SWITCHED SINGULAR LINEAR SYSTEMS AND REACHABILITY

J. Clotet
Dep. Matemàtica Aplicada I
Universitat Politècnica de Catalunya
Spain
josep.clotet@upc.edu

M.D. Magret
Dep. Matemàtica Aplicada I
Universitat Politècnica de Catalunya
Spain
maria.dolors.magret@upc.edu

Abstract
We consider switched singular linear systems and conditions for such a system to be reachable/controllable in the cases where some hypotheses hold.

Key words
Switched linear system, controllability.

1 Introduction

Switched singular linear systems arise from various fields such as electrical and electronic engineering, aeronautical or automotive. [Sun and Ge, 2005] is a nice and complete survey on these systems. A very complete survey of the methods which are required to study singular systems (traditional approaches are not suitable for their study) is [Dai, 1989] . Switched singular linear systems have been studied by B. Meng and F.J. Zhang ([Meng and Zhang, 2006], [Meng and Zhang, 2007]), who provided necessary conditions and sufficient conditions for reachability.

Our goal is, under the assumption of some special conditions, the algebraic characterization of reachability.

2 Preliminaries

First, we recall the concept of switched singular linear systems.

Definition 2.1. *A switched singular linear system Σ is a system which consists of several linear singular subsystems and a piecewise constant map σ taking values into the index set M which determines the switching between them.*

In the continuous case, such a system can be mathematically described by

$$\begin{cases} E_\sigma \dot{\mathbf{x}}(t) = A_\sigma \mathbf{x}(t) + B_\sigma u(t) \\ \mathbf{y}(t) = C_\sigma \mathbf{x}(t) \end{cases}$$

where $E_\sigma, A_\sigma \in M_n(\mathbb{R})$, $B_\sigma \in M_{n\times m}(\mathbb{R})$, $C_\sigma \in M_{p\times n}(\mathbb{R})$, $\operatorname{rk} E_\sigma < n$.

Definition 2.2. *Given an initial time t_0, a switching path is a function of time $\theta : [t_0, T) \longrightarrow M$, with $t_0 < T \leq \infty$ and the index set $M = \{1, \dots, \ell\}$.*

In the continuous-time case, two switching paths θ_1 and θ_2 over $[t_0, T)$ are said to be indistinguishable if the time set

$$\{t \in [t_0, T) \,|\, \theta_1(t) \neq \theta_2(t)\}$$

is a set of isolated real numbers.

Definition 2.3. *A switching path θ is said to be well-defined on $[t_0, T)$ if it is defined on $[t_0, T)$ and for all $t \in [t_0, T)$, both $\lim\limits_{s\longrightarrow t^+} \theta(s)$ and $\lim\limits_{s\longrightarrow t^-} \theta(s)$ exist and the set*

$$\left\{ t \in [t_0, T) \;\middle|\; \lim_{s\longrightarrow t^+} \theta(s) \neq \lim_{s\longrightarrow t^-} \theta(s) \right\}$$

is finite for any finite sub-interval of $[t_0, T)$ (in the case where $t = t_0$, we will consider $\lim\limits_{s\longrightarrow t_0^-} \theta(s) = \theta(t_0)$).

Time $t \in (t_0, T)$ such that $\lim\limits_{s\longrightarrow t^+} \theta(s) \neq \lim\limits_{s\longrightarrow t^-} \theta(s)$ is called a switching time. Let $t_1, t_2, \dots, t_\ell$ be the ordered switching times of θ. The sequence of ordered pairs

$$\{(t_0, \theta(t_0^+)), (t_1, \theta(t_1^+)), \dots, (t_\ell, \theta(t_\ell^+))\}$$

is said to be the *switching sequence* of θ over $[t_0, T)$.

Note that a switching sequence $\{(t_i, k_i)\}_{i=0}^{\ell}$ uniquely determines a switching path (up to possibly rearranging the value at the switching times) by the relationship:

$$\theta(t) = \begin{cases} k_0 & t \in [t_0, t_1) \\ k_1 & t \in [t_1, t_2) \\ \vdots & \\ k_\ell & t \in [t_\ell, T) \end{cases}$$

3 Our set-up

We will assume from now that $M = \{1, 2\}$ (an analogous reasoning might be applied to the case of more subsystems) and that the matrix pencils $\lambda E_1 + A_1$, $\lambda E_2 + A_2$ are regular and in the form

$$E_1 = \begin{pmatrix} I_{n_1} & \\ & \mathcal{N}_1 \end{pmatrix}, A_1 = \begin{pmatrix} G_1 & \\ & I_{n-n_1} \end{pmatrix}$$

$$E_2 = \begin{pmatrix} I_{n_2} & \\ & \mathcal{N}_2 \end{pmatrix}, A_2 = \begin{pmatrix} G_2 & \\ & I_{n-n_2} \end{pmatrix}$$

where $\mathcal{N}_1, \mathcal{N}_2$ are nilpotent matrices with nilpotent indices h_1, h_2. Let us denote by h the maximum of these nilpotent indices. We will finally assume that the function $u(t)$ is a h times piecewise continuous differentiable function.

We will write

$$B_1 = \begin{pmatrix} B_{1,1} \\ B_{1,2} \end{pmatrix}, B_2 = \begin{pmatrix} B_{2,1} \\ B_{2,2} \end{pmatrix}$$

Then we introduce the following notation, for $i = 1, 2$:

$$\overline{G}_i = \begin{pmatrix} G_i & 0 \\ 0 & 0 \end{pmatrix} \in M_n(\mathbb{R})$$

Let us denote by $\Phi(t, t_0, x_0, u, \sigma)$ the state trajectory at time t of the continuous-time switched singular linear system Σ starting from t_0 with initial value x_0, input u and switching well-defined path σ.

4 Reachable states

Let us remember the notion of reachability.

Definition 4.1. *System Σ is (completely) reachable if for any given initial time $t_0 \in \mathbb{R}$ and state $x_f \in \mathbb{R}^n$, there exists a real number $t_f > t_0$, a switching well-defined path $\sigma : [t_0, t_f] \longrightarrow M = \{1, 2\}$ and an input $u : [t_0, t_f] \longrightarrow \mathbb{R}^m$, such that:*

1. $(I_{n_i}|0)x_f = (I_{n_i}|0)\Phi(t_f, t_0, 0, u, \sigma)$, $i = \lim_{s \longrightarrow t_f^-} \sigma(s)$.

2. $(0|I_{n-n_i})x_f = -\sum_{j=0}^{h-1} \mathcal{N}_i^j B_{i,2} u^{(j)}(t_f)$, $i = \lim_{s \longrightarrow t_f^-} \sigma(s)$.

5 A previous result

In [Clotet, Ferrer and Magret, 2009], the authors determined the space of controllable and the space of reachable states and characterized (completely) controllable and (completely) reachable switched singular systems satisfying the "*equisingularity condition*" ($n_1 = n_2$).

Theorem. (see [Clotet, Ferrer and Magret, 2009]) For system Σ, the following conditions are equivalent:

(a) Σ is (completely) controllable.
(b) Σ is (completely) reachable.
(c) $\mathbb{R}^n = \Re \oplus < \mathcal{N}_1 | B_{1,2} >$ or $\mathbb{R}^n = \Re \oplus < \mathcal{N}_2 | B_{2,2} >$.

where $\Re = \sum_{p=1}^{n_1} \Re_p$, being

$$\Re_1 = Im\,[B_{1,1}, B_{2,1}]$$
$$\Re_2 = \Re_1 + G_1\Re_1 + G_2\Re_1 + \cdots + G_1^{n_1-1}\Re_1 + G_2^{n_1-1}\Re_1$$
$$\ldots$$
$$\Re_{p+1} = \Re_p + G_1\Re_p + G_2\Re_p + \ldots + G_1^{n_1-1}\Re_p + G_2^{n_1-1}\Re_p$$
$$\ldots$$

6 Characterization of reachable states

Let us define

$$\mathcal{V}_1 = \left(< \overline{G}_1 | B_1 > \oplus (\{0\} \times < \mathcal{N}_1 | B_{1,2} >)\right) + \left(< \overline{G}_2 | B_2 > \oplus (\{0\} \times < \mathcal{N}_2 | B_{2,2} >)\right)$$

where $< \overline{G}_i | B_i >$ ($i = 1, 2$) is the vector subspace spanned by

$$\begin{pmatrix} I_{n_i} & 0 \\ 0 & 0 \end{pmatrix} B_i, \begin{pmatrix} G_i & 0 \\ 0 & 0 \end{pmatrix} B_i, \begin{pmatrix} G_i^2 & 0 \\ 0 & 0 \end{pmatrix} B_i, \ldots$$

and $< \mathcal{N}_i | B_{i,2} >$ ($i = 1, 2$) is the vector subspace spanned by $B_{i,2}, \mathcal{N}_i B_{i,2}, \ldots, \mathcal{N}_i^{h-1} B_{i,2}$.

Similarly, for $k > 1$,

$$\mathcal{V}_k = \left(< \overline{G}_1 | \mathcal{V}_{k-1} > \oplus (\{0\} \times < \mathcal{N}_1 | B_{1,2} >)\right) + \left(< \overline{G}_2 | \mathcal{V}_{k-1} > \oplus (\{0\} \times < \mathcal{N}_2 | B_{2,2} >)\right)$$

where $< \overline{G}_i | \mathcal{V}_{k-1} >$ ($i = 1, 2$) is the vector subspace spanned by

$$\left\{ \begin{pmatrix} I_{n_i} & 0 \\ 0 & 0 \end{pmatrix} v, \begin{pmatrix} G_i & 0 \\ 0 & 0 \end{pmatrix} v, \begin{pmatrix} G_i^2 & 0 \\ 0 & 0 \end{pmatrix} v, \ldots \,\middle|\, v \in \mathcal{V}_{k-1} \right\}$$

Note that $\mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \cdots \subseteq \mathcal{V}_{n_0} = \mathcal{V}_{n_0+1} = \ldots$ where $n_0 = \max\{n_1, n_2\}$.

B. Meng and F.J. Zhang found necessary and sufficient conditions for a switched singular linear system to be (completely) controllable / (completely) reachable. Concretely, they obtained (adapted to our case) the following results.

Theorem. ([Meng and Zhang, 2007]) For system Σ,

(a) if Σ is (completely) controllable then $\mathcal{V}_n = \mathbb{R}^n$.
(b) if Σ is (completely) reachable, then $\mathcal{V}_n = \mathbb{R}^n$.

Theorem. ([Meng and Zhang, 2007]) For system Σ,

(a) if $\mathcal{V}_n = \mathbb{R}^n$ and $< \mathcal{N}_i|B_{i,2} >= \mathbb{R}^{n-n_i}$ for all $i \in M$, then Σ is (completely) controllable.
(b) if $\mathcal{V}_n = \mathbb{R}^n$ and $< \mathcal{N}_i|B_{i,2} >= \mathbb{R}^{n-n_i}$ for all $i \in M$, then Σ is (completely) reachable.

The main result is the following one.

Theorem 6.1. *Let us assume that $\mathcal{V}_1 = \mathbb{R}^n$ and there exists $i_0 \in M$ such that $< \mathcal{N}_{i_0}|B_{i_0,2} >= \mathbb{R}^{n-n_{i_0}}$. Then the switched singular linear system Σ is (completely) reachable.*

Proof. For a given switching sequence

$$\sigma = \{t_i, i+1\}_{i=0}^{1},\ t_0 < t_1 < t_2$$

we consider

$$\mathcal{R}_i = \{x = \Phi(t_i, t_0, 0, u, \sigma)\,|\, u : [t_0, t_2] \longrightarrow \mathbb{R}^m\},$$

$1 \le i \le 2$.

Let $\ell_i = t_i - t_{i-1}$, $i = 1, 2$. Then

$$\mathcal{R}_1 =< G_1|B_{1,1} > \oplus < \mathcal{N}_1|B_{1,2} >$$

according to [Dai, 1989]. On the other hand,

$$\mathcal{R}_2 = \begin{pmatrix} e^{G_2\ell_2}(I_{n_2}|0)\mathcal{R}_1 + < G_2|B_{2,1} > \\ < \mathcal{N}_2|B_{2,2} > \end{pmatrix}$$

because $\mathcal{R}_2$ is the set

$$\left\{ x = \begin{pmatrix} x_1(u,\sigma) \\ x_2(u) \end{pmatrix}, u : [t_0, t_2] \longrightarrow \mathbb{R}^m \right\}$$

where $x_1(u,\sigma)$ is $e^{G_2\ell_2}(I_{n_2}|0)\Phi(t_1^-, t_0, 0, u, \sigma) + \int_{t_1}^{t_2} e^{G_2(t_2-\tau)}B_{2,1}u(\tau)d\tau$, and
$x_2(u) = -\sum\limits_{j=0}^{h-1} \mathcal{N}_2^j B_{2,2}u^{(j)}(t_2)$. Then $\mathcal{R}_2$ is equal to

$$\begin{aligned}
&(e^{G_2\ell_2}(I_{n_2}|0)\mathcal{R}_1 + < G_2|B_{2,1} >) \oplus < \mathcal{N}_2|B_{2,2} > \\
&= \begin{pmatrix} I_{n_2} \\ 0 \end{pmatrix} e^{G_2\ell_2}(I_{n_2}|0)\mathcal{R}_1 + \begin{pmatrix} I_{n_2} \\ 0 \end{pmatrix} < G_2|B_{2,1} > \\
&\qquad + \begin{pmatrix} 0 \\ I_{n-n_2} \end{pmatrix} < \mathcal{N}_2|B_{2,2} >
\end{aligned}$$

Using a Lemma by Meng-Zhang, the dimension of $\mathcal{R}_2$ is greater or equal than

$$\begin{aligned}
&\dim(\begin{pmatrix} I_{n_2} \\ 0 \end{pmatrix}(I_{n_2}|0)\mathcal{R}_1 + \begin{pmatrix} I_{n_2} \\ 0 \end{pmatrix} < G_2|B_{2,1} > \\
&\qquad + \begin{pmatrix} 0 \\ I_{n-n_2} \end{pmatrix} < \mathcal{N}_2|B_{2,2} >) \\
&= \dim(((I_{n_2}|0)\mathcal{R}_1 + < G_2|B_{2,1} >) \oplus < \mathcal{N}_2|B_{2,2} >) \\
&= \dim(((I_{n_2}|0)\mathcal{R}_1 \oplus < \mathcal{N}_2|B_{2,2} >) \\
&\qquad + (< G_2|B_{2,1} > \oplus < \mathcal{N}_2|B_{2,2} >))
\end{aligned}$$

Since $\mathcal{R}_1 \subseteq ((I_{n_2}|0)\mathcal{R}_1 \oplus < \mathcal{N}_2|B_{2,2} >)$ (Meng-Zhang),

$$\begin{aligned}
\dim \mathcal{R}_2 &\ge \dim(\mathcal{R}_1 + (< G_2|B_{2,1} > \oplus < \mathcal{N}_2|B_{2,2} >)) \\
&= \dim((< G_1|B_{1,1} > \oplus < \mathcal{N}_1|B_{1,2} >) \\
&\qquad + (< G_2|B_{2,1} > \oplus < \mathcal{N}_2|B_{2,2} >)) \\
&= \dim \mathcal{V}_1 = n
\end{aligned}$$

Therefore, $\mathcal{R}_2 = \mathbb{R}^n$ and Σ is (completely) reachable.
□

7 Conclusion

In this paper a similar characterization of reachable switched singular linear systems to that in [Clotet, Ferrer and Magret, 2009] is obtained. Note that the hypothesis of "equisingularity" in [Clotet, Ferrer and Magret, 2009] is no longer in the statement. The controls are not assumed to be in the set of admissible controls, as in [Meng and Zhang, 2006].

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