APPLICATION OF SINGULAR SPACE-TIME TRANSFORMATION TO IMPULSIVE PEST MANAGEMENT

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Abstract
We consider three versions of a problem of impulsive pest control by its natural enemies. The model is of a prey-predator type. The control system is described by a measure differential equation and hence admits discontinuous solutions. The cost functional of the first problem represents a balance between a negative profit provided by a pest (prey) and a price for applying an additional population of its natural enemies (predators). The problem of keeping the pest quantity below a practically admissible threshold leads to a state-constrained model, while the most complicated version is due to the presence of a mixed constraint relating impulsive action (measure) and the left one-sided limit of a trajectory. We employ a conceptual approach for optimal impulsive pest control based on a certain singular space-time transformation and share our experience in its numerical implementation.

Key words
Optimal control, impulsive control, pest management problem, discontinuous time reparameterization.

1 Introduction

A number of intensive research of impulsive population models were launched recently and inspired by modern biological technologies in agriculture, see, e.g., [Cardoso and Takahashi, 2008, Code, 2009, Jiang and Lu, 2007, Tang and Cheke, 2005]. These studies form a popular trend in the mathematical theory of biological and ecological systems. Such models also arise in the study of problems of disease control (see [Verriest, Delmotte and Egerstedt, 2005]) as the cohort immunization makes changes in the epidemic process in the short term and results in an almost abrupt change in the velocity of propagation of the disease.

Our model is described by a measure differential equation, whose solutions are functions of bounded variation. For an introduction to mathematical theory of impulsive control with trajectories of bounded variation we refer to [Bressan and Rampazzo, 1994, Dykhta and Samsonyuk, 2009, Gurman, 1972, Miller, 1996, Miller and Rubinovich, 2001, Rishel, 1965, Warga, 1972, Warga, 1987, Zavalischin and Sesekin, 1997].

2 Optimal Impulsive Control Problems with Trajectories of Bounded Variation Subject to State and Nonstandard Mixed Constraints

This section contains some preliminary background from optimal impulsive control.

On a given finite time interval $[0, T]$ we consider the following variational problem (P) with state and mixed constraints:

$$I = F(x(T), \mu([0, T])) \to \inf, \qquad (1)$$

$$dx = f(x)dt + c\mu(dt), \quad x(0-) = x_0, \qquad (2)$$

$$\Phi(x(t)) \le 0 \text{ for all } t \in [0, T] \text{ and} \qquad (3)$$

$$Q(x(t-)) = 0 \text{ for } \mu\text{-almost all } t \in [0, T]. \qquad (4)$$

Here, $x(t-)$ denotes the left one-sided limit of function x at a point $t \in [0, T]$, μ is a finite regular nonnegative scalar measure on $[0, T]$. We are given vectors $c, x_0 \in \mathbb{R}^n$, and continuous functions $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}$, $\Phi : \mathbb{R}^n \to \mathbb{R}$ and $Q : \mathbb{R}^n \to \mathbb{R}_+$ ($\mathbb{R}_+$ is the nonnegative half-line).

We assume that f meets usual conditions (of sublinear growth and Lipschitz continuity), and $\Phi(x_0) \le 0$. As soon as a measure μ is fixed, the assumptions on f guarantees the existence and uniqueness of a solution to measure differential equation (2), and the solution is a right continuous function with a bounded variation on $[0, T]$.

Condition (4) is the so-called nonstandard mixed constraint, as it relates state trajectory and control measure, and practically can be viewed of as a jump permitting relation. The presence of such a condition imparts a hybrid feature of the addressed problem.

A couple $\sigma = (x, u)$ satisfying conditions (2)–(4) will be referred to as an admissible process for (P). We suppose that admissible processes exist.

To our knowledge, for problem (P) there are no direct variational techniques, or computational algorithms beyond a straightforward discretization.

Now we formulate the result on the problem's transformation to a conventional (non-impulsive) problem of optimal control, which can be treated by means of standard analytical and numerical methods.

On a time interval $[0, S]$, $S \geq T$, consider the following variational problem (RP):

$$J = F(y_+(S), \eta_+(S)) \to \inf;$$

$$\tfrac{d}{ds}\xi = \alpha, \quad \tfrac{d}{ds}\eta_\pm = (1-\alpha)\beta_\pm, \tag{5}$$

$$\tfrac{d}{ds}y_\pm = \alpha f(y_+) + (1-\alpha)\beta_\pm\, c; \tag{6}$$

$$\xi(0) = \eta_\pm(0) = 0, \quad y_+(0) = y_-(0); \tag{7}$$

$$\xi(S) = T,\ y_+(S) = y_-(S),\ \eta_+(S) = \eta_-(S); \tag{8}$$

$$\eta_- - \eta_+ \leq 0; \tag{9}$$

$$\int_0^S \Psi(s)ds = 0; \tag{10}$$

$$\alpha \in [0,1], \quad \beta_\pm \geq 0, \ \ \beta_+ + \beta_- = 1. \tag{11}$$

Here (α, β), $\beta = (\beta_+, \beta_-)$, are controls. Trajectories (ξ, y, η), $y = (y_+, y_-)$, $\eta = (\eta_+, \eta_-)$, are functions absolutely continuous on $[0, S]$, $y_\pm \in \mathbb{R}^n$, and $\eta_\pm \in \mathbb{R}_+$. The integrand from (10) is as follows:

$$\Psi = \alpha\left(\eta_+ - \eta_- + \rho(y_+ - y_-)\right) + \\ +(1-\alpha)\beta_+ Q_-(y_-),$$

where $\rho : \mathbb{R}^n \to \mathbb{R}$ is an arbitrarily chosen nonnegative function vanishing only at zero.

A collection $\varsigma = (y, \xi, \eta, \alpha, \beta; S)$ satisfying (5)–(11) is said to be an admissible process for (RP).

For problem (P), given a measure μ, we define the function

$$\Upsilon(t) = t + 2\mu([0,t]),\ t \in [0,T], \tag{12}$$

and let $\upsilon :\ [0, T + 2\mu([0,T])] \to [0,T]$ be its inverse.

In problem (RP), for given a control (α, β) satisfying (11) and such that the respective solution ξ of (5), (7) satisfies (8), we introduce the map $\Xi : [0,T] \to [0,S]$ by the formulas

$$\Xi(t) = \inf\{s \in [0,T] |\ \xi(s) > t\},\ t \in [0,T), \tag{13}$$

$$\Xi(T) = S. \tag{14}$$

Theorem 2.1 ([Goncharova and Staritsyn, 2015]). *1) For any (P)-admissible process σ, there exists an admissible for (RP) process $\varsigma = (y, \xi, \eta, \alpha, \beta; S)$ with $S = T + 2\mu([0,T])$, such that*

$$\upsilon(s) = \xi(s),\ s \in [0,S];$$

$$x(t) = y_\pm(\Upsilon(t)),\ \mu([0,t]) = \eta_\pm(\Upsilon(t)),\ t \in [0,T].$$

2) For any (RP)-admissible process ς, there exists a measure μ such that the process $\sigma = (x[\mu], \mu)$ is (P)-admissible, and

$$y_\pm(\Xi(t)) = x(t),\ \eta_\pm(\Xi(t)) = \mu([0,t]),\ t \in [0,T].$$

Here $x[\mu]$ is the solution to the measure differential equation under a measure μ.

3) Optimal solutions for problems (P) and (RP) can exist only simultaneously. For optimal processes σ^* and ς^*, we get $I(\sigma^*) = J(\varsigma^*)$.

The direct and inverse transforms are implemented as follows:

Given an admissible in (P) process σ, the passage $(P) \to (RP)$ consists in extending the instants $\tau \in D_\mu(T)$ of impulses to intervals $\Omega_\tau = [\Upsilon(\tau-), \Upsilon(\tau)]$, where $\Upsilon(\cdot)$ is given by (12). Denote $\Omega_{\tau+} = \Upsilon(\tau-) + [0, T_\tau[$, $\Omega_{\tau-} = \Omega_\tau \setminus \Omega_{\tau+}$, $T_\tau = \mu(\{\tau\})$, $\Omega = \cup_{\tau\in D_\mu(T)}\Omega_\tau$, $\Omega_\pm = \cup_{\tau\in D_\mu(T)}\Omega_{\tau\pm}$.

A desired (RP)-admissible process ς form Theorem 2.1 corresponds to controls

$$\alpha(s) = \pi(\Upsilon^{-1}(s)),\ s \in [0,S]\setminus\Omega,$$

$$\alpha(s) = 0,\ s \in \Omega, \text{ and}$$

$$\beta_\pm(s) = \begin{cases} 1/2, & s \in [0,S]\setminus\Omega, \\ 1, & s \in \Omega_\pm, \\ 0, & s \in \Omega_\mp. \end{cases}$$

Here, $\Upsilon^{-1}(\cdot)$ is the function inverse for $\Upsilon(\cdot)$, and by $\pi(\cdot)$ is the Radon-Nikodym derivative of the Lebesgue measure λ with respect to $(\lambda + 2\mu)(\cdot)$.

The inverse transformation $(RP) \to (P)$ of a process ς is based on the discontinuous time reparameterization $\Xi(\cdot)$, defined by (13), (14). This function is right continuous, monotone nondecreasing, and pseudo-inverse with respect to $\xi(\cdot)$.

A control process ς corresponding to σ, admissible for (P) and meeting the relations of the second assertion in Theorem 2.1, is produced by a differential measure $\mu(dt) = dF_\mu(t)$ with the distribution function F_μ of the form

$$F_\mu(0-) = 0, \quad F_\mu(t) = \eta_+(\Xi(t)).$$

3 Conceptual Approach for Numerical Implementation

Based on Theorem 2.1, we follow the algorithm:

- Problem (P) is reduced to problem (RP) with bounded controls by virtue of the time reparameterization.
- An appropriate numerical algorithm for optimal control (and a suitable software) is applied to solve the reduced problem.
- The result of the previous stage is interpreted in terms of problem (P) by applying the inverse time transformation.

4 Pest Control

The pest population dynamics is given by the following model of Lotka-Volterra type:

$$dx_1 = x_1(b_1 + a_{11}x_1 + a_{12}x_2)dt + c_1\,\mu(dt), \quad (15)$$
$$x_1(0-) = x_1^0, \quad (16)$$
$$dx_2 = x_2(b_2 + a_{21}x_1 + a_{22}x_2)dt + c_2\,\mu(dt), \quad (17)$$
$$x_2(0-) = x_2^0. \quad (18)$$

Here, $x = (x_1, x_2)$ is a vector of the populations of preys and predators (individuals per square unit), $x^0 = (x_1^0, x_2^0)$ are the initial populations, $A(t) = \{a_{ij}(t)\}_{i,j=1,2}$, $b(t) = (b_1(t), b_2(t))$, a_{ij} and b_i are the coefficients of the species interaction with $a_{ii}(t) < 0$, $a_{12}(t) < 0$, $a_{21}(t) > 0$. The terms $c_i\,\mu(dt)$, $i = 1, 2$, in (15), (17) correspond to an artificial (possibly, instantaneous) increase of the population of predators in order to terminate the pest population growth. The control period is $[0, T]$.

Problem 1: Consider the following cost functional

$$I = q_1 \int_0^T x_1(t)dt + q_2\mu([0, T]), \quad (19)$$

which represents a balance between the negative profit provided by the pest activity over the control period and the total price we pay for applying an additional population of the predators. Here, the scalar coefficients $q_i > 0$ are conditional prices. We are to minimize cost (19) subject to constraints (15)–(18).

Consider the model (15)–(18) identified by [Cardoso and Takahashi, 2008] for the interaction of caterpillars *Anticarsia gematalis*, parasitizing soy plant, and their natural predators wasps. We are given the following set of parameters:

$$A = \begin{pmatrix} -0.001 & -0.02 \\ 0.0029 & 0 \end{pmatrix}, \; b = (0.16, -0.19), \quad (20)$$
$$c = (0, 1).$$

The state $\bar{x} = (\bar{x}_1, \bar{x}_2) \approx (65.5172, 4.7241)$ is the only equilibrium point making a practical sense. However, from the economical point of view the level $\bar{x}_1$ is not satisfactory and should be reduced.

A similar problem is addressed by [Code, 2009], where replenishment of the predator population is produced by applying pure impulses at specified times — once per twenty days, — and by a control one mean the number of individuals introduced into the ecosystem at each of these moments. To study the problem the Maximum Principle was used, and a search for numerical solutions was performed by using a discrete scheme of dynamic optimization like in [Cardoso and Takahashi, 2008].

As a class of controls, we now consider Lebesgue–Stieltjes measures. They cover the above case of purely impulsive controls, however, the instants of impulses may be not fixed but subject to optimization.

To solve the problem we apply our control optimization scheme based on the problem transformation by means of discontinuous time change. The reduced optimization problem is a state constrained problem. To find its solution one should use an appropriate software. We employed OPTCON III (a software, designed in ISDCT SB RAS). For $q_1 = q_2 = 1$, different values of x^0, and T within the interval of 150–200 days, we obtained solutions in the form of purely impulsive controls. In fact, the number of impulses and their localization heavily depend on the initial data. Figs. 1, 3 illustrate the dynamics of the populations (preys are black, predators are red) for $x^0 = (10, 0)$. Figs. 2, 4 show the population evolutions for $x^0 = (10, 1)$.

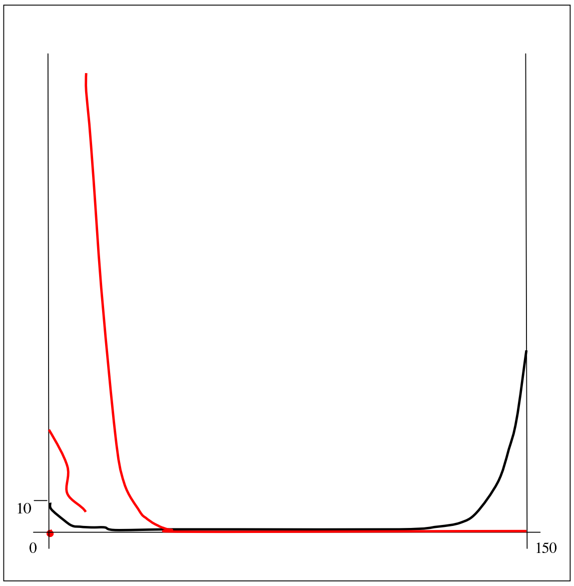

Figure 1. Problem 1. Population dynamics for $T = 150$, $x^0 = (10, 0)$.

Problem 2: We are to keep the population density x_1 below the critical value: $x_1 \le h$ over the whole control period.

Here we have a state constrained problem of impulsive control. Now it is natural to set $q_1 = 0$, $q_2 = 1$, and we use the data (20) as before. Numerical simulations were carried out for different levels $h \in [20, 50]$ and initial values $x_2^0 < x_1^0 \le h$. The period T is taken in the interval from 150 to 200 days. All the control strategies that we obtained as quasi-optimal solutions have a common pattern: We do nothing until the instant $t = \tau < T$ when the population x_1 reaches the critical level h. As soon as this happens, we introduce an external population of the predators in quantity

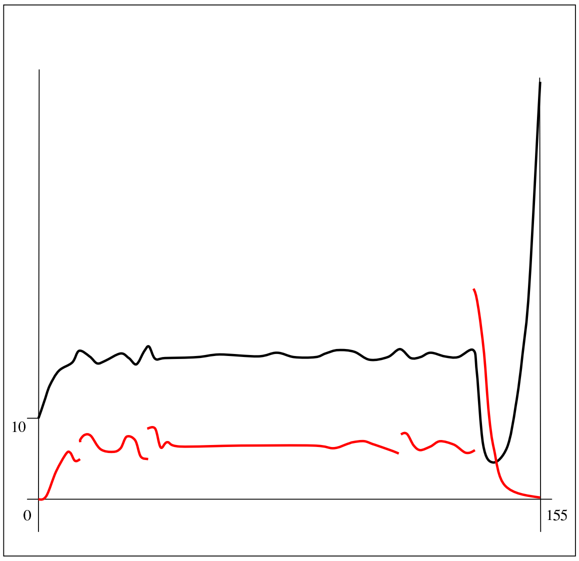

Figure 2. Problem 1. Population dynamics for $T = 155$, $x^0 = (10, 1)$.

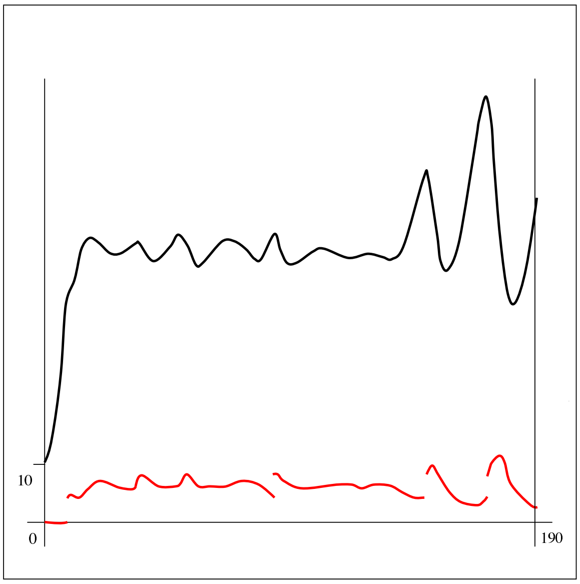

Figure 3. Problem 1. Population dynamics for $T = 190$, $x^0 = (10, 0)$.

$\nu_\tau = \mu(\{\tau\})$. Then we continuously replenish the population x_2 with new individuals in order to maintain the level $x_1 = h$. This way, x_2 keeps to the constant value $\widetilde{x}_2 = x_2^0 + \nu_\tau$. The optimal trajectories are presented in Fig. 5

The optimal control measure is of the form

$$\mu(dt) = m(t)\chi_{(\tau,T]}(t)\,dt + \nu_\tau\delta(t-\tau)\,dt,$$

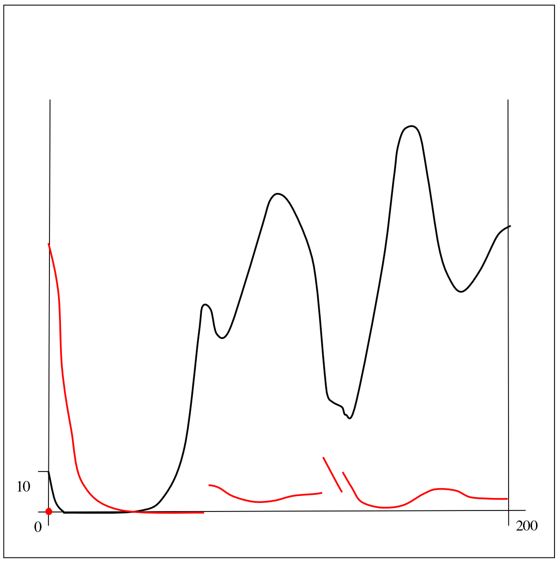

Figure 4. Problem 1. Population dynamics for $T = 200$, $x^0 = (10, 1)$.

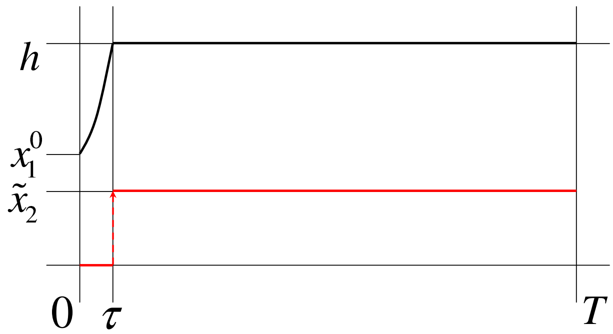

Figure 5. Problem 2. Optimal trajectories.

where $\chi_{(\tau,T]}$ is the characteristic function of the set $(\tau, T]$. The absolutely continuous (w.r.t. the Lebesgue measure λ) component of μ is nontrivial. Its density m can be found from the control system as follows:

$$m(t) = -\widetilde{x}_2(b_2 + a_{21}h + a_{22}\widetilde{x}_2),\ t \in [0, T].$$

For instance, given $h = 20$ and $x_2^0 = 1$, we get $\tau \approx 5.281$, $\nu_\tau \approx 7$, $m(t) \approx 0.924$, $t \in [0, T]$, and $I \approx 187.181$. In other words, we are to introduce about 187 predators per square unit in the course of the control period. For $x_2^0 = 0$, we can calculate the moment τ explicitly:

$$\tau = \frac{1}{b_1} \ln \frac{h\,(b_1 + a_{11}x_1^0)}{x_1^0\,(b_1 + a_{11}h)}.$$

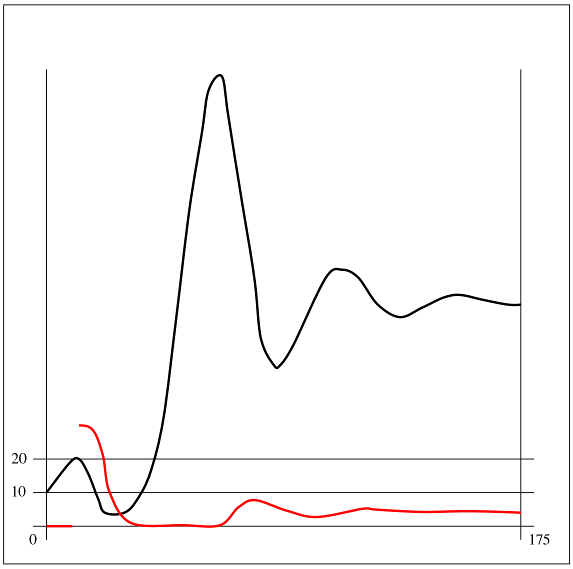

Figure 6. Problem 3. Population dynamics for $T = 175$, $x^0 = (10, 0)$.

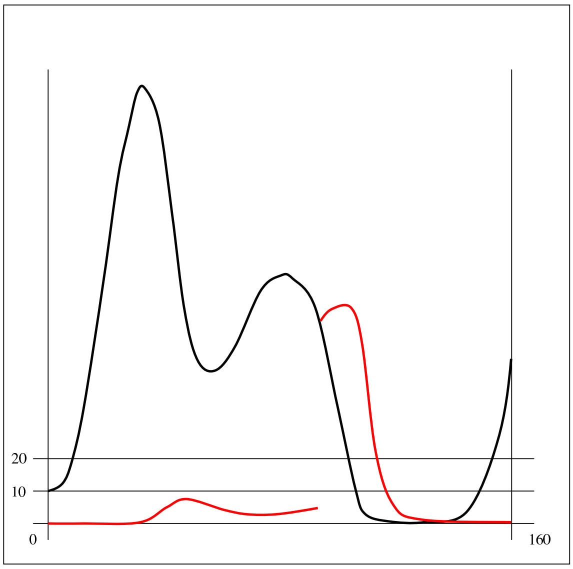

Figure 7. Problem 3. Population dynamics for $T = 160$, $x^0 = (10, 1)$.

Problem 3. Minimize (19) subject to (15)–(18) and an additional mixed constraint

$$l(x_1(t-), t) \leq 0 \text{ for } \mu\text{-almost all } t \in [0, T].$$

The practical motivation can be as follows: Sometimes it may be disadvantageous (or, actually, impossible) to introduce predators into the ecosystem until the population of preys reaches some "critical mass". For instant, being without the required amount of food, predators can migrate, or, even worse, cause damage to the ecosystem. Here, a scalar continuous function $l(x_1, t)$ defines an upper bound for x_2 depending on the current moment t and the level of x_1. Note that the inequality constraint can be always equivalently reformulated as an equality constraint like (4) with a certain nonnegative continuous (or even smooth) function.

Numerical implementation is performed for $h = 20$, $q_1 = q_2 = 1$, and the same data A, b, and c. The dynamics of the competing species is shown on Fig. 6, 7. For $T = 175$ and the initial population values $x^0 = (10, 0)$, the proposed control strategy consists in applying a single impulse at the moment when x_1 first reaches the prescribed level h. The impulse action results in recruiting about 30 individuals per square unit. If $T = 160$ and $x^0 = (10, 1)$, the best strategy we obtained is to introduce ≈ 65 individuals per square unit after the lapse of more than one hundred days.

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