \varepsilon h_1(x, y, z, \lambda, \varepsilon) \end{cases} \tag{5}$$

where $x, y \in \mathbf{R}^2$ and $z \in \mathbf{R}$. We assume that $f(x, y, \varepsilon)$, $f_1(x, y, z, \lambda, \varepsilon)$, $g(x, y, z)$, $g_1(x, y, z, \lambda, \varepsilon)$, $h(z)$ and $h_1(x, y, z, \lambda, \varepsilon)$ are C^2-functions bounded together with their derivatives and the following conditions hold:

1) $g(x, 0, 0) = 0$, $g_1(x, 0, 0, 0) = 0$

Taking x as *slow variable* and (y, z) as *fast variable*, it follows from 1) that when $\varepsilon = \lambda = 0$ system (5) has the centre manifold $(y, z) = v(x) = 0$. Moreover, according to 1), this centre manifold persists when $\lambda \neq 0$, that is $v(x, \lambda, 0) = 0$.

Then we assume

2) $h(0) > 0$ and $\dot{y} = g(0, y, 0)$ has a homoclinic orbit $y_0(t)$ to the fixed point $y = 0$ and for any $x \in \mathbf{R}^2$, $g_y(x, 0, 0)$ has the eigenvalues $\pm\alpha(x)$ with $\alpha(x) > \alpha > 0$.

Conditions 1) and 2) imply that (ii) is satisfied. Moreover 2) implies that the fast system

$$\begin{cases} \dot{y} = g(0, y, z) \\ \dot{z} = z h(z) \end{cases} \tag{6}$$

has the fixed point $(y, z) = (0, 0)$ and the homoclinic orbit to it: $\gamma_0(t) = (y_0(t), 0)$. Moreover the Jacobian matrix of (6) at $(y, z) = (0, 0)$ has two positive eigenvalues ($\alpha(0)$ and $h(0)$) and the negative eigenvalue $-\alpha(0)$. Thus the intersection of the stable and unstable manifold of $(y, z) = (0, 0)$ is one–dimensional and then condition (iv) is satisfied.

Next we assume

3) $x = 0$ is a focus of equation $\dot{x} = f(x, 0, 0)$, that is $f(0, 0, 0) = 0$ and $f_x(0, 0, 0)$ has the eigenvalues $\mu + i\omega$, $\mu, \omega > 0$.

Hence condition (iii) is satisfied. To apply our Theorem we need that conditions (v), (vi) and (4) are satisfied. Since when $\varepsilon = 0$ the equation on the centre manifold $(y, z) = v(x, \lambda, 0) = 0$ is always $\dot{x} = f(x, 0, 0)$, we see that $\xi_0(\lambda, 0) = 0$. Hence (v) and (4) read respectively:

$$\int_{-\infty}^{\infty} \psi^*(t) g_1(0, y_0(t), 0, 0) dt \neq 0 \tag{7}$$

and

$$\int_{-\infty}^{\infty} f(0, y_0(t), 0) dt \neq 0. \tag{8}$$

Thus we obtain the following:

Proposition 1. *Assume $f(x, y, z, \lambda, \varepsilon)$, $g(x, y, z)$, $g_1(x, y, z, \varepsilon)$, $h(z)$ and $h_1(x, y, z, \lambda, \varepsilon)$ are C^2- functions bounded with their derivatives and that conditions 1)–3) and (7), (8) hold together with*

$$\int_{-\infty}^{\infty} \psi^*(t) g_x(0, y_0(t), 0) dt \neq 0.$$

Then system (5) has a Shil'nikov saddle–focus orbit with the induced chaotic behaviour.

References

Battelli, F. and Palmer, K. J., Heteroclinic connections in singularly perturbed systems, *Disc. Cont. Dynam. Sys.*, to appear

Battelli, F. and Palmer, K. J., Transverse intersection of invariant manifolds in singular systems, *J. Diff. Equations*, **177**, (2001), 77-120

Battelli, F. and Palmer, K. J., Connections to fixed points and Shil'nikov saddle-focus homoclinic orbits in singularly perturbed systems, *Ukrainian Mathematical Journal*, to appear

Deng, B., On Shil'nikov's Homoclinic-Saddle-Focus Theorem, *J. Diff. Equations* **102**, (1993), 305-329

Deng, B. and Hines, G. Food chain chaos due to Shil'nikov's orbits, *Chaos* **12**, (2002), 533-538.

Feng, Z.C. and Wiggins, S., On the existence of chaos in a class of two degree of freedom, damped, parametrically forced mechanical systems with broken $O(2)$ symmetry, Z- Angew. Math. Phys. **44**, (1993), 201-248

Hastings, S. P., Single and multiple pulse waves for FitzHugh Nagumo equation, *SIAM J. Appl. Math.*, **42** (1982), 247-260.

Rodriguez, J. A., Bifurcation to homoclinic connections of the focus-saddle type, Arch. Rat. Mech. Anal. **93**, (1986), 81-90.

Shil'nikov, L. P., A contribution to the problem of the structure of an extended neighborhood of a rough equilibrium state of saddle-focus type, Math. USSR Sbornik **10**, (1970), 91-102