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DESIGN OF FEEDBACK CONTROLS FOR DYNAMICAL SYSTEMS BY USING COMMON LYAPUNOV FUNCTIONS

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Abstract
We address the problem of synthesizing a bounded feedback control of a linear dynamical system satisfying the Kalman controllability condition. An approach is developed which makes it possible to construct feedback control laws transferring the system to the origin in finite time. The approach is based on methods of stability theory. The construction utilizes the notion of a common Lyapunov function. It is shown that the constructed control remains effective in the presence of uncontrollable perturbations of the system. As an illustration, results of numerically modelling the dynamics of a second-order system controlled by the law proposed in the paper are presented.

Key words
Linear control system, bounded feedback control, common Lyapunov function, double pendulum.

1 Introduction

One of the basic problems of the control theory and practice is bringing a system from one given state to another one in finite time. In the case of a linear dynamic system the problem can be rather easily solved by means of an open-loop control [Kalman, 1961]. However, closed-loop control has obvious advantages: it can cope with unknown disturbances and uncertain parameters of the model. In many cases, for nonlinear dynamical systems it suffices to solve the problem locally, in a vicinity of the target point because it is often easy to reach the vicinity in finite time. If our target is an equilibrium point it is natural to linearize the system. It might happen that the feedback control for linearized system solves the initial nonlinear problem. In this context, the problem of damping of a set of pendulums by single control provides us with a good example. Another example is the problem of bringing a multi-link pendulum to the upper equilibrium state by a control torque applied to suspension point. In both cases it is well known how to bring these systems in the vicinity of the target [Shiriaev et al., 1999]. It remains to bring the systems to the target state precisely from the vicinity of the target in finite time.

The problem of design of a bounded feedback control bringing a linear system to zero has been studied, in particular, in [Korobov, 1979], and this paper is a starting point for our one, though our arguments can be hardly put into a direct correspondence with that of [Korobov, 1979].

In principle, given a bound for control, one can get to the zero by using the minimum time control [Pontryagin et al., 1962; Kalman, Falb and Arbib, 1969; Sontag, 1990]. The obvious drawback of this approach consists in great difficulties of implementation: the amount of computations required is prohibitive for a numerical simulation. We need, therefore, that the feedback control to be devised should be easily implementable (constructive). One can see a posteriori that our control algorithm does not require much memory or computational power. To implement it one needs just basic operations of linear algebra plus finding the only root of a scalar monotone function of one variable. Our control is more smooth than the minimum–time one: its only singular point is zero, while the singular locus of optimal control is a singular hypersurface. In comparison to [Korobov, 1979], the present paper suggests a somewhat different approach, which is much simpler and makes it possible to construct controls locally equivalent to optimal ones. This means that the time $\tau(x)$ required for our control to bring a given state x to 0 is not much greater than the minimal one $\tau_{\min}(x)$: the ratio $\frac{\tau(x)}{\tau_{\min}(x)}$ is bounded as x runs over a neighborhood of zero.

2 Statement of the Problem

Suppose that the linear autonomous control system

$$\dot{x} = Ax + Bu, \quad x \in \mathbf{R}^n, \quad u \in U = \mathbf{R}^m \qquad (1)$$

satisfies the Kalman controllability condition [1,2]. We want to build a bounded feedback control $u = u(x)$ such that, for any sufficiently small $x_0 \in R^n$, the solution of system (1) with initial state $x(0) = x_0 \in R^n$ reaches the point $x = 0$ in finite time.

Note that, given a bound $|u| \leq C$ on control, it is impossible, in general, to steer any given initial state into the origin. However, the above local problem always has a solution.

We simplify our control system (1). Note that the feedback control problem does not change essentially under transformation $A \mapsto A + BC,\ u \mapsto u - Cx$ of (1) corresponding to an extra linear feedback. Moreover, for any invertible matrix D the gauge transformation $A \mapsto D^{-1}AD,\ B \mapsto D^{-1}B,\ u \mapsto u$ does not affect the problem. By using these transformations one can bring system (1) to the canonical Brunovsky form [Brunovsky, 1970] — a set of independent subsystems of the form $z^{(k)} = u,\ z, u \in \mathbf{R}^1$. Now it suffices to bring each subsystem $z^{(k)} = u$ to zero by a bounded feedback control.

Thus, the initial problem reduces to the case

$$A = \begin{pmatrix} 0\ 1\ 0 \dots 0 \\ 0\ 0\ 1 \dots 0 \\ \vdots\ \vdots\ \vdots\ \ddots\ \vdots \\ 0\ 0\ 0 \dots 1 \\ 0\ 0\ 0 \dots 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \tag{2}$$

The problem is to construct a control $u = u(x)$ subject to the constraint

$$|u| \leq 1 \tag{3}$$

under which any solution reaches the point 0 in finite time.

3 The First Control Method

Consider a scalar function $T(x) > 0$, which will be defined below, and the diagonal matrices

$$\delta(T) = \begin{pmatrix} T^{-n} & 0 & 0 \dots & 0 \\ 0 & T^{-n+1} & 0 \dots & 0 \\ \vdots & \vdots & \vdots\ \ddots & \vdots \\ 0 & 0 & 0 \dots & T^{-1} \end{pmatrix}$$

$$M = \begin{pmatrix} -n & 0 & 0 \dots & 0 \\ 0 & -n+1 & 0 \dots & 0 \\ \vdots & \vdots & \vdots\ \ddots & \vdots \\ 0 & 0 & 0 \dots & 0 \\ 0 & 0 & 0 \dots & -1 \end{pmatrix}$$

The matrix $\delta(T)$ has the properties

$$\begin{aligned} &\delta A \delta^{-1} = T^{-1}A,\ \delta B = T^{-1}B \\ &\tfrac{d}{dT}\delta = T^{-1}M\delta \end{aligned} \tag{4}$$

We make the change of variables

$$y = \delta(T)x \tag{5}$$

and, using relation (3), rewrite Eq. (1) in the form

$$\dot{y} = T^{-1}\left(Ay + Bu + M\dot{T}y\right) \tag{6}$$

Choose a vector $a \in \mathbf{R}^n, a^\top = (a_1, \dots, a_n)$, so that the matrix

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_1 & a_2 & a_3 & \dots & a_n \end{pmatrix}$$

is stable. The elements $-a_i,\ i = 1, \dots, n$, of the vector $-a$ are the coefficients of the characteristic polynomial of the matrix A_1; therefore, for $-a_i$ we can take the coefficients of any Hurwitz polynomial.

Here we present the main novelty of the paper: a construction of a common Lyapunov function for two specific stable matrices. Our feedback controls are based on the existence of this function.

Theorem 1 *The vector a can be chosen so that there exist symmetric positive definite matrices Q, P and R satisfying the relations*

$$QA_1 + {A_1}^*Q = -P, \quad QM + MQ = -R \tag{7}$$

Equalities (7) mean that the matrix Q is a common solution of the two Lyapunov equations (7), that is, determines the "common" Lyapunov function $V(x) = (Qx, x)$ for the two systems of differential equations

$$\dot{x} = A_1 x, \quad \dot{x} = Mx$$

with stable matrices A_1 and M (here and in what follows, $(\cdot, \cdot)$ denotes inner product).

Now we can define a bounded feedback control u which brings the system (1),(2) to zero in finite time. We set

$$u = (a, \delta(T)x) = (a, y) \tag{8}$$

where the vector a is chosen in Theorem 1. Then Eq. (6) takes the form

$$\dot{y} = T^{-1}\left(A_1 y + M\dot{T}y\right) \tag{9}$$

Consider the function $T(x)$ implicitly defined by the relation

$$(Q\delta(T)x, \delta(T)x) = C, \quad x \neq 0 \tag{10}$$

The positive constant C is specified below.

It follows from the definition of the matrix $\delta(T)$ that, for any fixed x, the function $\Phi(T) = (Q\delta(T)x, \delta(T)x)$ has the properties

$$\lim_{T\to 0}\Phi(T) = \infty, \quad \lim_{T\to\infty}\Phi(T) = 0 \tag{11}$$

The second equality in (7) and the positive definiteness of the matrix R imply

$$\frac{d}{dT}\Phi(T) = -T^{-1}(R\delta x, \delta x) < 0$$

Therefore, the function $\Phi(T)$ monotonically decreases on the half-axis $T \in [0,\infty]$, which leads us to conclude that the equation (10) with respect to T has a unique positive solution for any $x \in \mathbf{R}^n$ such that $x \neq 0$.

Note that the function $T(x) > 0$ is analytic in $\mathbf{R}^n \setminus 0$, because relation (10) is a polynomial equation of order $2n$ with respect to T whose coefficients depend analytically on the components of the vector x. This function can be continuously extended to zero as $T(0) = 0$, because

$$\lim_{|x|\to 0} T(x) = 0$$

In our constructions, the function $T(x)$ plays the role of a Lyapunov function.

Differentiating T subject to (1) and taking into account (5), we obtain

$$\dot{T} = -\frac{(Py,y)}{(Ry,y)} \tag{12}$$

It follows that the derivative of the function T satisfies the inequality

$$\dot{T} \leq -\frac{p}{r} < 0 \tag{13}$$

where p and r are the minimum and maximum eigenvalues of the matrices P and R, respectively.

It follows from inequality (13) that the function $T(x)$ vanishes in finite time. This means that any trajectory of system (1), (2) reaches the origin in finite time. Moreover, the time of motion of $\tau(x)$ from x to zero is estimated as $\tau(x) = O(T(x))$. In its turn, $T(x)$ can be estimated as $O(\tau_{\min}(x))$ so that the time required for getting into zero is of the same order of magnitude as the minimal one.

To satisfy constraint (3), we define, at the very beginning, the constant C in Eq. (10) as

$$C = q_- |a|^{-2}$$

Here, q_- is the minimum eigenvalue of the matrix Q. The inequality

$$q_- |y|^2 \leq (Qy,y) \tag{14}$$

and relations (5), (8), and (10) imply

$$|u|^2 \leq |a|^2|y|^2 \leq |a|^2\frac{(Qy,y)}{q_-} = |a|^2\frac{C}{q_-} = 1$$

This means that control (8) satisfies constraint (3) on the entire phase space.

Thus, the procedure for constructing a control consists of the following steps:

(i) the choice of a vector a and a matrix Q according to the theorem 1;

(ii) the solution of the polynomial equation (10) with respect to T;

(iii) the calculation of the inner product (8).

Remark 1. Note that, for a system in the canonical Brunovsky form, the control problem stated above has only one essential parameter n, which is the space dimension. Therefore, for each number n, the choice of a vector a and a matrix Q can be implemented numerically, by using, e.g., the MATLAB procedure for solving systems of linear matrix inequalities.

Remark 2. Note that the proposed control is global: it is bounded in the whole phase space and brings any initial state of system (1),(2) to zero in finite time. It also remains effective for the system

$$z^{(n)} = u + v \tag{15}$$

provided that the perturbations v satisfy the inequality

$$|v| \leq v_0, \quad v_0 < \frac{pq_-^{1/2}}{q_+^2}$$

where q_+ is the maximum eigenvalue of the matrix Q.

4 The Second Control Method

One can generalize the above method of control as follows: as above, suppose that the vector a is such that the matrix A_1 is stable and the control function u is defined by (8). There exist symmetric positive definite matrices Q and P for which

$$QA_1 + A_1{}^*Q = -P \tag{16}$$

We define the function $T(x)$ by condition

$$T^{-2\beta}(Q\delta(T)x, \delta(T)x) = 1, \quad \beta > 0 \tag{17}$$

Introduction of the new parameter β does not spoil our previous arguments essentially. Denote

$$S(\beta) = Q(\beta I - M) + (\beta I - M)Q$$

where I is the identity matrix. For sufficiently large β, the symmetric matrix $S(\beta)$ is positive definite. This observation and relation (16) imply that the matrix Q determines the common Lyapunov function $V(x) = (Qx, x)$ for two systems of linear differential equations with stable matrices A_1 and $M - \beta I$.

The function

$$\Phi_\beta(T) = T^{-2\beta}\,(Q\delta(T)x, \delta(T)x)$$

tends to infinity as $T \to 0$, and to the zero as $T \to \infty$. Moreover,

$$\frac{d}{dT}\Phi_\beta(T) = -T^{-2\beta-1}(S(\beta)\delta(T)x, \delta(T)x)$$

Therefore, for sufficiently large β, the derivative of the function $\Phi_\beta(T)$ is negative, and the equation (17) with respect to T has a unique positive solution for an[y] $x \in \mathbf{R}^n$ such that $x \neq 0$. Similarly to our argument[s] in the previous section we conclude that the functio[n] $T(x) > 0$ is analytic and can be continuously extende[d] to zero as $T(0) = 0$.

Let us differentiate T subject to system (1). Takin[g] into account (16), we obtain

$$\dot{T} = -\frac{(Py, y)}{(S(\beta)y, y)} \le -\frac{p}{s(\beta)} < 0 \tag{18}$$

where p and $s(\beta)$ are the minimum and maximu[m] eigenvalues of the matrices P and $S(\beta)$, respectivel[y]. The last inequality implies that the function T vanishes and the trajectory of system (1) reaches the origin in finite time.

Under the second control method, constraints (3) hold in the neighborhood of zero

$$U = \{x \in \mathbf{R}^n : T^{2\beta}(x) \le q_-|a|^{-2}\}$$

Remark 3. The second method cannot be not applied to system (15), because under this method, the control function tends to zero as the trajectory approaches the terminal state and, therefore, cannot cope with finite perturbations. However, the second method of control has an advantage in that it still works under smooth perturbations

$$\dot{x} = Ax + f(x) + Bu, \; f(x) = O(|x|^2) \tag{19}$$

of the control system. Thus, the second approach is locally applicable to a nonlinear control system

$$\dot{x} = F(x) + Bu, \; F(x) \in C^2 \tag{20}$$

which can be represented in form (19) in the vicinity of an equilibrium state.

5 Results of Computer Simulation

To illustrate the proposed approach let us consider the following simple system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = u, \quad |u| \le 1 \tag{21}$$

For the vector a and the matrix Q we take

$$a = \begin{pmatrix} -1 \\ -2 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Equation (10) takes the form

$$T^4 - 4x_2^2T^2 - 4x_1x_2T - 2x_1^2 = 0 \tag{22}$$

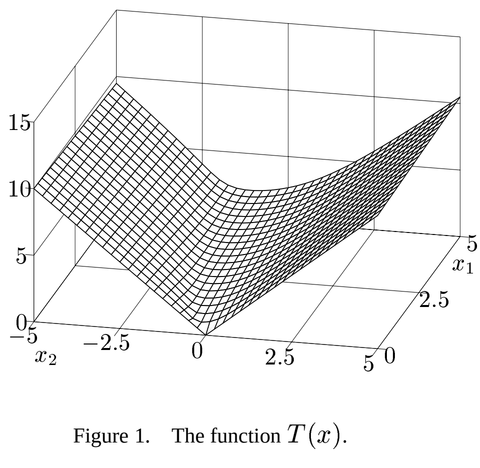

Figure 1. The function $T(x)$.

Fig. 1 shows the graph of the function $T(x)$, and Fig. 2, the graph of the control function $u(x)$. Both functions correspond to the first method of control. We see that the obtained control law is qualitatively close to the time-optimal one [Pontryagin et al., 1962].

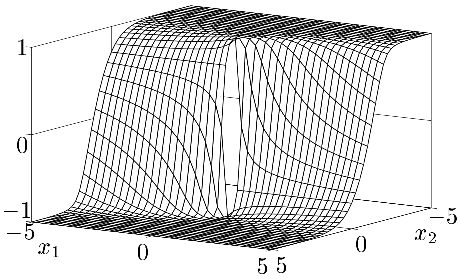

Figure 2. The control function $u(x)$.

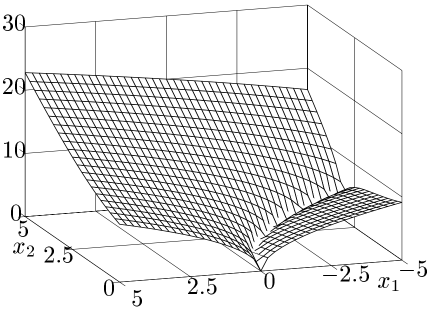

Figure 3. The time of motion $\tau(x)$.

Fig. 3 shows the graph of the function $\tau(x)$ whose value at each point $x = (x_1, x_2)$ of the phase space is equal to the time of motion of the system from this point to the origin.

For comparison, fig. 4 shows the graph of the Bellman function $\tau_{\min}(x)$, whose value at each point is equal to the minimum possible time of motion from this point to the origin. It is seen from the graphs that the time of motion of the system controlled by the law proposed in this paper is approximately 1.5 times longer than the minimal time.

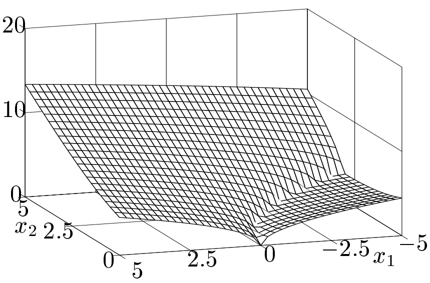

Figure 4. The minimum possible time of motion $\tau_{\min}(x)$.

Fig. 5 presents the graph of the control function $u(x)$ constructed by using the second method. This graph shows that the control does not meet constraint (3) outside a vicinity of the zero.

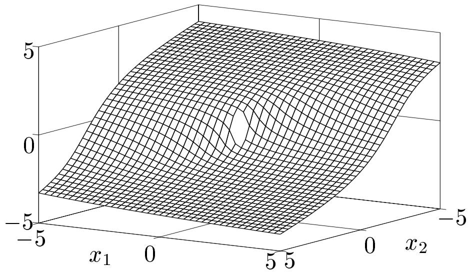

Figure 5. The control function $u(x)$ constructed by using the second method.

6 Conclusion

The paper presents two methods of bounded feedback control for a linear dynamical system satisfying the Kalman controllability condition. The controls obtained bring the system to the origin in finite time. The approach is based on methods of stability theory and utilizes the notion of a common Lyapunov function. The first control is locally equivalent to optimal one, and is global for linear system in the canonical Brunovsky form. The constructed controls remain effective in the presence of uncontrollable perturbations of the system. The second method can be applied for control of a nonlinear dynamic system in a vicinity of equilibrium.

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References

Brunovsky, P. (1970). A classification of linear controllable systems. *Kibernetika*, 6, pp. 176–188.

Kalman, R. E. (1961). In *Proceedings of the First International Congress of the International Federation of Automatic Control (IFAC).* Moscow. V. 2.

Kalman, R. E., Falb, P. and Arbib M. (1969). *Topics in Mathematical System Theory.* McGraw-Hill, New York.

Korobov, V. I. (1979). A general approach to the solution of the bounded control synthesis problem in a controllability problem. *Mat. Sb. (N.S.)*, 109(151):4(8), pp. 582–606.

Pontryagin, L. S., Boltyanskii, V. G., Gamkrelidze, R. V., Mishechenko, E. F. *The Mathematical Theory of Optimal Processes.* New York/London. John Wiley & Sons.

Shiriaev, A.S.,Ludvigsen, H., Egeland, O., Fradkov, A.L. Swinging Up of Non-affine in Control Pendulums. In: *Proc. of the American Control Conference*. San Diego, California, pp. 4039–4044.

Sontag, E. D. (1990). *Mathematical control theory: Deterministic finite-dimensional systems*. Texts Appl. Math. Springer-Verlag.