FEEDFORWARD STABILIZATION OF KAPITZA OSCILLATOR DRIVEN BY PERIODICAL KICKING PULSES

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Abstract. Here we propose the modification of Kapitza procedure of averaging to stabilize the oscillator driven by periodical external field. We demonstrate the effect of stabilization with relatively low frequency of the field oscillation to compare with sin- or cos-periodical force.

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1. Introduction

Kapitza pendulum driven by rapidly oscillating periodical force can have different structure of its stable points [1]. A set of control models have been applied to stabilize [2]-[3] and synchronize [4] the oscillations by special shapes of non-linear periodical excitation.

Here we propose the modified Kapitza method of averaging and apply it to stabilize the oscillator in rapidly changing external field in the frame of open-loop control technique. We involve one-dimensional T-periodical force f with the zero average meaning: $\bar{f} = 0$.

In the Section 2 we present the modified Kapitza averaging procedure. Then in Section 3 we apply it to find out the stable points of the oscillator in the periodical field of kicking pulses. We choose a special shape of the pulses to stabilize the pendulum in non-trivial stable points with the frequency that should not be very high.

2. Modified Kapitza method for external oscillating force

Let's discuss the one-dimensional motion of a classical particle in the time-independent potential $U(x)$ and under a periodical force:

$$f(x,t) = \sum_{k=1}^{\infty} \left[a_k(x)\cos(k\omega t) + b_k(x)\sin(k\omega t) \right] \ , \tag{1}$$

which varies in time with a high frequency ω (a_k, b_k are functions of the co-ordinates only). If we put T_U for a character time of the motion which the particle would execute in the field U alone, then by a "high" frequency $\omega \equiv 2\pi/T$ we mean such that $\omega >> 2\pi/T_U$. In (1) a_k and b_k are the Fourier coefficients given by

$$\begin{aligned} a_k(x) &= \frac{2}{T}\int_0^T f(x,t)\cos k\omega t \ dt \ ; \\ b_k(x) &= \frac{2}{T}\int_0^T f(x,t)\sin k\omega t \ dt \ . \end{aligned} \tag{2}$$

The equation of the particle motion is:

$$m\ddot{x} = -\frac{dU(x)}{dx} + f(x,t) \ . \tag{3}$$

Following the notation by [5], we present the movement as a smooth path and at the same time execute small oscillations of frequency ω about the path: $x(t) = X(t) + \xi(t)$, where $\xi(t)$ corresponds to these small oscillations. The mean value of the function $\xi(t)$ over its period T is zero, and the function $X(t)$ changes only slightly in that time. Denoting this average by a bar, we therefore have $\bar{x} = X(t)$. Now Taylor's expansion in powers of ξ up to the first order term provides us:

$$\frac{dU}{dx} = \frac{dU}{dX} + \xi\frac{d^2U}{dX^2} \ . \tag{4}$$

Substituting (4) in (3) we have:

$$m\ddot{X} + m\ddot{\xi} = -\frac{dU}{dX} - \xi\frac{d^2U}{dX^2} + f(X,t) + \xi\frac{df}{dX} \ . \tag{5}$$

This equation involves both oscillatory and "smooth" terms, which must evidently be separately equal. For the oscillating terms we can put simply

$$m\ddot{\xi} = f(X,t) \tag{6}$$

and the smooth term is

$$m\ddot{X} = -\frac{dU}{dX} - \xi\frac{d^2U}{dX^2} + \xi\frac{df}{dX} \ .$$

Integrating Eq.(6) with the function f given by (1) and regarding X as a constant, we get

$$\xi = -\frac{1}{m\omega^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\left(a_k\cos k\omega t + b_k\sin k\omega t\right)$$

Next we average equation (5) with respect to the time interval $[0,T]$: $1/T\int_0^T ...\, dt$, we denote it by a overline. Since $\bar{\xi}=0$ and $\bar{f}=0$,

$$m\ddot{X} = -\frac{dU}{dX} + \overline{\xi\frac{df}{dX}} \tag{7}$$

and

$$\frac{df}{dX} = \sum_{k=1}^{\infty}\left(\acute{a_k}\cos k\omega t + \acute{b_k}\sin k\omega t\right) \ ,$$

where $\acute{a_k} = da_k/dX$ and $\acute{b_k} = db_k/dX$. Then we apply the time averaging:

$$\overline{\xi\cdot\frac{df}{dX}} = -\frac{1}{m\omega^2}\sum_{k,j=1}^{\infty}\left[\frac{a_k\acute{a_j}}{k^2}\cdot\overline{\cos k\omega t\cos j\omega t} + \right.$$
$$+\frac{b_k\acute{a_j}}{k^2}\cdot\overline{\sin k\omega t\cos j\omega t} + \frac{a_k\acute{b_j}}{k^2}\cdot\overline{\cos k\omega t\sin j\omega t} +$$
$$\left. +\frac{b_k\acute{b_j}}{k^2}\cdot\overline{\sin k\omega t\sin j\omega t}\right] \ .$$

Since

$$\begin{aligned}\overline{\sin k\omega t\cos j\omega t} = \overline{\cos k\omega t\sin j\omega t} &= 0 \ ; \\ \overline{\cos k\omega t\cos j\omega t} = \overline{\sin k\omega t\sin j\omega t} &= \frac{1}{2}\end{aligned}$$

if $k=j$, and all of them are zero if $k\neq j$; we have

$$\overline{\xi\cdot\frac{df}{dX}} = -\frac{1}{4m\omega^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\left(\frac{da_k^2}{dX} + \frac{db_k^2}{dX}\right) \ . \tag{8}$$

Substituting (8) in (7),

$$m\ddot{X} \quad = \quad -\frac{dU}{dX} - \frac{1}{4m\omega^2}\sum_{k=1}^{\infty}\frac{1}{k^2}\frac{d(a_k^2+b_k^2)}{dX} \ . \tag{9}$$

Eq.(9) involves only the function $X(t)$. It can be written as

$$m\ddot{X} = -\frac{dU_{eff}}{dX} \ ,$$

where the effective potential energy is defined as

$$U_{eff} = U + \frac{1}{4m\omega^2}\sum_{k=1}^{\infty}\frac{(a_k^2+b_k^2)}{k^2} \ . \tag{10}$$

If $a_k, b_k = 0$ for any $k \geq 2$, our Eq.(10) coincides with the result of [5].

3. Kapitza oscillator stimulated by kicking pulses

Now we apply Eq.(10) to check the stable points of Kapitza pendulum whose point of support oscillates horizontally. If the force f is given by:

$$f \quad = \quad m\omega^2\cos\phi\sin\omega t \ ,$$

the effective potential energy is [5]:

$$U_{eff} = mgl\left(-\cos\phi + \frac{\omega^2}{4gl}\cos^2\phi\right) \ .$$

The positions of stable equilibrium correspond to the minima of U_{eff} which has extremum at $\phi = 0\,,\ \pi\,,\ \pm\arccos 2gl/\omega^2$. Vertically upward $\phi = \pi$ is not a stable point. If $\omega^2 < 2gl$, the downward point $\phi = 0$ is stable. If $\omega^2 > 2gl$ the point given by $\cos\phi = 2gl/\omega^2$ is stable.

Now let's introduce the rectangular-shape force:

$$f(t) = m\omega^2\cos\phi\cdot R(t,n) \quad ,$$

where the function R is T-periodical: $R(t+T,n) \equiv R(t,n)$; and for one its period:

$$R(t,n) \quad = \quad \begin{cases} 1 & 0 \leq t < \tau \ ; \\ -(n-1) & \tau \leq t < T \ . \end{cases} \tag{11}$$

Here: $\tau \equiv \left(1-\frac{1}{n}\right)T$. We choose such a form of $R(t,n)$ to satisfy the condition $\bar{f} = 0$, i.e. $\bar{R} = \int_0^T R(t,n)dt = 0$.

With (2) the Fourier coefficients are given by

$$\begin{aligned} a_k &= m\omega^2 \cos\phi \cdot \frac{n}{\pi k} \sin k\omega\tau \ ; \\ b_k &= m\omega^2 \cos\phi \cdot \frac{n}{\pi k}(1-\cos k\omega\tau) \ . \end{aligned} \tag{12}$$

Then Eq.(10) becomes:

$$U_{eff} = U + \frac{m\omega^2 n^2 \cos^2\phi}{2\pi^2} \sum_{k=1}^{\infty} \frac{(1-\cos k\omega\tau)}{k^4} \ ,$$

or, substituting τ and using $\cos k\omega\tau = \cos(2\pi k/n)$:

$$U_{eff} = U + m\omega^2\cos^2\phi \cdot S_n = mgl\left(-\cos\phi + S_n \cdot \frac{\omega^2}{gl}\cos^2\phi\right)$$

with the notation:

$$S_n \equiv \frac{n^2}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^4} \sin^2\left(\frac{\pi k}{n}\right) \ .$$

In particular, if we take $n = 2, 3, 4$, then

$$\begin{aligned} S_2 &= \frac{4}{\pi^2}\sum_{j=0}^{\infty}\frac{1}{(2j+1)^4} = \frac{\pi^2}{24} \simeq 0.411 \ ; \\ S_3 &= \frac{9}{\pi^2}\cdot\frac{3}{4}\sum_{j=0}^{\infty}\left[\frac{1}{(3j+1)^4} + \frac{1}{(3j+2)^4}\right] = \\ &= \frac{1}{72\pi^2}\left[\Psi^{(3)}\left(\frac{1}{3}\right) + \Psi^{(3)}\left(\frac{2}{3}\right)\right] \simeq 0.731 \ ; \\ S_4 &= \frac{16}{\pi^2}\sum_{j=0}^{\infty}\left[\frac{1}{2}\cdot\frac{1}{(4j+1)^4} + 1\cdot\frac{1}{(4j+2)^4} + \frac{1}{2}\cdot\frac{1}{(4j+3)^4}\right] = \\ &= \frac{1}{192\pi^2}\left[\Psi^{(3)}\left(\frac{1}{4}\right) + 2\Psi^{(3)}\left(\frac{2}{4}\right) + \Psi^{(3)}\left(\frac{3}{4}\right)\right] \simeq 0.925 \ , \end{aligned}$$

where $\Psi^{(m)}(z)$ is the polygamma function:

$$\Psi^{(m)}(z) = \sum_{j=0}^{\infty} \frac{(-1)^{m+1} m!}{(j+z)^{m+1}} \ .$$

Thus, we get the non-trivial stable point at $\pm\arccos(gl/2S_n\omega^2)$; and for $n = 2$ the points $\pm\arccos(1.217gl/\omega^2)$ are stable, if $\omega^2 > 1.217gl$;

for $n = 3$ the points $\pm\arccos(0.684gl/\omega^2)$ are stable, if $\omega^2 > 0.684gl$;
for $n = 4$ the points $\pm\arccos(0.541gl/\omega^2)$ are stable, if $\omega^2 > 0.541gl$.

With increasing n the coefficient S_n becomes greater and, thus, we can stabilize the oscillator with the comparatively low frequency ω.

The same effect we observe for the case of vertical modulation:

$$f = m\omega^2 \sin\phi \cdot \sin\omega t \ . \tag{13}$$

Here the inverse point $\phi = \pi$ is stable under the condition $\omega^2 > 2gl$ [5]. If in the place of (13) we apply the force

$$f = m\omega^2 \sin\phi \cdot R(t, n) \ ,$$

we reproduce conditions of the upper point stability as: $\omega^2 > 1.217gl$ for $n = 2$, $\omega^2 > 0.684gl$ for $n = 3$ and $\omega^2 > 0.541gl$ for $n = 4$ correspondingly. In the last case we can achieve the upper point stabilization with the kicking external force frequency that is half than for the sine-shape.

4. Conclusions

We can apply the open-loop (feedforward) algorithm to control the position of the non-trivial stable point for horizontally modulated Kapitza oscillator. In both cases of horizontal and vertical modulations we can stabilize the oscillator in non-trivial stable point with the frequency ω that is sufficiently less than in the case of harmonic modulation. For this purpose we have to apply the special shape of the control external force in the form of kicking pulses (11).

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