DELAYED FEEDBACK CONTROL OF DELAYED CHAOTIC SYSTEMS: NUMERICAL ANALYSIS OF BIFURCATION

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Abstract

In this paper, we consider the problem of controlling chaos in scalar delayed chaotic systems. It is revealed that delayed feedback in the form proposed by Pyragas may cause delay in bifurcation. Also, it is shown that many choice of feedback gain and time delay make stable periodic solution for chaotic system which is fictitious. Finally, the period of these fictitious periodic orbits are estimated.

Key words

Delayed feedback, Bifurcation, Delayed chaotic systems.

1 Introduction

The delayed feedback control (DFC) method has received considerable attention recently since it was proposed [Pyragas, 1992]. It provides an alternative effective method for feedback control of chaos [Pyragas, 1993,94,95, 2002].

It was a well established fact for decades that time delay reduces the efficiency of a control scheme. Therefore, it was quite a surprise and has been pointed out that delay may be suitable to generate control force for stabilizing periodic solutions. The main idea of these methods relies on the fact that unstable periodic orbits (UPOs) embedded in a chaotic attractor can be stabilized by applying a time-dependent perturbation in the form of feedback to some accessible system parameters. These schemes have been successfully utilized to control chaos in various dynamic systems [Pyragas 1993; Bielawski, 1994; Hikahira, 1996]. The basic idea of DFC method is to realize an active continuous control of a dynamic system by applying a feedback signal which is proportional to the difference between the dynamical variable $x(t)$ and its delayed value:

$$u = k(x_T - x) \tag{1}$$

where, T is the delay time and k is the feedback gain and $x_T = x(t-T)$. If the delay time T coincides with the period of UPO, then the feedback vanishes on this UPO. This means that the feedback in the form (1) does not change the solution of the system.

The task of stability analysis and controller design for delayed feedback systems is not easy. Nevertheless, some analytical results have been obtained recently. The task of stability analysis and controller design of DFC is not easy. Nevertheless, a full analytical eigenmode expansion of the linear delayed systems and a weakly nonlinear analysis has been given in [Amann 2007]. In [Giannakopoulos 1999] local and global Hopf bifurcation of scalar delayed model is studied. DFC with multiple delays has also been considered in [Ahlborn 2004, 2005]. Also, [Schuster 1999] contains a large number of relevant articles.

More recently, the DFC method has been used to control the time delay chaotic systems, where the controller time delay can be different from the system time delay [Gaun 2003; Park 2004; Sun 2004]. However, almost all of the aforementioned works only provide methods of determining the feedback gain k to stabilize the unstable fixed points UFPs. In fact, the control signal depends on both the feedback gain k and the controller time delay. As a parameter of the delayed feedback controller, T should also be considered in the controller design. To the best of our knowledge, there is a few results in the current literature to adjust T [Guan 2007] for stabilizing UPFs and in the existing methods the controller time delays is usually chosen adaptively and not analytically.

In this paper, we use analytical and numerical approach to study the behavior of delayed chaotic system under DFC. The areas of the parameter plane where no stable oscillations occur, i.e., where an unstable fixed point is stabilized, are found. It reveals that the bifurcation diagram has not the leaf structure as obtained in [Balanov, 2005] for Rossler system. Also it is shown that a cascade of period doubling occurs when the parameter of controller are varied. Finally, we estimate how the parameters of delayed feedback influence the periods of limit cycles in the closed loop system.

2 Free System Analysis

In this paper we consider a class of delayed systems described by the following equation:

$$\dot{x} = -x + f_b(x_\tau) \quad (2)$$

where, $x \in R$ is state variable and $b, \tau > 0$ are system parameters. This class of systems has been studied by many authors [Tian 1998 and References therein]. It has been observed that such delayed systems behave chaotically when the nonlinear function $f_b(x)$ satisfies a set of conditions. Some of these conditions are as follows:

i. $f_b(0) = 0$, or $x_{\text{eq1}} = 0$ is an unstable equilibrium point of (1);

ii. $x_{\text{eq2}} = f_b(x_{\text{eq2}}) > 0$, where x_{eq2} is another unstable equilibrium point of (1);

iii. $\forall x \quad x\, f_b(x) \geq 0$;

iv. $\exists\, r \in [0, x_{\text{eq2}}] \ni \forall x \geq 0 \quad f_b(x) \leq f_b(r) = M$.

v. $\mu < -1$, where $\mu = \left.\dfrac{\mathrm{d}\, f_b(y)}{\mathrm{d}\, y}\right|_{y = x_{\text{eq2}}}$.

Other conditions are given in [Tian, 1998]. Fig. 1 illustrates the conditions on the function $f_b(x)$ graphically. Some typical models that satisfy the above conditions are logistic, Ikeda and Mackey-Glass models. These systems may have chaotic attractors for some values of parameters.

If we linearize free system (2) around x_{eq2}, we have the following characteristic equation

$$\Delta(\lambda) = \lambda + 1 - \mu e^{-\tau\lambda} = 0 \quad (3)$$

By simple calculation one can find that for

$$\begin{cases} \omega_0 = \sqrt{\mu^2 - 1} \\ \tau_j = \dfrac{1}{\omega_0}\big((j+1)\pi - \text{Arc}\tan(\omega_0)\big) \end{cases} \quad (4)$$

system (2) undergoes to a sequence of Hopf bifurcation and $\lambda = \pm i\omega_0$ are roots of (3). So x_{eq2} is an unstable saddle focus. The changes in the qualitative behavior of the attractor as the parameter τ is varied are as follows. The instability occurs at $\tau = \tau_0$; for $\tau_0 < \tau < \tau_1$ there is a stable limit cycle and for $\tau > \tau_1$ a period-doubling bifurcation sequence, which routs to chaos is observed [Giannakopouls, 1999].

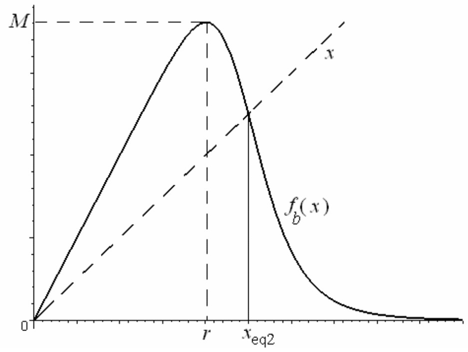

Fig1. A plot of $f_b(x)$.

3 Behavior of Controlled System

In this section, we analyze the delayed model (5) under DFC described by

$$\dot{x} = -x + f_b(x_\tau) + k(x_T - x) \quad (5)$$

3.1 Analytical results

The linearization of (2) around x_{eq2} is obtained as

$$\dot{x} = -(1+k)x + \mu x_\tau + k\, x_T$$

and also the characteristic equation is

$$\Delta(\lambda, k, T) = \lambda + (1+k) - \mu e^{-\tau\lambda} - k\, e^{-T\lambda} = 0 \quad (6)$$

If we choose $T = \tau$, then we have the following theorem.

Theorem 1. The equilibrium point x_{eq2} is stable if $T = \tau$ and $2k > -(1+\mu)$.

Proof: It is straightforward from a theorem in [Hu, 2002].

Now, it is shown that the closed system (5) is stabilizable for enough small T. If one chooses $T << 1$, then $x_T \approx x - T\dot{x}$ and the closed loop equation can be rewritten as

$$(1 + kT)\dot{x} = -x + \mu x_\tau \quad (7)$$

By the following characteristic equation

$$\Delta(\lambda, k, T) = (1 + kT)\lambda + 1 - \mu e^{-\tau\lambda} \quad (8)$$

Then the following theorem is obtained.

Theorem 2: All roots of (8) are on the left half plane if $kT > |\mu|\tau - 1$.

Proof: Let $\lambda = s + i\omega$ be a root of (8). Putting it into (8) and separating real and imaginary parts, it yields:

$$\mathrm{Re}(\Delta) = (1 + kT)s + 1 - \mu e^{-\tau s}\cos(\tau\omega)$$
$$\mathrm{Im}(\Delta) = (1 + kT)\omega + \mu e^{-\tau s}\sin(\tau\omega)$$

Without lost of generality assume $\omega \geq 0$. Obviously $\omega = 0$ is a root of (8). Let

$$P(\omega) = \frac{\partial\,\mathrm{Im}(\Delta)}{\partial\omega} = 1 + kT + \mu\tau e^{-\tau s}\cos\omega\tau$$

So $P(\omega) > 0$ if $kT > |\mu|\tau - 1$, $\mathrm{Im}(\Delta)$ is a strictly increasing function of ω and it has no root except $\omega = 0$. Similarly $\mathrm{Re}(\Delta)$ is a positive strictly increasing function of s for $s \geq 0$. Therefore (8) has no root in the right half plane.

Similarly one can obtain that for large T, if $1 + \mu\tau < 0$, the equilibrium point x_{eq2} is unstable for all value of k .

3.2 Numerical analysis of bifurcation

We use the logistic model as a paradigmatic chaotic model to which we apply delayed feedback control. It exhibits chaotic oscillations born via a cascade of period-doubling bifurcations:

$$\dot{x} = -x + bx_\tau(1 - x_\tau) + k(x_T - x) \qquad (9)$$

Consider the parameter $b = 8.4$. Where $k = 0$, by using Poincare section $\dot{x} = 0$, the bifurcation diagram versus τ is represented in Fig. 2. The instability occurs at $\tau_0 \approx 0.27$, for $\tau_0 < \tau < 0.47$, there is a stable limit cyclic. A periodic doubling bifurcation sequence and chaos are observed at $\tau > 0.47$. The chaotic attractor, mentioned in [Jiang, 2006] for $\tau = 0.5$, is shown in Fig. 3. Unstable periodic orbits embedded into the chaotic attractor with period two(green), four (red) and eight (blue) are shown, too.

To adjust controller parameters T and k , we use this fact that the period of periodic orbits (stable or unstable) born at bifurcation critical value $\tau = \tau_j$ is $T = T_m = 2m\pi/\omega_0$. Also, if $\lambda = i\omega_0$ is a root (3) for $\tau = \tau_j$, then it is a root of (6) if $T = T_m$ for any integer m . So we propose to choose such T to stabilize UPO.

It is easy to show that if $\tau = \tau_j$ is a Hopf bifurcation critical value for (2), one can choose k such that the periodic solution born from this Hopf bifurcation became stable. This means that if for $\tau = \tau_j$ system (2) has an UPO, the closed loop system (5) has a stable periodic orbit.

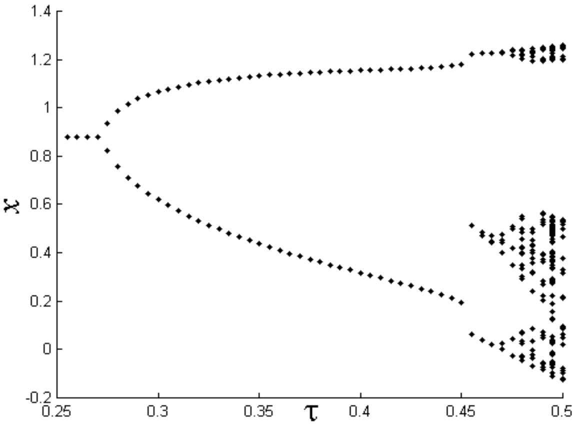

Fig. 2. Bifurcation diagram of free logistic model versus τ by using Poincare section $\dot{x} = 0$.

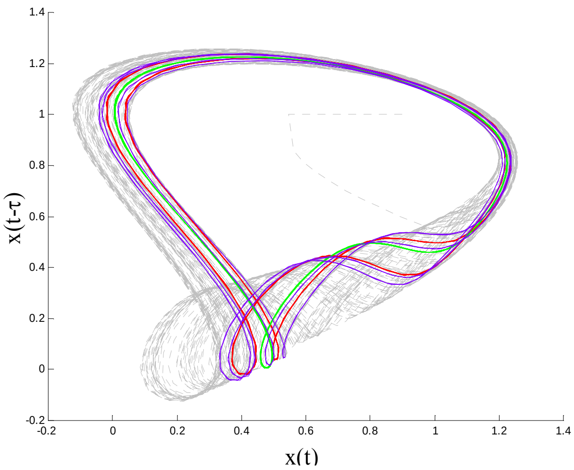

Fig. 3. Periodic orbits embedded into the chaotic attractor with period 2(green), 4 (red) and 8 (blue).

Calculation of bifurcation diagram of (5) with $T = T_1 = 0.994$ and $k = 1.5$ is plotted against time delay τ to view how the delay feedback causes delay in bifurcation. For example period doubling occurs in free system near $\tau = 0.47$, but it occurs for $\tau \approx 1.1$ in (9). It reveals that the period one is stabilize for a wide range of τ where it was stable only for $\tau < 0.47$ in free system (2). Fig. 4 also shows that for $\tau \approx 0.27$, which is a critical value of free system, the closed loop system also undergoes a Hopf bifurcation. This is obtained from (6): if one chooses $T = 2\pi/\omega_0$, then the critical values $\tau = \tau_j$ for (6) is as the same as (3). It seems that other high periodic orbits can be stabilized by choosing suitable T and k .

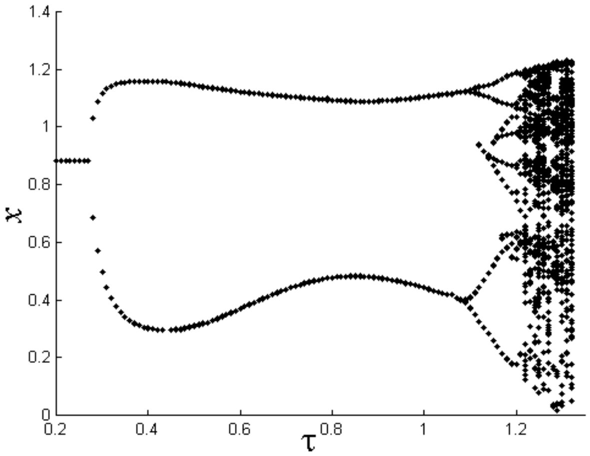

Fig. 4. Bifurcation diagram of the closed loop model versus τ for $T = T_1$ and $k = 1.5$.

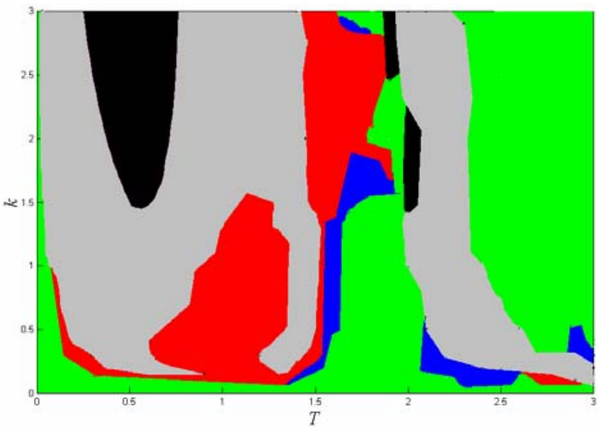

Fig. 5. Bifurcation diagram for closed loop model in the plane (T, k): black area mark of a stable fixed point, grey: stable period one, red stable period two, blue stable period four, green other behaviors such as higher periods, tours and chaos. It is drown approximately by simulations.

In Fig. 5 the bifurcation diagram of the closed loop model in terms of T and k is shown. The black areas show the stability region of fixed point. From theorem 2, a conservative lower bound 2.2 is obtained for kT for enough small T.

By simulation stability guarantees for $kT > 0.75$ and $T < 0.6$. This may destroy the leaf structure of bifurcation diagram (T, k). From Fig. 5 one can see that the delayed feedback eliminate chaos for a large value of T (for example for $0.05 < T < 1.5$ for different k), and it may induced new chaotic motion. For large k and T the trajectories of (9) goes to infinity.

To get general idea about the influence of feedback gain k, it is efficient to plot the bifurcation of x versus k.

From Fig. 5 and Fig. 7 one can find that for small k and some fixed T an inverse period doubling occurs and chaos born from cascade period doubling. But for large k or T (right and top of Fig. 5) other dynamics such as tours make chaos.

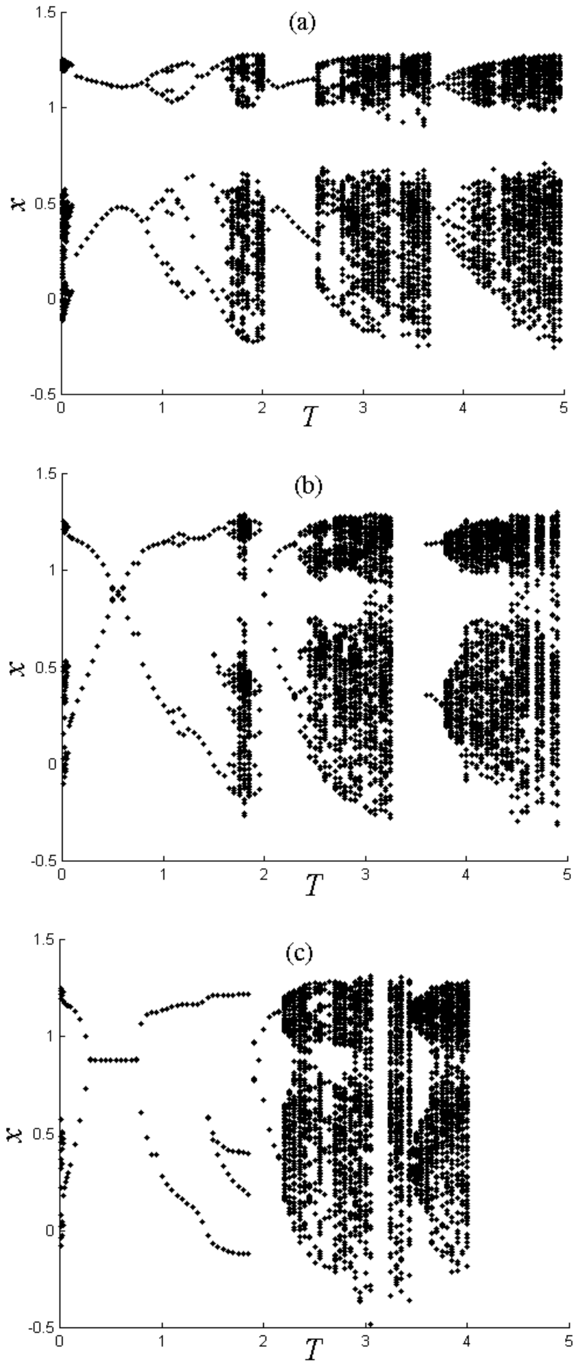

Fig. 6. Bifurcation diagram for the closed loop model versus T for (a) $k = 0.8$, (b) $k = 1.5$, (c) $k = 2.5$.

In Fig. 7 one can see that suitably chosen T can broaden the range of allowed k. In order to illustrate it we plot two one-parameter bifurcation diagrams versus k for two values $T = T_1$ and $T = 1.8$. It is clear that for $k > 1.3$ a periodic-one solution is obtained with a small change in the amplitude of oscillation. In other words, feedback gain in a certain range has small effects on periodic solution if T is chosen appropriately. Also inverse period doubling can be seen for the smaller k.

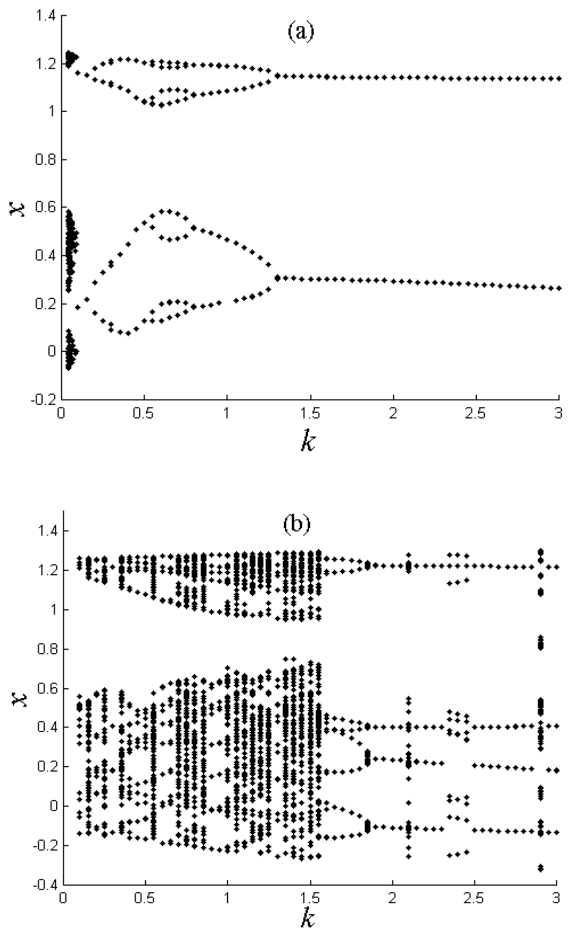

Fig. 7. Bifurcation diagram for the closed loop model versus k for (a) $T = T_1$ and (b) $T = 1.8$.

3.3 Period of Orbits

In order to find the period of the controlled orbit $\Theta(k,T)$, besides the controller parameters k and T, we find it for several values of k and T for period one and two. These are shown in Fig. 8. It seems that $\Theta(k,T)$ is a linear function of T and is a function of $\frac{1}{k}$. So we use the following approximation

$$\Theta(k,T) = gT + h \tag{10}$$

where g and h are functions of $\frac{1}{k}$. It is in a good agreement with that mentioned in [Balanov, 2005].

Eq. (10) and Fig. 8 mean that stable periodic solutions may not have the period of DFC and many fictitious periodic solutions are appeared.

4 Conclusion

In this paper, we have studied the behavior of scalar delayed chaotic model under DFC. The closed loop model of such systems has two delays and we use numerical method for its analysis. A method to find the controller delay is proposed, which is based on bifurcation analysis of the open loop system. Numerically, it is shown that a good choice of T can reduced the effect of k. Simulations show that T and $\frac{1}{k}$ effect the period of fictitious periodic solutions.

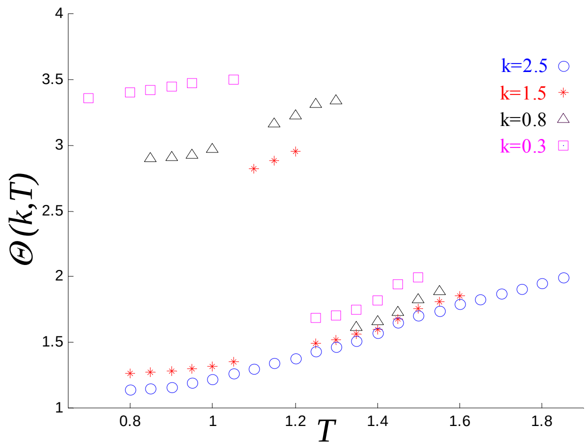

Fig. 8. Period of resulting stable orbit for different values of k and T.

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