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ON ANALYSIS OF STOCHASTIC SYSTEMS WITH DELAYS

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Abstract

Problems concerned with an analysis of stochastic differential equations with various forms of delays and fluctuations are considered. There are nonlinear difference-differential and linear neutral delay differential equations with multiple constant lags and linear equations with a variable delay perturbed by continuous fluctuations and linear parametric system under white noise and Poisson excitations among them. The main idea of study consists of an extension of the phase space. Chains of deterministic equations without delays satisfied by moments of phase vectors with increasing length are presented. At the end of paper the technique is applied to study a sensitivity of linear stochastic system response to deterministic parameters.

Key words

Stochastic systems, delay, moments, computer algebra.

1 Introduction

Difference-differential equations (DDEqs) [Bellman, Cooke, 1962; Hale, 1977] have been attracting an increased interest both from theoretical and practical viewpoints since the middle of the last century. Such equations are encountered in those areas where the properties of an object depend on the hereditary effect, and serve as models for different processes, viz., automatic control for technical devices and engineering procedures, development of economic and social systems, combustion in liquid jet engines, neutron moderation, effects of radiations, a radio-location, radar and radio-navigation, autonomous vessel course stabilization, oscillations in vacuum-tube generators, struggle for existence in biology, etc.

Such phenomena arise as a result of transport, technological, information, and inertial delays (in long-distance transmission of matter, energy, signals, information), finiteness of speed of charge carriers, and a lag of response delay in man-machine systems. Delays in systems induce new effects, for example, self-excitation of oscillations, increased readjustment, and instability of objects, etc.

As developments of methods for deterministic systems have become important for theory and practice as nowadays significant interest is paid to stochastic DDEqs (SDDEqs) of various types.

Our scheme for study of such systems is based on an extension of the phase space [Poloskov, 2002]. We apply this scheme to nonlinear stochastic difference-differential equations with multiple constant delays [Poloskov, 2006] (Section 2). An example (Section 3) shows the scheme afoot. A tool in our calculations is the computer algebra system *Mathematica* [Wolfram, 2003], a well-known powerful instrument for different sciences. Some other recent results are examined in Section 4.

2 Systems with Multiple Constant Delays

Let us consider a system of the Stratonovich SDDEqs

$$\begin{aligned} &\dot{\boldsymbol{x}}(t) = \boldsymbol{f}_\nu(\boldsymbol{x}(t), \boldsymbol{x}_\tau(t), \boldsymbol{x}_{2\tau}(t), ..., \boldsymbol{x}_{\nu\tau}(t), t) + \\ &\quad + G_\nu(\boldsymbol{x}(t), \boldsymbol{x}_\tau(t), \boldsymbol{x}_{2\tau}(t), ..., \boldsymbol{x}_{\nu\tau}(t), t)\, \boldsymbol{\xi}(t), \quad (1) \\ &t > t_\nu = t_0 + \nu\,\tau. \end{aligned}$$

Here $\boldsymbol{x} \in \mathbb{R}^n$ is the phase vector, $\boldsymbol{\xi} \in \mathbb{R}^m$ is a vector of independent Gaussian white noises ($\mathbf{M}[\boldsymbol{\xi}(t)] = 0$, $\mathbf{M}[\boldsymbol{\xi}(t)\boldsymbol{\xi}^T(t')] = \mathsf{E} \cdot \delta(t - t')$), τ is a constant delay, $\nu > 0$ is an integer, $\boldsymbol{f} = \{f_i\}^T : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}^n$ and $G = \{g_{ij}\} : \mathbb{R}^n \times [t_0, \infty) \to \mathbb{R}^n \times \mathbb{R}^m$ are deterministic vector- and matrix-function respectively, $\boldsymbol{x}_{q\tau} = \boldsymbol{x}_{q\tau}(t) = \boldsymbol{x}(t - q\tau)$, T is a symbol of the transposition, $\mathbf{M}$ stands for the mathematical expectation, E is the identity matrix.

We suppose that the phase vector $\boldsymbol{x}(t)$ being characterized by the probability density function (PDF) $p(\boldsymbol{x}, t)$ on the intervals$(t_0, t_1]$, $(t_1, t_2]$, ..., $(t_{\nu-1}, t_\nu]$ satisfies the following systems of stochastic differential equations (SDEqs)

$$\dot{\boldsymbol{x}} = \boldsymbol{f}_0(\boldsymbol{x},t) + G_0(\boldsymbol{x},t)\,\boldsymbol{\xi}(t), \tag{2}$$

$$\dot{\boldsymbol{x}} = \boldsymbol{f}_1(\boldsymbol{x},\boldsymbol{x}_\tau,t) + G_1(\boldsymbol{x},\boldsymbol{x}_\tau,t)\,\boldsymbol{\xi}(t), \tag{3}$$

$$\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots$$

$$\dot{\boldsymbol{x}} = \boldsymbol{f}_{\nu-1}(\boldsymbol{x},\boldsymbol{x}_\tau,\boldsymbol{x}_{2\tau},...,\boldsymbol{x}_{(\nu-1)\tau},t) + \tag{4}$$

$$+G_{\nu-1}(\boldsymbol{x},\boldsymbol{x}_\tau,\boldsymbol{x}_{2\tau},...,\boldsymbol{x}_{(\nu-1)\tau},t)\,\boldsymbol{\xi}(t),$$

$$\boldsymbol{f}_q = \{f_{qi}\}^T, \quad G_q = \{g_{qij}\},$$

$$q = 1,2,...,\nu-1.$$

Let's assume that the PDF of $\boldsymbol{x}$ is equal to $\bar{p}^0(\boldsymbol{x})$ at $t = t_0$.

If to look at Eqs (1)-(4) from the point of view of general theory for stochastic processes, one can draw a conclusion that the random vectors $\boldsymbol{x}$, which satisfy these Eqs, aren't the Markovian vector random processes due to presence of delay. Hence to calculate probabilistic characteristics of the vectors $\boldsymbol{x}$ such as the mean value vector, the matrix of covariances etc., the well-known analytical apparatus of the Markovian processes [Dimentberg, 1980; Gardiner, 1985; Risken, 1996] based on the Fokker–Planck–Kolmogorov Eqs (FPK Eqs) [Malanin and Poloskov, 2001; Malanin and Poloskov, 2005] can not be applied.

To study a random change of the vector $\boldsymbol{x}(t)$ for $t > t_0$, we use our scheme for analysis of different SDEqs with delay which is based on the idea of a transformation of the non-Markov vector process to a Markovian one. For this purpose, we expand the phase space of the system and introduce the following notation:

$$s \in [0,\tau], \quad t_q = t_0 + q\cdot\tau, \quad q = 0,1,2,...,$$

$$s_q = s + t_q, \quad \boldsymbol{x}_q(s) = \boldsymbol{x}(s_q), \quad \boldsymbol{\xi}_q(s) = \boldsymbol{\xi}(s_q),$$

$$p_q(\boldsymbol{x}_q,s) = p(\boldsymbol{x}_q,s_q), \quad p_0(\boldsymbol{x}_0,0) = \bar{p}^0(\boldsymbol{x}_0),$$

$$\Delta_q = [t_{q-1},t_q], \quad \boldsymbol{z}_0 = \boldsymbol{x}_0, \quad \boldsymbol{z}_1 = \mathrm{col}(\boldsymbol{x}_1,\boldsymbol{x}_0),$$

$$\boldsymbol{z}_2 = \mathrm{col}(\boldsymbol{x}_2,\boldsymbol{x}_1,\boldsymbol{x}_0),..., \quad \boldsymbol{\xi}_q(0) = \boldsymbol{\xi}_{q-1}(\tau),$$

$$\boldsymbol{y}_q \equiv \boldsymbol{x}_q(0) = \boldsymbol{x}_{q-1}(\tau), \;\; p_q(\boldsymbol{x}_q,0) = p_{q-1}(\boldsymbol{x}_q,\tau),$$

$$\mathrm{col}(\boldsymbol{x}_N,\boldsymbol{x}_{N-1},...,\boldsymbol{x}_0) = \{x_{N1},x_{N2},...,x_{Nn},$$

$$x_{N-1,1},x_{N-1,2},...,x_{N-1,n},...,x_{01},x_{02},...,x_{0n}\}^T.$$

Using this notation, we construct a chain of FPK-like Eqs for the PDFs of the vectors $\boldsymbol{z}_0$, $\boldsymbol{z}_1$, $\boldsymbol{z}_2$, ..., $\boldsymbol{z}_N$, ... belonging to the family of embedded phase spaces $\mathbb{R}^n \subset \mathbb{R}^{2n} \subset \mathbb{R}^{3n} \subset ... \subset \mathbb{R}^{n(N+1)} \subset ...$.

Let's consider a sequence of segments $\{\Delta_i\}$.

0^0. Let's start from the segment Δ_0. The random vector $\boldsymbol{x}_0(s)$ defined on Δ_0 satisfies the system

$$\dot{\boldsymbol{x}}_0(s) = \boldsymbol{f}_0(\boldsymbol{x}_0(s),s_0) + G_0(\boldsymbol{x}_0(s),s_0)\,\boldsymbol{\xi}_0(s). \tag{5}$$

The PDF $p_0(\boldsymbol{z}_0,t)$ of the vector $\boldsymbol{z}_0(t)$ is governed by the equation

$$\frac{\partial p_0}{\partial s} = \mathbf{L}_0 p_0, \tag{6}$$

where

$$\mathbf{L}_0 p_0 = \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2(b^*_{0ij}p_0)}{\partial z_{0i}\partial z_{0j}} - \sum_{i=1}^{n}\frac{\partial(a^*_{0i}p_0)}{\partial z_{0i}},$$

$$a^*_{0i} = f^*_{0i} + \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{m}\frac{\partial g^*_{0ik}}{\partial z_{0j}}g^*_{0jk},$$

$$b^*_{0ij} = \sum_{k=1}^{m} g^*_{0ik}g^*_{0jk},$$

$$\boldsymbol{f}^*_0(\boldsymbol{z}_0,s) = \boldsymbol{f}_0(\boldsymbol{z}_0,s_0), \quad G^*_0(\boldsymbol{z}_0,s) = G_0(\boldsymbol{z}_0,s_0).$$

1^0. Let's consider the intervals Δ_0 and Δ_1. It is possible to present the system of SDEqs for calculation of the vector $\mathrm{col}(\boldsymbol{x}_1,\boldsymbol{x}_0)$ as follows

$$\dot{\boldsymbol{x}}_0(s) = \boldsymbol{f}_0(\boldsymbol{x}_0(s),s_0) + G_0(\boldsymbol{x}_0(s),s_0)\,\boldsymbol{\xi}_0(s),$$

$$\dot{\boldsymbol{x}}_1(s) = \boldsymbol{f}_1(\boldsymbol{x}_1(s),\boldsymbol{x}_0(s),s_1) + \tag{7}$$

$$+G_1(\boldsymbol{x}_1(s),\boldsymbol{x}_0(s),s_1)\,\boldsymbol{\xi}_1(s).$$

Therefore the PDF for the vector $\boldsymbol{z}_1$ satisfies the FPK Eq

$$\frac{\partial p_1}{\partial s} = \mathbf{L}_1 p_1, \tag{8}$$

with

$$\mathbf{L}_1 p_1 = \frac{1}{2}\sum_{i,j=1}^{2n}\frac{\partial^2(b^*_{1ij}p_1)}{\partial z_{1i}\partial z_{1j}} - \sum_{i=1}^{2n}\frac{\partial(a^*_{1i}p_1)}{\partial z_{1i}},$$

$$a^*_{1i} = f^*_{1i} + \frac{1}{2}\sum_{j=1}^{2n}\sum_{k=1}^{2m}\frac{\partial g^*_{1ik}}{\partial z_{1j}}g^*_{1jk},$$

$$b^*_{1ij} = \sum_{k=1}^{2m} g^*_{1ik}g^*_{1jk},$$

$$\boldsymbol{f}^*_1(\boldsymbol{z}_1,s) = \mathrm{col}\big(\boldsymbol{f}_0(\boldsymbol{x}_0,s_0),\boldsymbol{f}_1(\boldsymbol{x}_1,\boldsymbol{x}_0,s_1)\big),$$

$$G^*_1(\boldsymbol{z}_1,s) = \mathrm{diag}\big(G_0(\boldsymbol{x}_0,s_0),G_1(\boldsymbol{x}_1,\boldsymbol{x}_0,s_1)\big).$$

$$\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots$$

ν^0. Now let's pay attention to the time intervals Δ_0, Δ_1, ..., Δ_ν and construct the set of SDEqs for the vector $\boldsymbol{z}_\nu$ by the way

$$\dot{\boldsymbol{x}}_0(s) = \boldsymbol{f}_0(\boldsymbol{x}_0(s),s_0) + G_0(\boldsymbol{x}_0(s),s_0)\boldsymbol{\xi}_0(s),$$

$$\dot{\boldsymbol{x}}_1(s) = \boldsymbol{f}_1(\boldsymbol{x}_1(s),\boldsymbol{x}_0(s),s_1) +$$

$$+G_1(\boldsymbol{x}_1(s),\boldsymbol{x}_0(s),s_1)\boldsymbol{\xi}_1(s),$$

$$\dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots \tag{9}$$

$$\dot{\boldsymbol{x}}_{\nu-1}(s)=\boldsymbol{f}_{\nu-1}(\boldsymbol{x}_{\nu-1}(s),...,\boldsymbol{x}_0(s),s_{\nu-1})+$$

$$+G_{\nu-1}(\boldsymbol{x}_{\nu-1}(s),...,\boldsymbol{x}_0(s),s_{\nu-1})\boldsymbol{\xi}_{\nu-1}(s),$$

$$\dot{\boldsymbol{x}}_\nu(s)=\boldsymbol{f}_\nu(\boldsymbol{x}_\nu(s),\boldsymbol{x}_{\nu-1}(s),...,\boldsymbol{x}_0(s),s_\nu)+$$

$$+G_\nu(\boldsymbol{x}_\nu(s),\boldsymbol{x}_{\nu-1}(s),...,\boldsymbol{x}_0(s),s_\nu)\boldsymbol{\xi}_\nu(s).$$

The PDF of $\boldsymbol{z}_\nu$ is governed by the equation

$$\frac{\partial p_\nu}{\partial s}=\mathbf{L}_\nu p_\nu, \tag{10}$$

where

$$\mathbf{L}_\nu p_\nu=\frac{1}{2}\sum_{i,j=1}^{n(\nu+1)}\frac{\partial^2(b^*_{\nu ij}p_\nu)}{\partial z_{\nu i}\partial z_{\nu j}}-\sum_{i=1}^{n(\nu+1)}\frac{\partial(a^*_{\nu i}p_\nu)}{\partial z_{\nu i}},$$

$$a^*_{\nu i}=f^*_{\nu i}+\frac{1}{2}\sum_{j=1}^{n(\nu+1)}\sum_{k=1}^{m(\nu+1)}\frac{\partial g^*_{\nu ik}}{\partial z_{\nu j}}g^*_{\nu jk},$$

$$b^*_{\nu ij}=\sum_{k=1}^{m(\nu+1)}g^*_{\nu ik}g^*_{\nu jk},$$

$$\boldsymbol{f}^*_\nu(\boldsymbol{z}_\nu,s)=\mathrm{col}\big(\boldsymbol{f}_0(\boldsymbol{x}_0,s_0),\boldsymbol{f}_1(\boldsymbol{x}_1,\boldsymbol{x}_0,s_1),$$

$$...,\boldsymbol{f}_\nu(\boldsymbol{x}_\nu,\boldsymbol{x}_{\nu-1},...,\boldsymbol{x}_0,s_\nu)\big),$$

$$G^*_\nu(\boldsymbol{z}_\nu,s)=\mathrm{diag}\big(G_0(\boldsymbol{x}_0,s_0),G_1(\boldsymbol{x}_1,\boldsymbol{x}_0,s_1),$$

$$...,G(\boldsymbol{x}_\nu,\boldsymbol{x}_{\nu-1},...,\boldsymbol{x}_0,s_\nu)\big).$$

$$\dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots$$

N^0. At this stage we consider the segments Δ_0, Δ_1, ..., Δ_N and obtain the set of SDEqs for the vector $\boldsymbol{z}_N$

$$\dot{\boldsymbol{x}}_0(s)=\boldsymbol{f}_0(\boldsymbol{x}_0(s),s_0)+G_0(\boldsymbol{x}_0(s),s_0)\boldsymbol{\xi}_0(s),$$

$$\dot{\boldsymbol{x}}_1(s)=\boldsymbol{f}(\boldsymbol{x}_1(s),\boldsymbol{x}_0(s),s_1)+$$

$$+G(\boldsymbol{x}_1(s),\boldsymbol{x}_0(s),s_1)\boldsymbol{\xi}_1(s),$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dot{\boldsymbol{x}}_{\nu-1}(s)=\boldsymbol{f}_{\nu-1}(\boldsymbol{x}_{\nu-1}(s),...,\boldsymbol{x}_0(s),s_{\nu-1})+$$

$$+G_{\nu-1}(\boldsymbol{x}_{\nu-1}(s),...,\boldsymbol{x}_0(s),s_{\nu-1})\boldsymbol{\xi}_{\nu-1}(s),$$

$$\dot{\boldsymbol{x}}_\nu(s)=\boldsymbol{f}_\nu(\boldsymbol{x}_\nu(s),\boldsymbol{x}_{\nu-1}(s),...,\boldsymbol{x}_0(s),s_\nu)+ \tag{11}$$

$$+G_\nu(\boldsymbol{x}_\nu(s),\boldsymbol{x}_{\nu-1}(s),...,\boldsymbol{x}_0(s),s_\nu)\boldsymbol{\xi}_\nu(s),$$

$$\dot{\boldsymbol{x}}_{\nu+1}(s)=\boldsymbol{f}_\nu(\boldsymbol{x}_{\nu+1}(s),\boldsymbol{x}_\nu(s),...,\boldsymbol{x}_1(s),s_{\nu+1})+$$

$$+G_\nu(\boldsymbol{x}_{\nu+1}(s),\boldsymbol{x}_\nu(s),...,\boldsymbol{x}_1(s),s_{\nu+1})\boldsymbol{\xi}_{\nu+1}(s),$$

$$\dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots$$

$$\dot{\boldsymbol{x}}_{N-1}(s)=\boldsymbol{f}_\nu(\boldsymbol{x}_{N-1}(s),...,\boldsymbol{x}_{N-\nu-1},s_{N-1})+$$

$$+G_\nu(\boldsymbol{x}_{N-1}(s),...,\boldsymbol{x}_{N-\nu-1}(s),s_{N-1})\boldsymbol{\xi}_{N-1}(s),$$

$$\dot{\boldsymbol{x}}_N(s)=\boldsymbol{f}_\nu(\boldsymbol{x}_N(s),\boldsymbol{x}_{N-1}(s),...,\boldsymbol{x}_{N-\nu}(s),s_N)+$$

$$+G_\nu(\boldsymbol{x}_N(s),\boldsymbol{x}_{N-1}(s)...,\boldsymbol{x}_{N-\nu}(s),s_N)\boldsymbol{\xi}_N(s).$$

In this case the PDF of the random vector $\boldsymbol{z}_N$ satisfies the equation

$$\frac{\partial p_N}{\partial s}=\mathbf{L}_N p_N. \tag{12}$$

This Eq includes the FPK operator

$$\mathbf{L}_N p_N=\frac{1}{2}\sum_{i,j=1}^{n(N+1)}\frac{\partial^2(b^*_{Nij}[\cdot])}{\partial z_{Ni}\partial z_{Nj}}-\sum_{i=1}^{n(N+1)}\frac{\partial(a^*_{Ni}[\cdot])}{\partial z_{Ni}}$$

with coefficients of drift

$$a^*_{Ni}=f^*_{Ni}+\frac{1}{2}\sum_{j=1}^{n(N+1)}\sum_{k=1}^{m(N+1)}\frac{\partial g^*_{Nik}}{\partial z_j}g^*_{Njk}$$

and diffusion

$$b^*_{Nij}=\sum_{k=1}^{m(N+1)}g^*_{Nik}g^*_{Njk},$$

where

$$\boldsymbol{f}^*_N(\boldsymbol{z}_N,s)=\mathrm{col}\big(\boldsymbol{f}_0(\boldsymbol{x}_0,s_0),\boldsymbol{f}_1(\boldsymbol{x}_1,\boldsymbol{x}_0,s_1),$$

$$...,\boldsymbol{f}_\nu(\boldsymbol{x}_N,\boldsymbol{x}_{N-1},...,\boldsymbol{x}_{N-\nu},s_N)\big),$$

$$G^*_N(\boldsymbol{z}_N,s)=\mathrm{diag}\big(G_0(\boldsymbol{x}_0,s+t_0),G_1(\boldsymbol{x}_1,\boldsymbol{x}_0,s_1),$$

$$...,G_\nu(\boldsymbol{x}_N,\boldsymbol{x}_{N-1},...,\boldsymbol{x}_{N-\nu},s_N)\big).\quad\square$$

To solve our task, let's define the following case for a classic method of steps [Bellman, Cooke, 1962] that is based on the obtained FPK Eqs.

Step 0. The PDF $p_{01}(\boldsymbol{x}_0,s)$ for the vector $\boldsymbol{x}_0(t)$ and the PDF $p_{02}(\boldsymbol{x}_0,s;\boldsymbol{y}_0,0)$ for the extended vector $\boldsymbol{z}_0^+=\mathrm{col}(\boldsymbol{x}_0,\boldsymbol{y}_0)$ satisfy the equation (6) with the initial conditions

$$\begin{aligned}&p_{01}(\boldsymbol{x}_0,0)=\bar{p}^0(\boldsymbol{x}_0),\\&p_{02}(\boldsymbol{x}_0,0;\boldsymbol{y}_0,0)=p_{01}(\boldsymbol{x}_0,0)\,\delta(\boldsymbol{x}_0-\boldsymbol{y}_0)\end{aligned} \tag{13}$$

correspondingly. Then to obtain moments $m^+_{0\alpha}(s)=\mathbf{M}\big[z_0^{+\alpha}\big]=\mathbf{M}\big[z_{01}^{+\alpha_1}z_{02}^{+\alpha_2}...z_{0,2n}^{+\alpha_{2n}}\big]$ of the vector $\boldsymbol{z}_0^+$ ($\alpha=\{\alpha_1,\alpha_2,...,\alpha_{2n}\}$, $\alpha_i\geq 0$, $|\alpha|=\alpha_1+\alpha_2+$

$... + \alpha_{2n} \leq K$) of different orders, it is possible to derive the following ordinary DEqs (ODEqs):

$$\dot{m}^{+}_{0\alpha}(s) = \sum_{i=1}^{2n} \alpha_i \mathbf{M}\big[a^{*}_{0i} z_0^{+\alpha-e_i}\big] + \\ + \frac{1}{2}\sum_{i=1}^{2n} \alpha_i(\alpha_i - 1)\mathbf{M}\big[b^{*}_{0ii} z_0^{+\alpha-2e_i}\big] + \\ + \sum_{i=1}^{2n-1}\sum_{j=i+1}^{2n} \alpha_i\alpha_j \mathbf{M}\big[b^{*}_{0ij} z_0^{+\alpha-e_i-e_j}\big], \tag{14}$$

where the initial conditions are defined by such relations as

$$m^{+}_{0\alpha}(0) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} z_0^{+\alpha} p_{02}(\boldsymbol{x}_0, 0; \boldsymbol{y}_0, 0)\, d\boldsymbol{x}_0\, d\boldsymbol{y}_0 = \\ = \int\limits_{\mathbb{R}^n} x_{01}^{\alpha_1+\alpha_{n+1}} ... x_{0n}^{\alpha_n+\alpha_{2n}}\, p_{01}(\boldsymbol{x}_0, 0)\, d\boldsymbol{x}_0. \tag{15}$$

Here the required moments $m_\beta(t) = \mathbf{M}\big[x^\beta\big]$ ($\beta = \{\beta_1, \beta_2, ..., \beta_n\}$) at $t \in \Delta_0$ are equal to $m^{+}_{0\beta}(t - t_0)$.

Step 1. The PDFs $p_{11}(\boldsymbol{x}_1, \boldsymbol{x}_0, s)$ and $p_{12}(\boldsymbol{x}_1, \boldsymbol{x}_0, s; \boldsymbol{y}_0, 0)$ of the vectors $\boldsymbol{z}_1$ and $\boldsymbol{z}_1^{+} = \mathrm{col}(\boldsymbol{z}_1, \boldsymbol{y}_0)$ are the solutions of Eq (8) under the initial conditions

$$p_{11}(\boldsymbol{x}_1, \boldsymbol{x}_0, 0) = p_{02}(\boldsymbol{x}_1, \tau; \boldsymbol{x}_0, 0) \tag{16}$$

and

$$p_{12}(\boldsymbol{x}_1, \boldsymbol{x}_0, 0; \boldsymbol{y}_0, 0) = p_{11}(\boldsymbol{x}_1, \boldsymbol{x}_0, 0)\, \delta(\boldsymbol{x}_0 - \boldsymbol{y}_0) \tag{17}$$

correspondingly. Therefore we can obtain the following ODEqs for the moments $m^{+}_{1\alpha}(s) = \mathbf{M}\big[\boldsymbol{z}_1^{+\alpha}\big]$ of the expanded phase vector $\boldsymbol{z}_1^{+}$ ($\alpha = \{\alpha_1, \alpha_2, ..., \alpha_{3n}\}$) and the initial conditions

$$\dot{m}^{+}_{1\alpha}(s) = \sum_{i=1}^{3n} \alpha_i \mathbf{M}\big[a^{*}_{1i} \boldsymbol{z}_1^{+\alpha-e_i}\big] + \\ + \frac{1}{2}\sum_{i=1}^{3n} \alpha_i(\alpha_i - 1)\mathbf{M}\big[b^{*}_{1ii} \boldsymbol{z}_1^{+\alpha-2e_i}\big] + \\ + \sum_{i=1}^{3n-1}\sum_{j=i+1}^{3n} \alpha_i\alpha_j \mathbf{M}\big[b^{*}_{1ij} \boldsymbol{z}_1^{+\alpha-e_i-e_j}\big], \tag{18}$$

$$m^{+}_{1\alpha}(0) = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} \boldsymbol{z}_1^{+\alpha} \times \\ \times p_{12}(\boldsymbol{x}_1, \boldsymbol{x}_0, 0; \boldsymbol{y}_0, 0)\, d\boldsymbol{x}_1\, d\boldsymbol{x}_0\, d\boldsymbol{y}_0 = \\ = \int\limits_{\mathbb{R}^n}\int\limits_{\mathbb{R}^n} x_{11}^{\alpha_1} ... x_{1n}^{\alpha_n}\, x_{01}^{\alpha_{n+1}+\alpha_{2n+1}} ... \times \\ \times x_{0n}^{\alpha_{2n}+\alpha_{3n}}\, p_{11}(\boldsymbol{x}_1, \boldsymbol{x}_0, 0)\, d\boldsymbol{x}_1\, d\boldsymbol{x}_0. \tag{19}$$

Then the required moments $m_\beta(t)$ are calculated as $m^{+}_{1\beta}(t - t_1)$ for $t \in \Delta_1$.

... ...

Step N. The main characteristics of the vectors $\boldsymbol{z}_N(s)$ and $\boldsymbol{z}_N^{+} = \mathrm{col}(\boldsymbol{z}_N, \boldsymbol{y}_0)$, i.e. the PDFs $p_{N1}(\boldsymbol{x}_N, \boldsymbol{x}_{N-1}, ..., \boldsymbol{x}_0, s)$ and $p_{N2}(\boldsymbol{x}_0, \boldsymbol{x}_1, ..., \boldsymbol{x}_N, s; \boldsymbol{y}_0, 0)$, satisfy Eq (12) under the initial conditions

$$p_{N1}(\boldsymbol{x}_N, \boldsymbol{x}_{N-1}, ..., \boldsymbol{x}_0, 0) = \\ = p_{N-1,2}(\boldsymbol{x}_N, \boldsymbol{x}_{N-1}, ..., \boldsymbol{x}_1, \tau; \boldsymbol{x}_0, 0) \tag{20}$$

and

$$p_{N2}(\boldsymbol{x}_N, \boldsymbol{x}_{N-1}, ..., \boldsymbol{x}_0, 0; \boldsymbol{y}_0, 0) = \\ = p_{N1}(\boldsymbol{x}_N, \boldsymbol{x}_{N-1}, ..., \boldsymbol{x}_0, 0)\delta(\boldsymbol{x}_0 - \boldsymbol{y}_0). \tag{21}$$

Then at this step, the moments $m^{+}_{N\alpha}(s) = \mathbf{M}\big[\boldsymbol{z}_N^{+\alpha}\big]$ of the vector $\boldsymbol{z}_N^{+}$ ($\alpha = \{\alpha_1, \alpha_2, ..., \alpha_{(N+2)n}\}$) can be found from ODEqs and the initial conditions in the form

$$\dot{m}^{+}_{N\alpha}(s) = \sum_{i=1}^{(N+2)n} \alpha_i \mathbf{M}\big[a^{*}_{Ni} \boldsymbol{z}_N^{+\alpha-e_i}\big] + \\ + \frac{1}{2}\sum_{i=1}^{(N+2)n} \alpha_i(\alpha_i - 1)\mathbf{M}\big[b^{*}_{Nii} \boldsymbol{z}_N^{+\alpha-2e_i}\big] + \\ + \sum_{i=1}^{(N+2)n-1}\sum_{j=i+1}^{(N+2)n} \alpha_i\alpha_j \mathbf{M}\big[b^{*}_{Nij} \boldsymbol{z}_N^{+\alpha-e_i-e_j}\big], \tag{22}$$

$$m^{+}_{N\alpha}(0) = \int\limits_{\mathbb{R}^n} (N+2) \int\limits_{\mathbb{R}^n} \boldsymbol{z}_N^{+\alpha} \times \\ \times p_{N2}(\boldsymbol{x}_N, ..., \boldsymbol{x}_0, 0; \boldsymbol{y}_0, 0)\, d\boldsymbol{x}_N ... d\boldsymbol{x}_0 d\boldsymbol{y}_0 = \\ = \int\limits_{\mathbb{R}^n} (N+1) \int\limits_{\mathbb{R}^n} x_{N1}^{\alpha_1} ... x_{Nn}^{\alpha_n} ... \times \\ \times x_{01}^{\alpha_{Nn+1}+\alpha_{(N+1)n+1}} ... x_{0n}^{\alpha_{(N+1)n}+\alpha_{(N+2)n}} \times \\ \times p_{N1}(\boldsymbol{x}_N, ..., \boldsymbol{x}_0, 0)\, d\boldsymbol{x}_N ... d\boldsymbol{x}_0. \tag{23}$$

As the result, the required moments $m_\beta(t)$ are calculated as $m^{+}_{N\beta}(t - t_N)$ for $t \in \Delta_N$.

3 Example

The scheme was applied to study a system in the form

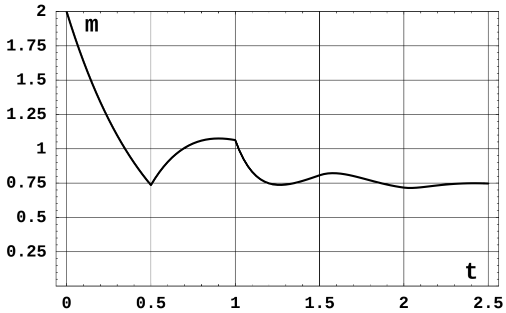

Figure 1. Time evolution of the mean value.

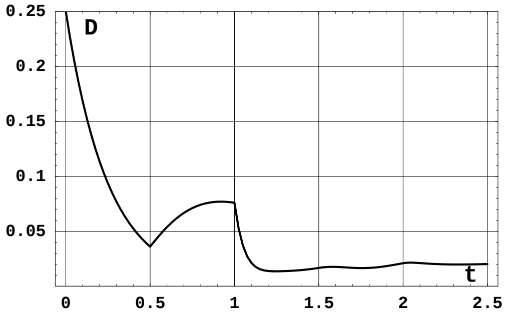

Figure 2. Time evolution of the covariance.

$$\dot{x}(t) + k_1\, x(t) = g_0\, \xi(t), \quad t \in (0, \tau], \quad x(0) = \bar{x}_0,$$

$$\dot{x}(t) + k_2\, x(t) + k_3\, x_\tau(t) = g_1\, \xi(t), \quad t \in (\tau, 2\tau],$$

$$\dot{x}(t) + k_2\, x(t) + k_3\, x_\tau(t) + k_4\, x_{2\tau}^3(t) = g_2\, \xi(t),$$

$$t > 2\tau,$$

where k_i ($i = 1, 2, 3, 4$), g_j ($j = 0, 1, 2$) are constants.

Moments until the forth order were calculated to take into account nonlinearity of the system. As it is known [Dimentberg, 1980], a finite subset of ODEqs satisfied by moments of the phase vectors of such objects aren't closed ones. To generate and to close these Eqs at each step, the cumulant closure was applied with the help of our *Mathematica* code package *ProbRel*. An algorithm of task solution has been implemented with the help of package *Mathematica* too.

Calculations were produced in assumption that the initial displacement $\bar{x}_0$ has the Gaussian distribution with the mean value α_0 and the covariance D_0. Parameters were as follows:

$$N = 4, \quad k_1 = 2, \quad k_2 = 1.25, \quad k_3 = -1.5,$$

$$k_4 = 1/3, \quad g_0 = g_1 = g_2 = 0.1, \quad \tau = 0.5,$$

$$\alpha_0 = 2, \quad D_0 = 0.25.$$

Notice that 209 nonlinear ODEqs were generated and integrated at the last step.

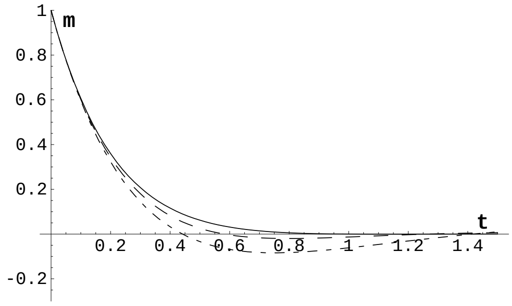

Figure 3. Time evolution of the mean value a = -3, b = -2, c = 0.1.

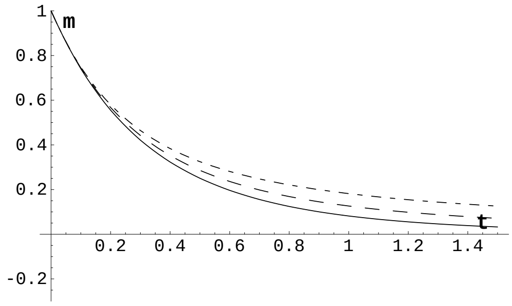

Figure 4. Time evolution of the mean value for a = -5, b = 2, c = 0.1.

Behavior of the mean value m and covariance D for the displacement x is shown in Figs.1 and 2.

4 Additional Models

The scheme considered above was applied to analyze a number of systems with single and multiple constant delays. Moreover this scheme is extendable for study of new classes of stochastic equations.

4.1 Linear SDEqs with a Single Variable Delay

In our research, systems of such equations had the following form

$$\dot{\boldsymbol{x}}(t) = \mathsf{P}(t)\, \boldsymbol{x}(t) + \mathsf{Q}(t)\, \boldsymbol{x}(t - \tau) + \tag{24}$$

$$+\boldsymbol{c}(t) + \mathsf{R}(t)\, \boldsymbol{\xi}(t), \quad t > t_0, \quad \boldsymbol{x}(t_0) = \boldsymbol{x}^0,$$

$$\dot{\boldsymbol{y}}(t) = \mathsf{P}_0(t)\, \boldsymbol{y}(t) + \boldsymbol{c}_0(t) + \mathsf{R}_0(t)\, \boldsymbol{\xi}(t), \tag{25}$$

$$t \in (\bar{t}_0, t_0), \quad \bar{t}_0 \le \min_{t \ge t_0}(t - \tau), \quad \boldsymbol{y}(\bar{t}_0) = \boldsymbol{y}^0,$$

where $\boldsymbol{x} \in \mathbb{R}^n$; $\mathsf{P}(t)$, $\mathsf{Q}(t)$, $\mathsf{R}(t)$, $\mathsf{P}_0(t)$, $\mathsf{R}_0(t)$ and $\boldsymbol{c}(t)$, $\boldsymbol{c}_0(t)$ are known matrix- and vector-functions; $\tau(t) \ge 0$.

The scheme presented in Section 2 was applied for Eqs (24)-(25) after some modification. This modification is based on replacement of $\tau(t)$ by a piecewise constant function $\bar{\tau}(t)$ with steps of an equal length τ^* which is selected to get a necessary accuracy. This replacement allows to transform the source system to the

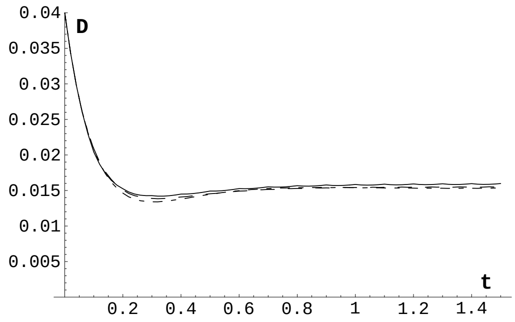

Figure 5. Time evolution of the covariance for a = -3, b = -2, c = 0.3.

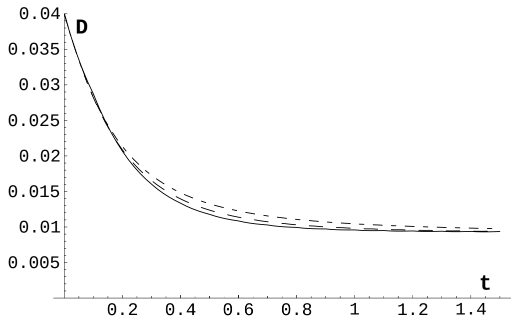

Figure 6. Time evolution of the covariance for a = -5, b = 2, c = 0.3.

other one with constant multiple delays but without any regular structure.

As an example, a random transient behaviour described by SDEq of a pantograph in the form

$$\dot{x}(t) = a\,x(t) + b\,x(q\,t) + c\,\xi(t), \quad 0 < t \le T,$$
$$x(0) = \bar{x}_0, \quad a, b, c, q = \text{const}, \quad 0 < q < 1$$

($\tau = t - q\,t = (1-q)\,t \ge 0$) was considered. The first moments of x are shown in Fig.3–6.

It is easy to see that the initial set for this equation consists of one point $x = 0$. 3004 linear ODEqs were numerically integrated at the last step.

4.2 Stochastic Neutral Differential Equations

We consider a full system of equations in the form

$$\dot{\boldsymbol{x}}(t) = \mathsf{P}_\nu(t)\,\boldsymbol{x}(t) + \tag{26}$$
$$+\mathsf{Q}_{\nu 1}(t)\,\boldsymbol{x}_\tau(t) + \mathsf{H}_{\nu 1}(t)\,\dot{\boldsymbol{x}}_\tau(t) + \ldots +$$
$$+\mathsf{Q}_{\nu\nu}(t)\,\boldsymbol{x}_{\nu\tau}(t) + \mathsf{H}_{\nu\nu}(t)\,\dot{\boldsymbol{x}}_{\nu\tau}(t) +$$
$$+\boldsymbol{c}_\nu(t) + \mathsf{R}_\nu(t)\,\boldsymbol{\xi}(t), \quad t > t_\nu;$$
$$\dot{\boldsymbol{x}}(t) = \mathsf{P}_{\nu-1}(t)\,\boldsymbol{x}(t) + \tag{27}$$
$$+\mathsf{Q}_{\nu-1,1}(t)\,\boldsymbol{x}_\tau(t) + \mathsf{H}_{\nu-1,1}(t)\,\dot{\boldsymbol{x}}_\tau(t) +$$
$$+ \ \ldots \ + \mathsf{Q}_{\nu-1,\nu-1}(t)\,\boldsymbol{x}_{(\nu-1)\tau}(t) +$$
$$+\mathsf{H}_{\nu-1,\nu-1}(t)\,\dot{\boldsymbol{x}}_{(\nu-1)\tau}(t) +$$
$$+\boldsymbol{c}_{\nu-1}(t) + \mathsf{R}_{\nu-1}(t)\,\boldsymbol{\xi}(t), \quad t_{\nu-1} < t \le t_\nu;$$
$$\ldots \ \ldots \ \ldots \ \ldots \ \ldots \ \ldots \ \ldots \ \ldots$$
$$\dot{\boldsymbol{x}}(t) = \mathsf{P}_1(t)\,\boldsymbol{x}(t) + \tag{28}$$
$$+\mathsf{Q}_{11}(t)\,\boldsymbol{x}_\tau(t) + \mathsf{H}_{11}(t)\,\dot{\boldsymbol{x}}_\tau(t) +$$
$$+\boldsymbol{c}_1(t) + \mathsf{R}_1(t)\,\boldsymbol{\xi}(t), \quad t_1 < t \le t_2;$$
$$\dot{\boldsymbol{x}}(t) = \mathsf{P}_0(t)\,\boldsymbol{x}(t) + \boldsymbol{c}_0(t) + \mathsf{R}_0(t)\,\boldsymbol{\xi}(t), \tag{29}$$
$$t_0 < t \le t_1, \quad \boldsymbol{x}(t_0) = \boldsymbol{x}^0.$$

As before, here $\boldsymbol{x} \in \mathbb{R}^n$ is the phase vector; $\mathsf{P}_{ij}(t)$, $\mathsf{Q}_{ij}(t)$, $\mathsf{R}_i(t)$ and $\boldsymbol{c}_i(t)$ are known matrix- and vector-functions; $\tau = \text{const} > 0$.

An application of our technique has allowed to obtain ODEqs for all necessary moments of $\boldsymbol{x}$. Notice that a system of these equations isn't in a normal form.

To demonstrate the scheme, the simple system

$$\dot{x}(t) = -p_1\,x(t) - q_1\,x(t-\tau) - h_1\,\dot{x}(t-\tau) +$$
$$+r_1\,\xi(t), \quad t > 0; \tag{30}$$
$$\dot{x}(t) = 0, \quad t \le 0; \tag{31}$$
$$m(-\tau) = m^0, \quad D(-\tau) = D^0.$$

is under consideration. A form of $m(t)$ for the system is shown in Fig.7 ($m^0 = 5$, $D^0 = 0.25$, $\tau = 0.5$, $p_1 = q_1 = h_1 = 1$, $r_1 = 0.4$).

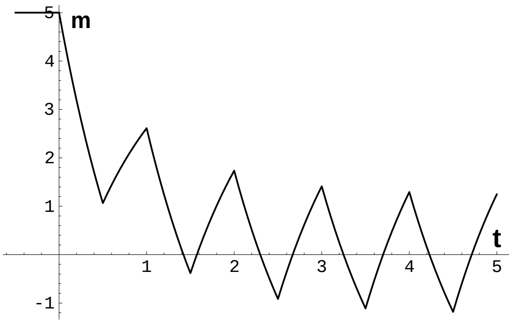

Figure 7. Time evolution of the mean value for neutral system.

4.3 Stochastic Systems under Continuous and Discrete Fluctuations

Our scheme was applied to a difference-differential system excited by continuous and discrete fluctuations [Poloskov, 2007a]

$$\dot{x}(t) = \alpha(t)\,x(t) + \beta(t)\,x(t)\,\xi(t) + \tag{32}$$
$$+\gamma(t)\,x(t-\tau) + \nu(t)\,x(t-\tau)\,\eta(t) +$$
$$+r(t), \quad t > t_1 = t_0 + \tau,$$
$$\dot{x}(t) = \bar{\alpha}(t)\,x(t) + \bar{\beta}(t)\,x(t)\,\xi(t) + r(t), \tag{33}$$
$$t_0 < t \le t_1, \quad x(t_0) = x_0,$$

where $\xi(t)$ and $\eta(t)$ are independent white noises, $r(t)$ is a Poisson noise, $\alpha(t)$, $\beta(t)$, $\bar{\alpha}(t)$, $\bar{\beta}(t)$, $\gamma(t)$, $\nu(t)$ are known functions. As before, ODEqs of the first moments were obtained but in contrast to previous results, a derivation of these Eqs was founded on a chain of Kolmogorov–Feller Eqs. In the case of absence of the Poisson noise, such equation has got the following form:

$$\frac{\partial p(x,t)}{\partial t} = \frac{1}{2}\,\frac{\partial^2\big[\beta^2\,x^2\,p(x,t)\big]}{\partial x^2} - \\ -\frac{\partial}{\partial x}\Big[\big(\alpha+\frac{\beta^2}{2}\big)\,x\,p(x,t)\big]-\mu(t)\,p(x,t) + \\ +\mu(t)\int\limits_{-\infty}^{+\infty} w(x-x',t)\,p(x',t)\,dx'\Big], \tag{34}$$

$$p(x,t_0) = p_0(x), \tag{35}$$

where $\mu(t)$ is an intensity of jumps, $w(\cdot,t)$ is a function describing a jump distribution,

$$\int\limits_{-\infty}^{+\infty} w(x-x',t)\,dx' = 1.$$

Calculations were produced for a number of values of parameters and forms of the function w.

4.4 Stochastic Sensitivity of Linear Dynamic Systems with Delay

As for study of indicated systems, the technique was applied to derive equations for the first moments of the phase vector and its functions of sensitivity with respect to a vector of deterministic parameters $\boldsymbol{\pi} \in \mathbb{R}^p$ at the point $\boldsymbol{\pi}_0$ for a system in the form

$$\dot{\boldsymbol{x}}(t;\boldsymbol{\pi}) = \mathsf{P}(t;\boldsymbol{\pi})\,\boldsymbol{x}(t;\boldsymbol{\pi}) + \\ +\mathsf{Q}(t;\boldsymbol{\pi})\,\boldsymbol{x}(t-\tau;\boldsymbol{\pi}) + \boldsymbol{c}(t;\boldsymbol{\pi}) + \\ +\mathsf{R}(t;\boldsymbol{\pi})\,\boldsymbol{\xi}(t), \qquad t > t_0+\tau, \tag{36}$$

$$\dot{\boldsymbol{x}}(t;\boldsymbol{\pi}) = \mathsf{P}_0(t;\boldsymbol{\pi})\,\boldsymbol{x}(t;\boldsymbol{\pi}) + \\ +\boldsymbol{c}_0(t;\boldsymbol{\pi}) + \mathsf{R}_0(t;\boldsymbol{\pi})\,\boldsymbol{\xi}(t), \quad t_0 < t \le t_0+\tau, \tag{37}$$

$$\boldsymbol{x}(t_0;\boldsymbol{\pi}) = \boldsymbol{x}_0^0(\boldsymbol{\pi}) \tag{38}$$

($\boldsymbol{\pi}_0$ is nominal values of parameters).

5 Conclusion

We presented the scheme that was developed to estimate characteristics of stochastic systems effected by different forms of delays. It is clear that this scheme can be used for different types of system with aftereffect.

6 Acknowledgements

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