ON THE ROLE OF FJÖRTOFT'S SPECTRAL NUMBER IN THE LINEAR INSTABILITY OF IDEAL FLOWS ON A SPHERE

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Abstract

The normal mode instability of steady solutions to the vorticity equation governing the motion of an ideal incompressible fluid on a rotating sphere is considered. All the types of known solutions are considered: the Legendre-polynomial (LP) flows, Rossby-Haurwitz waves, Wu-Verkley waves and modons. A conservation law for disturbances to each solution is derived and used to obtain a necessary condition for its instability. These conditions specify Fjörtoft's [1] spectral number of the amplitude of unstable modes. For the LP (zonal) flows, it complements the well-known Rayleigh-Kuo and Fjörtoft conditions. The maximum growth rate of modes is also estimated, and the orthogonality of any non-neutral or non-stationary mode to basic flow is shown in the energy inner product.

The analytical instability results obtained are especially useful for testing the computational programs and algorithms in the normal mode stability study. Note that Fjörtoft's spectral number plays a vital part in the linear instability problem.

1. Introduction

In hydrodynamics, the vorticity equation

$$\Delta\psi_t + J(\psi, \Delta\psi + 2\mu) = 0 \quad (1)$$

governs the motion of an ideal incompressible fluid on a rotating unit sphere *S*. In the non-dimensional equation (1), $\psi(t,x) \equiv \psi(t,\lambda,\mu)$ is the streamfunction, μ is the sine of latitude, λ is the longitude, $\Delta\psi$ and $\Omega = \Delta\psi + 2\mu$ are the relative and absolute vorticity, respectively, Δ is the Laplacian on *S* and

$$J(\psi, f) = (\vec{k} \times \nabla\psi)\cdot\nabla f \quad (2)$$

is the Jacobian ($\vec{k}$ is the unit normal to *S*).

Equation (1) is also used in meteorology, and the stability of its solutions is not only interesting hydrodynamic problem, but also important problem of atmospheric dynamics providing insight into deeper understanding of the low-frequency atmosphere variability and climate predictability [2-6]. The four classes of BVE solutions have been known by now: the simple zonal flows $\psi(\mu)$ depending only on μ, and the Rossby-Haurwitz (RH) waves [7-9], Wu-Verkley (WV) waves [10] and modons [11-16] whose streamfunction depends on t, λ, μ.

Let $C_0^\infty(S)$ denote the set of infinitely differential functions $f(x)$ such that

$$\int_S f(x)\,dx = 0 \quad (3)$$

We will use here the three inner products

$$\langle f, h\rangle_k = \int_S (-\Delta)^k f(x)\overline{h(x)}\,dx \quad (4)$$

and norms

$$\|f\|_k = \langle f, f\rangle_k^{1/2} \quad , \quad (k = 0,1,2), \quad (5)$$

besides, $\langle\cdot,\cdot\rangle \equiv \langle\cdot,\cdot\rangle_0$ and $\|\cdot\| \equiv \|\cdot\|_0$, for the L_2-space. The space

$$L_2(S) = H_1 \oplus H_2 \oplus \ldots \oplus H_n \oplus \ldots \quad (6)$$

is the orthogonal sum of subspaces

$$H_n = \{\ p(x):\ -\Delta\, p = n(n+1)p\ \} \quad (7)$$

of homogeneous spherical polynomials of degree *n* [22]. Each H_n is of dimension $2n+1$, and any its polynomial is the eigenfunction of the Laplacian $-\Delta$ corresponding to eigenvalue

$$\chi_n = n(n+1) \qquad (8)$$

The $2n+1$ spherical harmonics $Y_n^m(x)$ of degree *n* and zonal number m $(-n \le m \le n)$ form an orthonormal basis in H_n [23]. By (6), any $f(x)$ of $L_2(S)$ is represented by its Fourier series

$$f(x) = \sum_{n=1}^{\infty} f_n(x) = \sum_{m=-n}^{n} f_n^m Y_n^m(x)$$

where $f_n(x) \in H_n$ is the homogeneous spherical polynomial of degree *n*,

In order to estimate the distribution of kinetic energy between different scales of an ideal incompressible 2D flow, Fjörtoft [1] introduced the mean spectral number

$$\rho(\psi) = \|\psi\|_2 / \|\psi\|_1 \qquad (9)$$

of the stream function $\psi(t,x)$. This is a square root of the ration of the enstrophy

$$\tfrac{1}{2}\|\psi\|_2^2 \equiv \tfrac{1}{2}\|\Delta\psi\|^2 \qquad (10)$$

of flow to its kinetic energy

$$\tfrac{1}{2}\|\psi\|_1^2 \equiv \tfrac{1}{2}\|\nabla\psi\|^2 \qquad (11)$$

We will call $\|\cdot\|_1$ and $\langle\cdot,\cdot\rangle_1$ the energy norm and product, and $\|\cdot\|_2$ - the enstrophy norm.

2. Integral formulas related to Jacobian

It is well known [22] that

$$\int_S J(\psi, f)dS = 0$$

Let $G = \{x \in S : \mu \in (\mu_a, 1]\}$ be a part of *S*. If the latitudinal circle $\mu = \mu_a$ $(-1<\mu_a<1)$ is an isoline of $\psi(x)$ then

$$\psi_\lambda(\lambda, \mu_a) = 0 \qquad (12)$$

and we obtain

$$\int_G J(\psi, f)dS = 0 \qquad (13)$$

Obviously, equation (13) is also true for a periodic channel $G = \{(\lambda,\mu) \in S : \mu \in [\mu_a, \mu_b]\}$ on *S* provided that

$$\psi_\lambda(\lambda, \mu_a) = \psi_\lambda(\lambda, \mu_b) = 0 \qquad (14)$$

Since $J(\psi, f)h = J(\psi, fh) - J(\psi, h)f$, equation (13) leads to

$$\int_G J(\psi, f)hdS = -\int_G J(\psi, h)f\,dS \qquad (15)$$

if ψ satisfies (12) or (14). The substitution $h = f$ in (15) gives

$$\int_G J(\psi, f)f\,dS = 0 \qquad (16)$$

Let ψ be a real function and $f = f_r + \boldsymbol{i}f_i$. Then (15) leads to

$$\int_G J(\psi, f)\bar{f}\,dS = -\int_G J(f,\psi)\bar{f}\,dS = \int_G J(f,\bar{f})\psi\,dS$$

Since $J(f,\bar{f}) = -2\boldsymbol{i}J(f_r, f_i)$, we obtain

$$\mathrm{Re}\int_G J(\psi, f)\bar{f}\,dS = 0 \qquad (17)$$

In particular, for $G = S$, (16) and (17) lead to

$$\langle J(\psi, f), \bar{f}\rangle = 0\ , \quad \mathrm{Re}\langle J(\psi, f), f\rangle = 0 \quad (18)$$

Also, we will use the following assertion:

Lemma 1 [24]. *Let $\psi(x)$, $f(x)$ and $h(x)$ be sufficiently smooth complex functions on the sphere S, and let $F(\psi)$ be a continuously differentiable function. Then*

$$\langle J(\psi, f), h\rangle = -\langle J(f,\psi), h\rangle = ,$$
$$\langle J(f,\bar{h}), \bar{\psi}\rangle = -\langle J(\psi,\bar{h}), \bar{f}\rangle \qquad (19)$$

$$\langle J(\psi, h), \overline{F(\psi)}\rangle = 0, \qquad (20)$$

$$\mathrm{Re}\langle J(\psi, \mu), \psi\rangle = 0, \qquad (22)$$

$$\mathrm{Re}\langle J(\psi, \Delta\psi), \mu\rangle = 0,\ \mathrm{Re}\langle J(\psi, \mu), \Delta\psi\rangle = 0 \qquad (23)$$

If, $G=\{x\in S:\mu\in(\mu_a,1]\}$ *and* $\psi(x)$ *satisfies (15), or if* $G=\{(\lambda,\mu)\in S:\mu\in[\mu_a,\mu_b]\}$ *is a periodic channel and* $\psi(x)$ *satisfies (17) then*

$$\int_G J(\psi,f)h\,dS=-\int_G J(f,\psi)h\,dS=\int_G J(f,h)\psi\,dS=-\int_G J(\psi,h)f\,dS \quad (24)$$

3. Steady solutions to the BVE on a sphere

We will consider the exponential instability of the four types of steady solutions to (1):

1) The LP flow

$$\psi(\mu)=aP_n(\mu) \quad (25)$$

2) The RH wave

$$\psi(\lambda,\mu)=-\omega\mu+\psi_n(\lambda,\mu)\equiv-\omega\mu+\sum_{m=-n}^{n}\psi_n^m Y_n^m(\lambda,\mu) \quad (26)$$

where $\psi_n\in H_n$, $n\geq 2$, $\omega=2/(\chi_n-2)$, χ_n is given by (8), ψ_n^m is arbitrary for $m>0$, and $\psi_n^{-m}=(-1)^m\overline{\psi}_n^m$ for a real wave [8,24]. Each RH wave represents a super-rotation flow $\psi(\mu)=-\omega\mu$ perturbed by its $2n+1$ neutral modes of eigensubspace H_n corresponding to an isolated eigenvalue with finite multiplicity $2n+1$. Besides, the isolated eigenvalues have the finite accumulation point $\nu=\omega$ [21].

3) The WV wave

$$\psi(\lambda,\mu)=\begin{cases} X_i(\lambda,\mu)-\omega_i\mu+d_i\text{, in } S_{in} \\ X_o(\lambda,\mu)-\omega_o\mu+d_o\text{, in } S_{out}\end{cases} \quad (27)$$

where $S_{in}=\{(\lambda,\mu)\in S:\mu\in(-\mu_0,\mu_0)\}$ and $S_{out}=S\setminus S_{in}$, $0<\mu_0<1$, and ω_i, ω_o, d_i and d_o are certain constants [10]. We will refer to S_{in} and S_{out} as to the "inner" and "outer" regions of the WV wave on the sphere, respectively. The wave (27) is antisymmetric about the equator ($\mu=0$), and is a particular form of the solution given by Wu [25]. Both X_i and X_o are eigenfunctions of the Laplacian $-\Delta$ on S corresponding to the eigenvalues

$$\chi_\alpha=\alpha(\alpha+1) \text{ and } \chi_\sigma=\sigma(\sigma+1) \quad (28)$$

with real numbers α and σ, and $\omega_i=2/(\chi_\alpha-2)$, $\omega_o=2/(\chi_\sigma-2)$ for the steady WV wave. Note that, by construction,

$$\psi_\lambda(\lambda,\mu_0)=0 \text{ and } \psi_\lambda(\lambda,-\mu_0)=0 \quad (29)$$

4) Modon by Verkley [12-14] and Neven [15, 16]

$$\psi(\lambda,\mu)=\begin{cases} X_i(\lambda',\mu')-\omega_i\mu+d_i\text{, in } S_{in} \\ X_o(\lambda',\mu')-\omega_o\mu+d_o\text{, in } S_{out}\end{cases} \quad (30)$$

where (λ',μ') is a system of coordinates on the sphere whose north pole has coordinates $(\lambda,\mu)=(\lambda_0,\mu_0)$, $S_{in}=\{(\lambda',\mu')\in S:\mu'>\mu_a\}$ and $S_{out}=\{(\lambda',\mu')\in S:\mu'<\mu_a\}$ are the inner and outer regions of the modon separated by the circle $\mu'=\mu_a$, and ω_i, ω_o, d_i and d_o are some constants. The modon centre moves along the latitudinal circle $\mu=\mu_0$ with a constant velocity *C*. It is easy to show that

$$\psi_{\lambda'}(\lambda',\mu_a)=0 \quad (31)$$

for a steady dipole (*C*=0), monopole ($1-\mu_0^2=0$) or quadrupole modon [15], i.e., the boundary $\mu'=\mu_a$ between S_{in} and S_{out} is a streamline for such modons.

4. Conservation law for perturbations to LP flow and RH wave

Let $\psi(\lambda,\mu)$ be some LP flow (25) or steady RH wave. The following theorem is valid:

Theorem 1. *Any complex disturbance (as well as arbitrary real perturbation) to the LP flow (25) or RH wave (26) obeys the conservation law*

$$[\eta(t)-\chi_n K(t)]_t=0 \quad (32)$$

Thus, its energy

$$K(t)=\tfrac{1}{2}\|\psi'(t)\|_1^2 \quad (33)$$

and enstrophy

$$\eta(t)=\tfrac{1}{2}\|\psi'(t)\|_2^2 \quad (34)$$

decrease, remain constant or increase simultaneously according to

$$\eta_t(t) = \chi_n K_t(t) \qquad (35)$$

The law (32) was first obtained in [17] for infinitesimal perturbations to the LP flow. The Theorem 1 generalizes this law to arbitrary perturbations of LP flow and RH wave. Thus, although the energy norm $\|\cdot\|_1$ is generally weaker than enstrophy norm $\|\cdot\|_2$, both norms give the same results on linear and nonlinear stability of the LP flows and RH waves.

The corresponding assessment for a stationary WV wave or modon is

Theorem 2. *Any disturbance to steady WV wave (27), or modon (30) evolves so that*

$$\left[\chi_\alpha^{-1}\eta^{(i)} + \chi_\sigma^{-1}\eta^{(o)} - K\right]_t = 0 \qquad (36)$$

where $K(t)$ *is the perturbation energy (33),* $\chi_\alpha = \alpha(\alpha+1)$, $\chi_\sigma = \sigma(\sigma+1)$ *(see(28)), and*

$$\eta^{(i)}(t) = \tfrac{1}{2}\int_{S_{in}} |\Delta\psi'|^2 dS \ , \ \eta^{(o)}(t) = \tfrac{1}{2}\int_{S_{out}} |\Delta\psi'|^2 dS$$

are the parts of perturbation enstrophy (34) concentrated in S_{in} *and* S_{out}*, respectively.*

The laws (32) and (36) are related with the basic flow only by means of degrees of the spherical harmonics forming this flow (degree n in (25), (26), and degrees α and σ in (29), (30)).

5. Unified conservation law for infinitesimal perturbations

We can combine (32) and (36) into one law. Denoting by

$$\chi(t) = \rho^2(t) = \eta(t)/K(t) \qquad (37)$$

the square of Fjörtoft's spectral number (9) of disturbance ψ', and by

$$\delta(t) = \eta^{(o)}(t)/\eta(t) \ \text{ and } \ 1-\delta(t) = \eta^{(i)}(t)/\eta(t)$$

the portions of perturbation enstrophy corresponding to the regions S_{out} and S_{in}, respectively ($0 \le \delta \le 1$), the two laws (32) and (36) can be unified into one equation:

$$\left\{[p(t)-1]\,K(t)\right\}_t = 0 \qquad (38)$$

where $p(t)$ is the spectral characteristic of disturbance $\psi'(\lambda,\mu,t)$, besides,

$$p(t) = \chi(t)\,\chi_n^{-1} = \chi(t)[n(n+1)]^{-1} \qquad (39)$$

for LP flow (25) or steady RH wave (26), and

$$p(t) = \chi(t)\,[\delta\chi_\sigma^{-1} + (1-\delta)\chi_\alpha^{-1}] \qquad (40)$$

for steady WV wave (27) or modon (30).

Due to (38), all the disturbances $\psi'(\lambda,\mu,t)$ can be divided into three sets:

$$M_+ = \{\psi' : p(t) > 1\}$$
$$M_0 = \{\psi' : p(t) = 1\} \qquad (41)$$
$$M_- = \{\psi' : p(t) < 1\}$$

The set M_0 is a hypersurface separating the set M_- of large-scale disturbances and the set M_+ of small-scale disturbances.

Note that in the case of LP flows and RH waves, M_0 contains the subspace H_n of homogeneous spherical polynomials of degree *n*, besides, any perturbation of H_n is neutral [5]. Also, Theorem 1 asserts that sets (43) are invariant sets for arbitrary (not only infinitesimal) perturbations to the LP flow (25) and RH wave (26).

6. Conditions for instability of LP flows, RH waves, WV waves and modons

The existence of continuous spectrum and finite accumulation points makes increased demands to the accuracy of numerical algorithms used to construct normal modes, especially in the case of such not so smooth solutions as WV waves and modons [16, 19-21]. In this connection, analytical instability results are of great importance for checking numerical algorithms and computational programs. We now obtain a necessary condition for the normal mode (exponential) instability of solutions. Let ψ be one of the four steady BVE solutions: a LP flow, RH wave, WV wave, or modon. A normal mode of ψ can be written as

$$\psi'(t,\lambda,\mu)=\Psi(\lambda,\mu)\exp\{\nu t\}, \quad (42)$$

where

$$\Psi(\lambda,\mu)=R(\mu)\exp\{im\lambda\} \quad (43)$$

for any LP flow or monopole modon. Here i is the imaginary unit, $\nu=\nu_r+i\nu_i$, and Ψ is the mode amplitude. The real part ν_r of ν determines the growth (or decay) rate of the mode amplitude, whereas its imaginary part ν_i characterizes the mode frequency. Thus, a mode is unstable if $\nu_r>0$, decaying if $\nu_r<0$, and neutral if $\nu_r=0$.

Let

$$\chi_\Psi=\eta_\Psi/K_\Psi$$

where

$$K_\Psi=\tfrac{1}{2}\sum_{n=1}^{\infty}\chi_n\sum_{m=-n}^{n}\left|\Psi_n^m\right|^2,\ \eta_\Psi=\tfrac{1}{2}\sum_{n=1}^{\infty}\chi_n^2\sum_{m=-n}^{n}\left|\Psi_n^m\right|^2$$

are respectively the total energy and enstrophy of amplitude $\Psi(\lambda,\mu)$ to mode (42).

Theorem 3. *Let $n\ge 1$, and $0<|m|<n$. The spectral number χ_Ψ of amplitude $\Psi(\lambda,\mu)$ to any unstable mode (42),(43) of the LP flow (25) must satisfy the condition*

$$\chi_\Psi=\chi_n=n(n+1) \quad (44)$$

In Theorem 3, we consider only the case $0<|m|<n$, because any mode with $|m|\ge n$ and $m=0$ is stable [18, 24].

Theorem 4. *Let ψ be a steady RH wave (26), WV wave (27) or modon (30). Then the spectral number χ_Ψ of amplitude $\Psi(\lambda,\mu)$ to any unstable mode of ψ must satisfy the condition*

$$\chi_\Psi=\begin{cases} n(n+1) & \text{for RH wave} \\ [\delta\chi_\sigma^{-1}+(1-\delta)\chi_\alpha^{-1}]^{-1} & \text{for WV wave or modon}\end{cases} \quad (45)$$

Note that any mode of a monopole (zonal) modon with $|m|(|m|+1)>\max\{\chi_\alpha,|\chi_\sigma|\}$ is neutral [5].

The values χ_Ψ given by Theorems 3 and 4 serve as rather good tests for checking the precision of numerical instability results. For the LP flow, the new condition given by Theorem 3 complements the Rayleigh-Kuo condition in that while the latter is related to the structure of flow velocity (or enstrophy), the former establishes a strict dependence of the spectral number of growing disturbance on the flow scale (the polynomial degree *n*): the greater is the flow degree, the smaller is the geometric scale of unstable modes.

All the modes are divided into three sets (43), and the amplitude $\Psi(\lambda,\mu)$ of any unstable mode must belong to the hypersurface M_0.

7. Maximum growth rate of unstable modes

We estimate the maximum growth rate of unstable modes.

Theorem 5. *The maximum growth rate of an unstable mode of the LP flow (25) or RH wave (26) is limited:*

$$|\nu_r|\le\sqrt{n(n+1)}\max_S|\vec{u}| \quad (46)$$

where $\max_S|\vec{u}|$ is the maximum value of flow velocity.

Due to (46), the growth rate decreases with velocity $|\vec{u}|$ and number *n*, and given maximum velocity, the larger is the scale of basic flow the smaller is the growth rate of its unstable modes.

Theorem 6. *The maximum growth rate of an unstable mode (42) of steady WV wave (27) or modon (30) is limited:*

$$|\nu_r|\le\max_S|\vec{u}|\max\{\chi_\alpha,|\chi_\sigma|\}\,[\delta\chi_\sigma^{-1}+(1-\delta)\chi_\alpha^{-1}]^{1/2} \quad (47)$$

Thus, the growth (decay) rate of modes depends on the velocity maximum, the degrees α and σ of basic solution and the part δ of mode enstrophy concentrated in the region S_{out}.

8. Orthogonality of unstable modes to flow

We now establish the orthogonality of unstable modes to the basic flow.

Theorem 7. *Let ψ be the LP flow (25), RH wave (26), WV wave (27) or modon (30). Then the amplitude Ψ of each unstable, decaying, or non-stationary mode is orthogonal to the basic flow ψ in the W_2^1-inner product.*

The W_2^1-orthogonality means that the mode amplitude velocity $\vec{U} = \vec{k} \times \nabla\Psi$ is orthogonal to the solution velocity $\vec{u} = \vec{k} \times \nabla \psi$:

$$\left(\vec{U}, \vec{u}\right) \equiv \int_S \vec{U} \cdot \vec{u}\, ds = 0$$

Corollary 1. *Let ψ be the LP flow (25) or RH wave (26). Then amplitude Ψ of each unstable, decaying, or non-stationary mode is both L_2 and W_2^2-orthogonal to flow ψ:*

$$\nu\left\langle \Psi, \psi \right\rangle = 0 \ , \quad \nu\left\langle \Psi, \psi \right\rangle_2 = 0$$

Corollary 2. *The amplitude Ψ of each non-neutral or non-stationary mode of the RH wave is also orthogonal to the subspace H_1 of spherical polynomials of degree one*.

It immediately follows from these results that amplitude Ψ is orthogonal to a super-rotation flow

$$\nu\left\langle \Psi, \mu \right\rangle = 0 \ .$$

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