SINGULARITIES OF STABILITY BOUNDARIES AND PARADOX OF NICOLAI

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Abstract
We present a general approach to the paradox of Nicolai and related effects analyzed as a singularity of the stability boundary. We study potential systems with arbitrary degrees of freedom and two coincident eigenfrequencies disturbed by small non-conservative positional and damping forces. The instability region is obtained in the form of a cone having a finite discontinuous increase in the general case when arbitrarily small damping is introduced. This is a new destabilization phenomenon, which is similar to the effect of the discontinuous increase of the combination resonance region due to addition of infinitesimal damping. Then we consider the paradox of Nicolai: the instability of a uniform axisymmetric elastic column loaded by axial force and a tangential torque. It is shown that the paradox of Nicolai is related to the conical singularity of the stability boundary which transforms to a hyperboloid with the addition of small dissipation.

Key words
singularity, stability, nonconservative forces, destabilization paradox.

1 Introduction

In 1928 Evgenii L. Nicolai [Nicolai, 1928] formulated a problem of stability of an elastic column with equal basic moments of inertia loaded by a tangential torque and axial force. For the case of cantilever boundary conditions he found that there is no static form of equilibrium of the column except the straight one. Then he studied stability of the straight form of equilibrium using dynamic method and came to the conclusion that it is unstable for arbitrary small magnitude of the torque. This effect is called *the paradox of Nicolai*. For the stability study he used a discrete model with a lumped mass attached to the free end of a massless cantilever column. In the same paper Nicolai introduced a small viscous damping and found that it has a stabilizing effect. In his next paper [Nicolai, 1929], Nicolai introduced geometrical imperfections related to non-equal basic moments of inertia. He used the same discrete model for the stability study and came to the conclusion that the geometrical imperfections are also stabilizing. That was the beginning of the era of non-conservative stability problems. An account of the Nicolai papers is given in [Bolotin, 1963].

In 1950–60's Bolotin [Bolotin, 1963] and Ziegler [Ziegler, 1968] explained absence of static forms of the loss of stability in several contemporary problems by non-conservative nature of loading leading to dynamic forms of instability (flutter). A number of destabilization paradoxes due to dissipation have been discovered in such systems: Ziegler's destabilization paradox, destabilization effect for combination resonance, destabilization of a Hamiltonian system. Recently, these destabilization paradoxes have been associated with generic singularities of the stability boundary, see e.g., [Seyranian and Mailybaev, 2004; Krechetnikov and Marsden, 2007]. We continue this list by showing that the paradox of Nicolai is related to the conical singularity of the stability boundary. Singularities of stability boundaries were analyzed for different systems by [Arnold, 1978; Mailybaev and Seyranian, 1999a; Mailybaev and Seyranian, 1999b; Mailybaev and Seyranian, 2001; Seyranian and Mailybaev, 2001].

2 Destabilization of a conservative system by small circulatory forces

A linear vibrational system of arbitrary dimension with non-conservative positional forces can be written in the form

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\mathbf{q} = 0, \quad \mathbf{C} = \mathbf{P} + \mathbf{N}, \tag{1}$$

where $\mathbf{q}$ is the vector of generalized coordinates, $\mathbf{M}$ is the real symmetric positive definite mass matrix, the real matrices $\mathbf{P} = \mathbf{P}^T$ and $\mathbf{N} = -\mathbf{N}^T$ describe, respectively, potential and nonconservative (also called

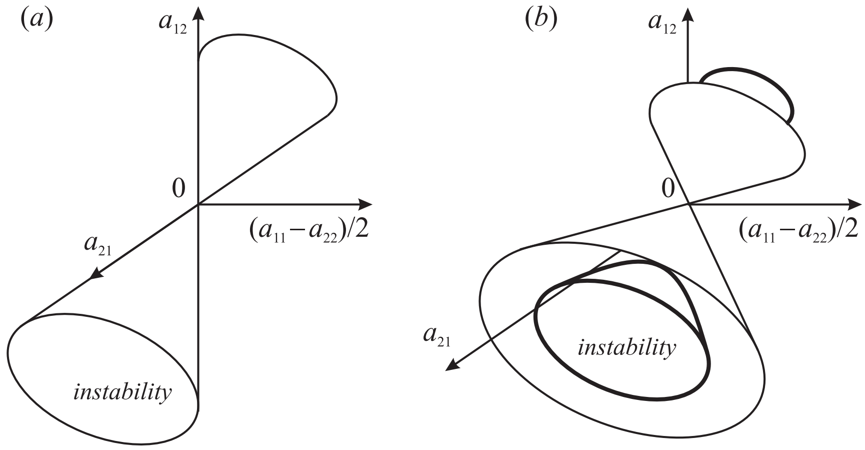

Figure 1. (a) The cone singularity of the instability domain in the case of a perturbed conservative system with a double frequency. (b) The stability boundary for a system with finite damping (bold lines) and infinitesimal damping (thin lines).

circulatory) forces. We study the case of a general small perturbation $\mathbf{M} = \mathbf{M}_0 + \delta\mathbf{M}$ and $\mathbf{C} = \mathbf{P}_0 + \delta\mathbf{C}$ of the stable conservative system with a double frequency $\omega_0 > 0$. The two linearly independent eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$ of the unperturbed system are determined by the equations and normalization conditions

$$\begin{aligned} \mathbf{P}_0\mathbf{u}_i &= \mu_0\mathbf{M}_0\mathbf{u}_i, \\ \mathbf{u}_i\mathbf{M}_0\mathbf{u}_j &= \delta_{ij}, \\ \mu_0 = \omega_0^2, &\quad i,j = 1,2. \end{aligned} \tag{2}$$

For small perturbations, we derive the asymptotic destabilization condition as

$$\left(\frac{a_{11}-a_{22}}{2}\right)^2 + a_{12}a_{21} < 0$$
$$\text{with} \quad a_{ij} = \mathbf{u}_i^T\delta\mathbf{C}\mathbf{u}_j - \omega_0^2\mathbf{u}_i^T\delta\mathbf{M}\mathbf{u}_j. \tag{3}$$

This inequality determines the internal part of a cone in the space $(a_{12}, a_{21}, (a_{11}-a_{22})/2)$, see Fig. 1(a).

When small damping forces $\delta\mathbf{D}\dot{\mathbf{q}}$ with positive definite symmetric matrix $\delta\mathbf{D}$ are added to the left-hand side of (1), the instability condition takes the form

$$\begin{aligned} &\left(\frac{a_{11}-a_{22}}{2}\right)^2 + a_{12}a_{21} + \\ &+\omega_0^2 d^2(1-\eta_1^2-\eta_2^2) < \\ &< \left(\eta_1\frac{a_{11}-a_{22}}{2} + \eta_2\frac{a_{12}+a_{21}}{2}\right)^2, \end{aligned} \tag{4}$$

where

$$\begin{aligned} d &= \frac{d_{11}+d_{22}}{2}, \\ \eta_1 = \frac{d_{11}-d_{22}}{d_{11}+d_{22}}, &\quad \eta_2 = \frac{2d_{12}}{d_{11}+d_{22}}. \end{aligned} \tag{5}$$

with the damping coefficients $d_{kj} = \mathbf{u}_k^T\delta\mathbf{D}\mathbf{u}_j$. If $\eta_1 \neq 0$ or $\eta_2 \neq 0$, the limiting instability region with infinitely small damping ($d \to 0$) is larger than the instability region with zero damping given by (3), so that instability region undergoes a finite (discontinuous) increase, Fig. 1(b). This destabilization phenomenon is similar to the discontinuous increase of a combination resonance region due to infinitesimal damping in the theory of parametric resonance, see, e.g., [Seyranian and Mailybaev, 2004].

3 Instability of a column loaded by an axial force and tangential torque

Consider a straight cantilever elastic column of length l loaded at the free end by a tangential torque L and an axial force P. The column has variable cross-section characterized by the mass per unit length $m(x)$, the matrix of moments of inertia $\mathbf{J}(x)$, Young's modulus E, the external and internal (the Kelvin–Voigt model) damping coefficients γ and η. Using variational analysis we derive the instability condition, similar to (4), for a straight and twisted equilibrium as

$$L^2 > b_1^2 + b_2^2 + \omega_0^2(d+\gamma/m_0)^2/\beta^2, \tag{6}$$

$$\begin{aligned} b_1 &= \frac{1}{2\beta}\int_0^l E(\delta J_{11}-\delta J_{22})w''^2 dx, \\ b_2 &= \frac{1}{\beta}\int_0^l E\delta J_{12}w''^2 dx, \\ \beta &= \int_0^l w'w''dx = \frac{w'^2(l)}{2}, \\ d &= \eta E J_0\int_0^l w''^2 dx, \end{aligned} \tag{7}$$

where $w(x)$ is the eigenmode evaluated for the uniform column with $m = m_0$, $\mathbf{J} = J_0\delta_{ij}$ and $L = 0$. The instability region is determined by the sum of instability regions (6) taken for all eigenmodes of the column.

Formula (6) shows that the perfect column with no damping ($b_1 = b_2 = d = \gamma = 0$) is destabilized by an arbitrarily small tangential torque L. This effect is known as the paradox of Nicolai [Nicolai, 1928; Bolotin, 1963]. The quantities b_1 and b_2 describe the effect of geometric imperfections of the column. When $d = \gamma = 0$ (no damping), the instability region corresponds to the interior of the cone in the space (b_1, b_2, L). Addition of damping has stabilizing effect, which corresponds to the degenerate case $\eta_1 = \eta_2 = 0$ in (4). As an example, we analyze numerically the case when a circular cross-section of a uniform column is slightly changed to elliptic cross-section. A remarkable feature of this analysis is that the critical axial moment L is determined by the second mode when only external damping is considered.

4 Conclusion

We developed a general approach to the paradox of Nicolai and related effects analyzed from the point of view of singularity theory. Geometrical interpretation of the obtained results is that the boundary of the instability region represents a conical surface in the reduced three-dimensional space of nonconservative disturbance parameters. It is shown that damping forces change the conical instability region to a hyperboloid with two sheets increasing or decreasing the instability region. We confirmed and extended the results of Nicolai showing that the uniform cantilever column with equal principal moments of inertia loaded by an axial force loses stability under the action of an arbitrary small tangential torque, but it is stabilized by small geometric imperfections and internal and external damping forces. The same result holds when the tangential torque is substituted by the axial torque, since the corresponding eigenvalue problems are adjoint.

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