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ELLIPSOIDAL ESTIMATES OF REACHABLE SETS OF IMPULSIVE CONTROL SYSTEMS WITH BILINEAR UNCERTAINTY

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Abstract
The paper deals with the state estimation problem for impulsive control systems under bilinear uncertainty. It is assumed that we know the bounding set for initial system states and any additional statistical information is not available. Also the matrix included in the differential equations of the system dynamics is uncertain and only bounds on admissible values of this matrix coefficients are known. Under such conditions the dynamical system is nonlinear and reachable set loses convexity property. We use Minkowski function to describe the trajectory tubes and their set-valued estimates. Basing on the techniques of approximation of the generalized trajectory tubes by the solutions of control systems without measure terms and using the techniques of ellipsoidal calculus we present here a state estimation algorithms for the studied impulsive control problem bilinear type. The motivations to consider set-membership approach in state estimation problems for dynamical systems with uncertainty may be found in many applied areas including engineering problems in physics and cybernetics.

Key words
Bilinear control systems, impulsive control, ellipsoidal calculus, trajectory tubes, estimation.

1 Introduction

The paper deals with the problem of state estimation for control problems and of the evaluation of related estimating sets describing uncertainty. We study the case when a probabilistic description of noise and errors is not available, but only a bound on them is known [Bertsekas and Rhodes, 1971; Kurzhanski and Valyi, 1997; Milanese, Norton, Piet-Lahanier and Walter, 1996; Schweppe, 1973; Walter and Pronzato, 1997]. Such models may be found in many applied areas ranged from engineering problems in physics and cybernetics [Ceccarelli and etc., 2006] to economics as well as to biological and ecological modeling when it occurs that a stochastic nature of the errors is questionable because of limited data or because of nonlinearity of the model.

Bilinear dynamical systems are a special kind of nonlinear systems representing a variety of important physical processes. A great number of results related to control problems for such systems has been developed over past decades, among them we mention here [Chernousko, 1996; Chernousko and Rokityanskii, 2000; Filippova and Lisin, 2000; Kurzhanski and Varaiya, 2014; Kurzhanski and Filippova, 1993; Mazurenko, 2012; Polyak, Nazin, Durieu and Walter, 2004]. A number of bilinear problems can be found in control of quantum systems, e.g. [Boscain, Chambrion and Sigalotti, 2013; Boussaïd, Caponigro and Chambrion, 2013; Boussaïd, Caponigro and Chambrion, 2012; Gough, 2008; Nihtila, 2010].

Reachable sets of bilinear systems in general are not convex, but have special properties (for example, are star-shaped). We, however, consider here the guaranteed state estimation problem and use ellipsoidal calculus for the construction of external estimates of reachable sets of such systems.

Unlike the classical estimation approach, set-membership estimation is not concerned with minimizing any objective function and instead of finding a single optimal parameter vector, a set of feasible parameters vectors, consistent with the model structure, measurements and bounded uncertainty characterization, should usually be found.

The solution of many control and estimation problems under uncertainty involves constructing reachable sets and their analogs. For models with linear dynamics under such set-membership uncertainty there are several constructive approaches which allow finding effective estimates of reachable sets. We note here two of the most developed approaches to research in this area. The first one is based on ellipsoidal calculus [Chernousko, 1994; Kurzhanski and Valyi, 1997] and the second one uses the interval analysis [Walter and Pronzato, 1997]. Among other interesting approaches to solving the problems of estimation of the dynamics of the control systems we also note results [Boyd, El Ghaoui, Feron and Balakrishna, 1994; Chernousko and Ovseevich, 2004; Goncharova and Ovseevich, 2010].

Using results of the theory of trajectory tubes of control systems and techniques of differential inclusions theory we find set-valued estimates of related reachable sets of such impulsive uncertain control system. The algorithms of constructing the external ellipsoidal estimates for studied systems are given. Numerical simulation results related to the proposed techniques and to the presented algorithms are also included.

2 Basic Notations

Let us introduce the following basic notations.

Let $\mathbb{R}^n$ be the n–dimensional vector space, $\operatorname{comp}\mathbb{R}^n$ be the set of all compact subsets of $\mathbb{R}^n$, $\operatorname{conv}\mathbb{R}^n$ be the set of all convex and compact subsets of $\mathbb{R}^n$, $\mathbb{R}^{n\times n}$ stands for the set of all real $n\times n$–matrices and $x'y = (x,y) = \sum_{i=1}^n x_i y_i$ be the usual inner product of $x,y\in\mathbb{R}^n$ with prime as a transpose, $\|x\| = (x'x)^{1/2}$ be the vector norm for $x\in\mathbb{R}^n$, $I\in\mathbb{R}^{n\times n}$ be the identity matrix, $\mathrm{Tr}(A)$ be the trace of $n\times n$-matrix A (the sum of its diagonal elements), $\operatorname{diag} b = \operatorname{diag}\{b_i\}$ be the diagonal matrix A with $a_{ii} = b_i$ where b_i are components of the vector b.

We denote by

$$B(a,r) = \{x\in\mathbb{R}^n : \|x-a\| \le r\}$$

the ball in $\mathbb{R}^n$ with a center $a\in\mathbb{R}^n$ and a radius $r>0$ and by

$$E(a,Q) = \{x\in\mathbb{R}^n :\ (Q^{-1}(x-a),(x-a)) \le 1\}$$

the ellipsoid in $\mathbb{R}^n$ with a center $a\in\mathbb{R}^n$ and with a symmetric positive definite $n\times n$-matrix Q.

For $x,y\in\mathbb{R}^n$ we will use the notation $x\cdot y' = Z$, where matrix

$$Z = \{z_{ij} = x_i y_j :\ 1\le i,j\le n\} \in \mathbb{R}^{n\times n}.$$

Denote by $h(A,B)$ the Hausdorff distance between sets $A,B\in\mathbb{R}^n$, $h(A,B) = \max\{h^+(A,B),\, h^-(A,B)\}$, with $h^+(A,B)$ and $h^-(A,B)$ being the Hausdorff semidistances between A and B, $h^+(A,B) = \sup\{d(x,B) : x\in A\}$, $h^-(A,B){=}h^+(B,A)$, $d(x,A){=}\inf\{\|x-y\| {:}\ y{\in}A\}$.

3 Linear Impulsive Control System

Let us first consider the following linear control system

$$dx(t) = A(t)x(t)dt + B(t)dv(t), \tag{1}$$
$$x\in\mathbb{R}^n,\quad x(t_0-0) = x_0\quad t\in[t_0;T].$$

Here the given matrix-function $A(t)\in\mathbb{R}^{n\times n}$ and vector-function $B(t)\in\mathbb{R}^n$ are continuous on $t\in[t_0,T]$. The impulsive function v is the function of bounded variation on $[t_0,T]$, monotonically increasing and right-continuous. We assume that for some $\mu>0$ and

$$\operatorname*{Var}_{t\in[t_0,T]} v(t) = \sup_{\{t_i\}} \sum_{i=1}^k |v(t_i) - v(t_{i-1})| \le \mu,$$

where $t_i : t_0 \le t_1 \le \ldots \le t_k = T$. Denote by $\mathcal{V}$ the class of all admissible controls $v(\cdot)$.

The initial condition $x(t_0-0)$ of the system (1) is unknown but bounded

$$x_0\in\mathcal{X}_0 = E(a_0,Q_0). \tag{2}$$

Let the function $x(\cdot) = x(\cdot;t_0,x_0,v(\cdot))$ be a *solution* of the system (1) with the initial state $x_0\in\mathcal{X}_0$ (2) and with admissible controls $v(t)\in\mathcal{V}$.

The *trajectory tube* $\mathcal{X}(\cdot)$ of the system (1)–(2) is defined as the following set

$$\mathcal{X}(\cdot) = \bigcup\big\{x(\cdot;\, t_0,x_0,v(\cdot)) :\ x_0{\in}\mathcal{X}_0,\ v(\cdot){\in}\mathcal{V}\big\}.$$

The *reachable set* is the cross-section $\mathcal{X}(t)$ of this set at the instant t ($t\in[t_0,T]$).

Let us introduce a new time variable [Rishel, 1965]:

$$\eta(t) = t + \int_{t_0}^{t} dv(s),$$

and a new state coordinate

$$\tau(\eta) = \inf\{t \mid \eta(t)\ge\eta\}.$$

Consider the following auxiliary equation:

$$\frac{d}{d\eta}\begin{pmatrix} z\\ \tau\end{pmatrix} \in H(\tau,z), \tag{3}$$
$$z(t_0) = x_0 \in \mathcal{X}_0 = E(a_0,Q_0),$$
$$\tau(t_0) = t_0,\quad t_0\le\eta\le T+\mu,$$
$$H(\tau,z) = \bigcup_{0\le\nu\le1}\left\{(1-\nu)\begin{pmatrix}A(\tau)z\\ 1\end{pmatrix} + \nu\begin{pmatrix}B(\tau)\\ 0\end{pmatrix}\right\}.$$

Denote by $w = \{z, \tau\}$ the extended state vector of the system (3) and by $W(\eta) = W(\eta; t_0, w_0, \mathcal{X}_0 \times \{t_0\})$ $(t_0 \le \eta \le T + \mu)$ the reachable set of the system (3).

Theorem 1. [Filippova, 2010] *The following inclusion holds true for* $\sigma > 0$:

$$W(t_0 + \sigma) \subseteq W(t_0, \sigma) + o(\sigma)B^{n+1}(0, 1), \qquad (4)$$

$$\lim_{\sigma \to +0} \sigma^{-1} o(\sigma) = 0,$$

$$W(t_0, \sigma) = \bigcup_{0 \le \nu \le 1} W(t_0, \sigma, \nu),$$

$$W(t_0, \sigma, \nu) = \begin{pmatrix} E\big(a^+(t_0, \sigma, \nu), Q^+(t_0, \sigma, \nu)\big) \\ t_0 + \sigma(1 - \nu) \end{pmatrix}.$$

Here

$$a^+(t_0, \sigma, \nu) = (I + \sigma(1-\nu)A(t_0))a_0 + \sigma\nu B(t_0),$$

$$Q^+(t_0, \sigma, \nu) = (I + \sigma(1-\nu)A(t_0))Q_0(I + \sigma(1-\nu)A(t_0))'$$

is true for all $\sigma > 0$.

Remark 1. [Filippova and Matviychuk, 2011] To determinate simpler estimate of the reachable set $W(t_0 + \sigma)$ we introduce small parameter $\varepsilon > 0$ and embed the degenerate ellipsoid $W(t_0, \sigma, \nu)$ in nondegenerate ellipsoid $E_\varepsilon\big(w(t_0, \sigma, \nu), O_\varepsilon(t_0, \sigma, \nu)\big)$:

$$W(t_0, \sigma, \nu) \subseteq E_\varepsilon\big(w(t_0, \sigma, \nu), O_\varepsilon(t_0, \sigma, \nu)\big),$$

$$w(t_0, \sigma, \nu) = \begin{pmatrix} a^+(t_0, \sigma, \nu) \\ t_0 + \sigma(1 - \nu) \end{pmatrix},$$

$$O_\varepsilon(t_0, \sigma, \nu) = \begin{pmatrix} Q^+(t_0, \sigma, \nu) & 0 \\ 0 & \varepsilon^2 \end{pmatrix}.$$

Thus, for all small $\varepsilon > 0$ we get

$$W(t_0, \sigma) \subset W_\varepsilon(t_0, \sigma),$$

$$W_\varepsilon(t_0, \sigma) = \bigcup_{0 \le \nu \le 1} E_\varepsilon\big(w(t_0, \sigma, \nu), O_\varepsilon(t_0, \sigma, \nu)\big)$$

and $\lim_{\varepsilon \to +0} h(W(t_0, \sigma), W_\varepsilon(t_0, \sigma)) = 0$. The passage to the family of nondegenerate ellipsoids enables one to use the algorithms of [Vzdornova and Filippova, 2006] and construct an external estimate of the union of the ellipsoids $W_\varepsilon(t_0, \sigma)$

$$W_\varepsilon(t_0, \sigma) \subset E_\varepsilon(w^+(\sigma), O^+(\sigma)).$$

The passage to the family of nondegenerate ellipsoids enables one to use the algorithms of [Filippova and Matviychuk, 2011; Matviychuk, 2012] and construct an external estimate $E_\varepsilon(w^+(\sigma), O^+(\sigma))$ of the union of ellipsoids $W_\varepsilon(t_0, \sigma)$. Therefore we get ellipsoidal estimates of the reachable set $W(t_0 + \sigma)$

$$W(t_0 + \sigma) \subset E_\varepsilon(w^+(\sigma), O^+(\sigma)) + o(\sigma)B(0, 1).$$

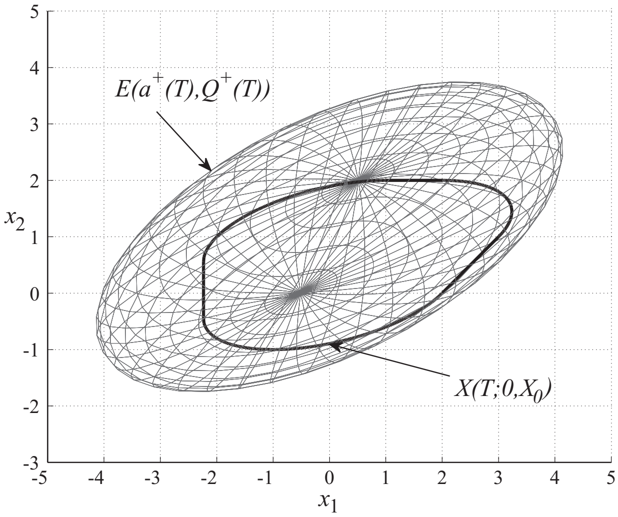

Figure 1. Reachable set $\mathcal{X}(T)$ of the linear impulsive control system (5) and its external estimate $E(a^+(T), Q^+(T))$.

The following lemma explains the construction of the differential inclusion (3).

Lemma 1. [Filippova and Matviychuk, 2011] The set $\mathcal{X}(T) = \mathcal{X}(T, t_0, \mathcal{X}_0)$ is the projection of $W(T + \mu)$ at the subspace of variables z: $\mathcal{X}(T) = \pi_z W(T + \mu)$.

The iterative algorithm based on Theorem 1 is given in [Filippova and Matviychuk, 2011] and may be used to produce the external ellipsoidal tube estimating the reachable sets of the system (1) on the whole time interval $t \in [t_0, T]$.

Example 1. Consider the following impulsive control system

$$\begin{cases} dx_1(t) = x_2(t)dt, \\ dx_2(t) = dv(t), \end{cases} \quad t_0 \in [0, 1], \qquad (5)$$

$$x_0 \in \mathcal{X}_0 = E(0, Q_0), \quad Q_0 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mu = 1.$$

Figure 1 illustrates the external estimation algorithm based on Theorem 1. The external ellipsoidal estimate $E(a^+(T), Q^+(T))$ and exact reachable set $\mathcal{X}(T)$ of the linear impulsive control system (5) are presented at Figure 1.

4 Bilinear System

Consider the bilinear control system [Kurzhanski and Filippova, 1993]

$$\dot{x} = A(t)\, x + u(t), \quad t \in [t_0, T], \qquad (6)$$
$$x_0 \in \mathcal{X}_0 = E(a_0, Q_0)$$

where the right-hand side in is a bilinear function of variables $\{x; A(\cdot)\}$ (a state vector $x \in \mathbb{R}^n$ and a matrix $A(\cdot) \in \mathbb{R}^{n\times n}$). The measurable matrix function $A(t) \in \mathbb{R}^{n\times n}$ in (6) is unknown but belongs to given set-valued constraints $\mathcal{A}$

$$A(t) \in \mathcal{A}, \quad t \in [t_0, T]. \tag{7}$$

Let us consider a special class of dynamical system (6), where for every t the values $\mathcal{A}$ are sets of all diagonal $n \times n$ matrices $A(t)$ with unknown but bounded diagonal elements $a_{ii}(t)$:

$$\begin{aligned} \mathcal{A} = \{A(t) \in \mathbb{R}^{n\times n} :\ & A(t) = \operatorname{diag} a, \\ & a = (a_1, \dots, a_n) \in \mathbf{A}_0\}, \end{aligned} \tag{8}$$

$$\mathbf{A}_0 = \{a \in \mathbb{R}^n :\ \sum_{i=1}^{n} |a_i|^2 \le 1\}.$$

It is assumed that control functions $u(t) \in \mathbb{R}^n$ in (6) are Lebesgue measurable on $[t_0, T]$. These controls are satisfying constraint for a.e. $t \in [t_0, T]$

$$u(t) \in \mathcal{U} = E(\hat{a}, \hat{Q}). \tag{9}$$

The differential control system (6) with constraints (8)–(9) describes a model of uncertain dynamical system with a unknown a priori matrix for which inclusion $A(t) \in \mathcal{A}$ is given.

Let the function $x(\cdot) = x(\cdot; t_0, x_0, A(\cdot), u(\cdot))$ be a *solution* of the system (6) with initial state $x_0 \in \mathcal{X}_0$, admissible control $u(t) \in \mathcal{U}$ and a matrix $A(t) \in \mathcal{A}$ satisfying (7)–(9). The *trajectory tube* $\mathcal{X}(\cdot)$ of the system (6)–(9) is defined as the following set

$$\begin{aligned} \mathcal{X}(\cdot) = \bigcup \big\{ x(\cdot) = x\big(\cdot;\, t_0, x_0, A(\cdot), u(\cdot)\big) :\ & x_0 \in \mathcal{X}_0, \\ & A(\cdot) \in \mathcal{A},\ u(\cdot) \in \mathcal{U} \big\}. \end{aligned}$$

The *reachable set* is the cross-section $\mathcal{X}(t)$ of this set at the instant t ($t \in [t_0, T]$).

Note that the reachable sets $\mathcal{X}(t)$ need not be convex for considering bilinear system. However, these sets have other geometrical properties.

A set $Z \subseteq \mathbb{R}^n$ is called *star-shaped* (with center c) if $c + \lambda(Z - c) \subseteq Z$ for all $\lambda \in [0, 1]$.

The set of all star-shaped compact subsets $Z \subseteq \mathbb{R}^n$ with center c will be denoted by $\operatorname{St}(c, \mathbb{R}^n)$, $\operatorname{St}\mathbb{R}^n = \operatorname{St}(0, \mathbb{R}^n)$.

Assumption 1. *For every $t \in [t_0, T]$ the inclusion $0 \in \mathcal{U}$ is true. The inclusion $0 \in \mathcal{X}_0$ is true.*

Theorem 2. [Kurzhanski and Filippova, 1993] *Under Assumption 1 the reachable sets $\mathcal{X}(t)$ are star-shaped and compact sets for all $t \in [t_0, T]$ ($\mathcal{X}(t) \in \operatorname{St}\mathbb{R}^n$).*

We need the following notation

$$\mathcal{M} * X = \{z \in \mathbb{R}^n :\ z = Mx,\ M \in \mathcal{M},\ x \in X\},$$

where $\mathcal{M} \in \operatorname{conv}\mathbb{R}^{n\times n}$, $X \in \operatorname{conv}\mathbb{R}^n$.

Then the evolution equation that describes the dynamics of star-shaped trajectory tubes is described in the following theorem.

Theorem 3. [Filippova and Lisin, 2000] *The trajectory tube $\mathcal{X}(t)$ of the bilinear differential system (6) with constraints (7)–(9) is the unique solution to the evolution equation*

$$\lim_{\sigma\to+0} \sigma^{-1} h\big(\mathcal{X}(t+\sigma), (I + \sigma\mathcal{A}) * \mathcal{X}(t) + \sigma\mathcal{U}\big) = 0, \tag{10}$$

with initial condition

$$\mathcal{X}(t_0) = \mathcal{X}_0, \quad t \in [t_0, T].$$

We will denote the *Minkowski function* of a set $M \in \operatorname{St}\mathbb{R}^n$ by

$$h_M(z) = \inf\{t > 0 :\ z \in tM,\ z \in \mathbb{R}^n\}.$$

Now we need calculate the Minkowski function of a reachable set $\mathcal{X}(t_0 + \sigma)$ or its approximation $(I + \sigma\mathcal{A}) * \mathcal{X}_0 + \sigma\mathcal{U}$.

We will assume further that Assumption 1 is satisfied.

Let $\rho(l|M)$ be the support function of a convex compact set $C \in \operatorname{conv}\mathbb{R}^n$, i.e.,

$$\rho(l|C) = \max\{(l, c) :\ c \in C,\ l \in \mathbb{R}^n\}.$$

Theorem 4. [Filippova and Lisin, 2000] *For every $z \in \mathbb{R}^n$ such that $z_i \ne 0$ ($i = 1, \dots, n$) the following formula is true:*

$$\begin{aligned} h_{\mathcal{A}_0 * \mathcal{X}_0}(z) = \min \Big\{ & \max_{l\ne 0} \frac{1}{\rho(l|\mathcal{X}_0)} \sum_{i=1}^{n} l_i z_i a_i^{-1} : \\ & a \in \mathbf{A}_0,\ a_i \ne 0,\ i = 1, \dots, n \Big\}. \end{aligned}$$

Corollary 1. [Matviychuk, 2016] *For every $z \in \mathbb{R}^n$ such that $z_i \ne 0$ ($i = 1, \dots, n$) the following formula is true:*

$$\begin{aligned} h_{(I+\sigma\mathcal{A}_0)*\mathcal{X}_0}(z, \sigma) = \min \Big\{ & \max_{l\ne 0} \frac{1}{\rho(l|\mathcal{X}_0)} \sum_{i=1}^{n} \frac{l_i z_i}{1 + \sigma a_i} : \\ & a \in \mathbf{A}_0,\ i = 1, \dots, n \Big\}. \end{aligned} \tag{11}$$

The next theorem describes discrete internal ellipsoidal estimates of reachable sets $\mathcal{X}(t)$ of the uncertain bilinear system (6) with constraints (7), $\mathcal{X}_0 = E(0, Q_0)$ and $\mathcal{U} = \{0\}$.

Theorem 5. *Let $\mathcal{X}_0 = E(0, Q_0)$ and $\mathcal{U} = \{0\}$, then following external estimate holds*

$$\mathcal{X}(t_0+\sigma) \subseteq B(0, r(\sigma)) + o(\sigma)B(0,1),$$
$$r(\sigma) = \max_z \|z\| \left(h_{(I+\sigma\mathcal{A})*\mathcal{X}_0}(z,\sigma)\right)^{-1}. \quad (12)$$

Proof. Here we use the properties of the Minkowski function. Formulas in (12) may be derived by direct calculation. ▲

By using equality (11) for the given sets $\mathcal{A}$ and $\mathcal{X}_0$ we get the Minkowski function of set $(I + \sigma\mathcal{A}) * \mathcal{X}_0$ [Matviychuk, 2016]

$$h_{(I+\sigma\mathcal{A})*\mathcal{X}_0}(z,\sigma) = \Big(\|Q_0^{-\frac{1}{2}} z\|^2 - 2\sigma\big(\sum_{i=1}^{n} w_i^4(z)\big)^{\frac{1}{2}}\Big)^{\frac{1}{2}} + $$
$$+ o(\sigma)\|Q_0^{-\frac{1}{2}} z\|, \quad (13)$$
$$w(z) = Q_0^{-\frac{1}{2}} z, \quad \lim_{\sigma\to+0} \sigma^{-1} o(\sigma) = 0.$$

Theorem 6. *Let $\mathcal{X}_0 = E(a_0, Q_0)$ and $\mathcal{U} = \{0\}$. Then for all $\sigma > 0$ the following inclusion holds*

$$\mathcal{X}(t_0+\sigma) \subseteq E(a_0, Q_1(\sigma)) + o(\sigma)B(0,1), \quad (14)$$
$$\lim_{\sigma\to+0} \sigma^{-1} o(\sigma) = 0,$$

where

$$Q_1(\sigma) = \mathrm{diag}\{(p^{-1}+1)\sigma^2(a_i^0)^2 + (p+1)r^2(\sigma)\},$$

$a_0 = \{a_i^0\}$, $r(\sigma)$ is defined in Theorem 5 and p is the unique positive root of the equation

$$\sum_{i=1}^{n} \frac{1}{p+\alpha_i} = \frac{n}{p(p+1)},$$

with $\alpha_i \geq 0$ $(i = 1, ..., n)$ being the roots of the following equation $\prod_{i=1}^{n} \big(\sigma^2(a_i^0)^2 - \alpha r^2(\sigma)\big) = 0$.

Proof. From Theorem 3 we have the funnel equation for small σ $(t = t_0 + \sigma)$

$$h\big(\mathcal{X}(t_0+\sigma), (I+\sigma\mathcal{A}) * \mathcal{X}_0\big) = o(\sigma), \quad (15)$$
$$\lim_{\sigma\to+0} \sigma^{-1} o(\sigma) = 0.$$

Note that

$$(I+\sigma\mathcal{A}) * \mathcal{X}_0 = (I+\sigma\mathcal{A}) * E(a_0, Q_0) =$$
$$= a_0 + \sigma\mathcal{A} * a_0 + (I+\sigma\mathcal{A}) * E(0, Q_0).$$

The set $\mathcal{A} * a_0$ is convex, therefore

$$\rho(l|\mathcal{A} * a_0) = \max_{A\in\mathcal{A}} l'Aa_0 = \big(\sum_{i=1}^{n} l_i^2 (a_i^0)^2\big)^{\frac{1}{2}} =$$
$$= \rho\big(l|E(0, \mathrm{diag}\{(a_i^0)^2\})\big).$$

Taking into account results of the Theorem 5 we have

$$(I+\sigma\mathcal{A}) * \mathcal{X}_0 = E(a_0, \sigma^2 \mathrm{diag}\{(a_i^0)^2\}) +$$
$$+ (I+\sigma\mathcal{A}) * E(0, Q_0) \subseteq$$
$$\subseteq E\big(a_0, \sigma^2 \mathrm{diag}\{(a_i^0)^2\}\big) + B(0, r(\sigma)),$$

where $r(\sigma)$ is defined in Theorem 5. Now we find external ellipsoidal estimate of the sum of two ellipsoids $E(a_0, \sigma^2 \mathrm{diag}\{(a_i^0)^2\})$ and $B(0, r(\sigma))$ [Chernousko, 1994; Kurzhanski and Valyi, 1997] and get the external estimate (14). ▲

Theorem 7. *Let $\mathcal{X}_0 = E(a_0, Q_0)$ and $\mathcal{U} = E(\hat{a}, \hat{Q})$. Then for the trajectory tube $\mathcal{X}(t)$ of the system (6) with constraints (7)–(9) all $\sigma > 0$ the following inclusion holds*

$$\mathcal{X}(t_0+\sigma) \subseteq E(a^+(\sigma), Q^+(\sigma)) + o(\sigma)B(0,1), \quad (16)$$
$$\lim_{\sigma\to+0} \sigma^{-1} o(\sigma) = 0,$$

where

$$a^+(\sigma) = a_0 + \sigma\hat{a},$$
$$Q^+(\sigma) = (p^{-1}+1)Q_1(\sigma) + (p+1)\sigma^2\hat{Q},$$

here $Q_1(\sigma)$ is defined in Theorem 6 and p is the unique positive root of the equation

$$\sum_{i=1}^{n} \frac{1}{p+\alpha_i} = \frac{n}{p(p+1)},$$

with $\alpha_i \geq 0$ $(i = 1, ..., n)$ being the roots of the following equation $|Q_1(\sigma) - \alpha\sigma^2\hat{Q}| = 0$.

Proof. The proof of this theorem follows from the previous Theorem 5–6 and based the procedure of external ellipsoidal estimate of the sum of two ellipsoids given in [Chernousko, 1994; Kurzhanski and Valyi, 1997]. ▲

Algorithm 1. The time segment $[t_0, T]$ is subdivided into subsegments $[t_i, t_{i+1}]$ where $t_i = t_0 + ih$ $(i = 1, \ldots, m)$, $h = (T - t_0)/m = \sigma$, $t_m = T$.

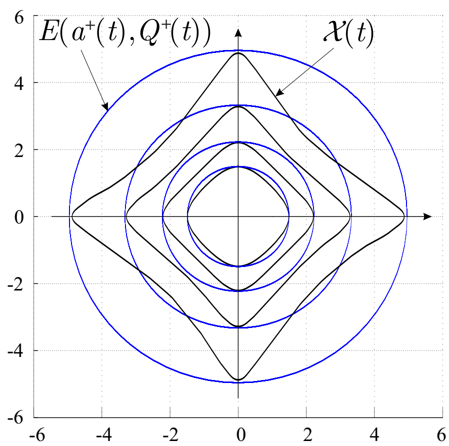

Figure 2. Reachable sets $\mathcal{X}(t)$ and their external estimates $E(a^-(t), Q^-(t))$ for $t = 0.4; 0.8; 1.2; 1.6$.

• For given $\mathcal{X}_0 = E(a_0, Q_0)$ and $\mathbf{A}_0 = B(0, 1)$ we find the external estimate $E(a^+(\sigma), Q^+(\sigma))$ by Theorem 7 such that

$$\mathcal{X}(t_1) = \mathcal{X}(t_0 + \sigma) \subseteq E(a^+(\sigma), Q^+(\sigma)).$$

• Consider the system on the next subsegment $[t_1, t_2]$ with $E(a^+(\sigma), Q^+(\sigma))$ as the initial ellipsoid at instant t_1.

The following steps repeat the previous iteration.

At the end of the process we will get the external estimate of the tube $\mathcal{X}(\cdot)$ of the system (6) with constraints (7)–(9), with accuracy tending to zero when $m \to \infty$.

Example 2. Consider the following bilinear system

$$\dot{x} = A(t)x, \quad t \in [0, 1.6], \qquad (17)$$
$$x_0 \in \mathcal{X}_0 = B(0, 1), \quad \mathcal{U} = \{0\}.$$

Here the uncertain bounded matrix function $A(t) \in \mathcal{A}$ where

$$\mathcal{A}=\left\{A(t) : A(t) = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \, a_1^2+a_2^2 \le 1, \, t \in [0, 2]\right\}.$$

The reachable sets $\mathcal{X}(t)$ and their external ellipsoidal estimates $E(a^-(t), Q^-(t))$ calculated by the Algorithm 1 are given in Figure 2 and Figure 3.

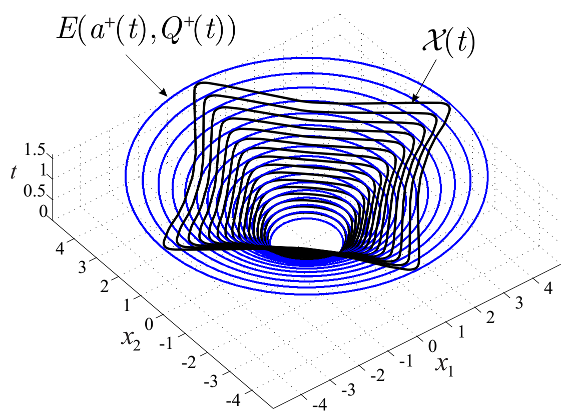

Figure 3. Trajectory tube $\mathcal{X}(t)$ and its ellipsoidal estimating tube $E(a^-(t), Q^-(t))$ for the bilinear system with uncertain initial states.

5 Impulsive Control System with Bilinear Uncertainty

Consider the following impulsive bilinear control system ($t_0 \le t \le T$)

$$dx(t){=}(A(t)x(t){+}u(t))dt{+}Bdv(t), \qquad (18)$$
$$x(t_0 - 0) = x_0 \in \mathcal{X}_0 = E(a_0, Q_0), \quad t \in [t_0, T]$$
$$A(t) \in \mathcal{A}, \quad u(t) \in \mathcal{U} = E(\hat{a}, \hat{Q}), \quad v(t) \in \mathcal{V},$$

where $x(t) \in \mathbb{R}^n$ and $B \in \mathbb{R}^n$, the set $\mathcal{A}$ defined in (8), the set $\mathcal{V}$ defined in Section 3.

The *trajectory tube* $\mathcal{X}(\cdot)$ of the system (18) is defined as the following set

$$\mathcal{X}(\cdot) = \bigcup \{x(\cdot) = x(\cdot;\, t_0, x_0, A(\cdot), u(\cdot), v(\cdot)) :$$
$$x_0{\in}\mathcal{X}_0,\ A(\cdot){\in}\mathcal{A},\ u(\cdot){\in}\mathcal{U},\ v(\cdot){\in}\mathcal{V}\}$$

and the *reachable set* is the cross-sections $\mathcal{X}(t)$ of this set at the instant t ($t \in [t_0, T]$).

We follow the scheme of the Section 3 of the present paper in considering the extended differential inclusion under additional assumptions:

$$\frac{d}{d\eta}\begin{pmatrix} z \\ \tau \end{pmatrix} \in H(\tau, z), \qquad (19)$$
$$z(t_0) = x_0 \in \mathcal{X}_0 = E(a_0, Q_0),$$
$$\tau(t_0) = t_0, \quad t_0 \le \eta \le T + \mu,$$

$$H(\tau, z) = \bigcup_{0 \le \nu \le 1} \left\{ \nu \begin{pmatrix} B \\ 0 \end{pmatrix} + \right.$$
$$\left. +(1-\nu)\begin{pmatrix} A(\tau)z + E(\hat{a}, \hat{Q}) \\ 1 \end{pmatrix} \right\}.$$

Denote by $w = \{z, \tau\}$ the extended state vector of the system (19) and by $W(\eta) = W(\eta; t_0, w_0, \mathcal{A}, \mathcal{X}_0 \times$

$\{t_0\})$ $(t_0 \le \eta \le T+\mu)$ the reachable set of the system (19).

Theorem 8. *The following inclusion holds true for* $\sigma{>}0$:

$$W(t_0+\sigma) \subseteq W(t_0,\sigma) + o(\sigma)B(0,1),$$
$$\lim_{\sigma\to+0} \sigma^{-1} o(\sigma) = 0.$$

Here

$$W(t_0,\sigma) = \bigcup_{0\le\nu\le1} W(t_0,\sigma,\nu),$$
$$W(t_0,\sigma,\nu) = \begin{pmatrix} E\big(a^*(\sigma,\nu),Q^*(\sigma,\nu)\big) \\ t_0+\sigma(1-\nu)\end{pmatrix}, \qquad (20)$$
$$a^*(\sigma,\nu) = a_0 + \sigma(1-\nu)\hat a + \sigma\nu B,$$
$$Q^*(\sigma,\nu) = (p^{-1}+1)\tilde Q(\sigma,\nu) + (p+1)\sigma^2(1-\nu)^2\hat Q, \qquad (21)$$

where

$$\tilde Q(\sigma,\nu) = \operatorname{diag}\{(q^{-1}+1)\sigma^2(1-\nu)^2(a_i^0)^2 + \\ +(q+1)r^2(\sigma,\nu)\},$$
$$r(\sigma,\nu) = \max_z \|z\| \left(h_{(I+\sigma(1-\nu)\mathcal{A})*\mathcal{X}_0}(z,\sigma)\right)^{-1},$$

$q = q(\sigma,\nu)$ *is the unique positive root of the equation*

$$\sum_{i=1}^n \frac{1}{q+\alpha_i} = \frac{n}{q(q+1)},$$

and $\alpha_i = \alpha_i(\sigma,\nu) \ge 0$ *satisfy the equation* $\prod_{i=1}^n \left(\sigma^2(1-\nu)^2(a_i^0)^2 - \alpha r^2(\sigma,\nu)\right) = 0.$
In equation (21) $p = p(\sigma,\nu)$ *is the unique positive root of the equation*

$$\sum_{i=1}^n \frac{1}{p+\lambda_i} = \frac{n}{p(p+1)},$$

where $\lambda_i = \lambda_i(\sigma,\nu) \ge 0$ *satisfy the equation* $|\tilde Q(\sigma,\nu) - \lambda\sigma^2(1-\nu)^2\hat Q| = 0.$

Proof. The proof of this theorem is based on a combination of the techniques described above and on the results of the paper [Filippova and Matviychuk, 2011; Filippova and Matviychuk, 2014]. ▲

Remark 2. The set $W(t_0,\sigma,\nu)$ in (20) is degenerate ellipsoid in the extended space $\mathbb{R}^{n+1}$ for each value of the parameter ν, but the set $W(t_0,\sigma) = \cup\{W(t_0,\sigma,\nu)|\ 0\le\nu\le1\}$ may be not an ellipsoid. By analogy with linear case (in Remark 1), we fix an arbitrary small parameter $\varepsilon>0$ and embed the degenerate ellipsoid $W(t_0,\sigma,\nu)$ in nondegenerate ellipsoid $E_\varepsilon\big(w(t_0,\sigma,\nu),O_\varepsilon(t_0,\sigma,\nu)\big)$ so that to satisfy the inclusion

$$W(t_0,\sigma,\nu) \subseteq E_\varepsilon\big(w(t_0,\sigma,\nu),O_\varepsilon(t_0,\sigma,\nu)\big),$$
$$w(t_0,\sigma,\nu) = \begin{pmatrix} a^+(t_0,\sigma,\nu) \\ t_0+\sigma(1-\nu)\end{pmatrix},$$
$$O_\varepsilon(t_0,\sigma,\nu) = \begin{pmatrix} Q^+(t_0,\sigma,\nu) & 0 \\ 0 & \varepsilon^2\end{pmatrix}.$$

Therefore, the inclusion

$$W(t_0,\sigma) = \bigcup_{0\le\nu\le1} W_\varepsilon(t_0,\sigma), \subseteq$$
$$\subseteq W_\varepsilon(t_0,\sigma) = \bigcup_{0\le\nu\le1} E_\varepsilon\big(w(t_0,\sigma,\nu),O_\varepsilon(t_0,\sigma,\nu)\big)$$

is also valid for any $\varepsilon > 0$. The passage to the union to family of nondegenerate ellipsoids allow to use algorithms from papres [Filippova and Matviychuk, 2011; Matviychuk, 2012] and construct an external estimate $E_\varepsilon(w^+(\sigma),O^+(\sigma))$ of the union of ellipsoids $W_\varepsilon(t_0,\sigma)$. Therefore we get ellipsoidal estimates of the reachable set $W(t_0+\sigma)$

$$W(t_0+\sigma) \subset E_\varepsilon(w^+(\sigma),O^+(\sigma)) + o(\sigma)B(0,1).$$

The following iterative algorithm based on Theorem 8 may be used to produce the external ellipsoidal tube estimating the reachable sets of the system (18) on the whole time interval $t\in[t_0,T]$.

Algorithm 2. The time segment $[t_0,T+\mu]$ is subdivided into subsegments $[t_i,t_{i+1}]$ where $t_i = t_0+ih$ $(i=1,\dots,m)$, $h=(T+\mu-t_0)/m$, $t_m = T+\mu$. Subdivide the segment $[0,1]$ into subsegments $[\nu_j,\nu_{j+1}]$ where $\nu_i = ih_*$, $h_* = 1/m$, $\nu_0=0$, $\nu_m=1$.

• Take $\sigma = h$ and for the given $\mathcal{X}_0 = E(a_0,Q_0)$ define by Theorem 8 the sets $W(\sigma,\nu_i)$ $(i=0,\dots,m)$.

• Find ellipsoid $E_\varepsilon(w_1(\sigma),O_1(\sigma))$ in $\mathbb{R}^{n+1}$ such that $W(\sigma,\nu_i) \subseteq E_\varepsilon(w_1(\sigma),O_1(\sigma))$ $(i=0,\dots,m)$. At this step we find the ellipsoidal estimate for the union of a finite family of ellipsoids [Filippova and Matviychuk, 2011; Matviychuk, 2012].

• Find the projection of $E(a_1,Q_1) = \pi_z E_\varepsilon(w_1(\sigma),O_1(\sigma))$ by Lemma 1.

• Consider the system on the next subsegment $[t_1,t_2]$ with $E(a_1,Q_1)$ as the initial ellipsoid at instant t_1.

• The following steps are repeated previous iteration.

At the end of the process we will get the external estimate $E(a^+(t),Q^+(t))$ of the reachable set of the impulsive control system (18) with bilinear uncertainty.

6 Conclusions

The paper deals with the problems of state estimation for uncertain impulsive control systems for which we assume that the initial state is unknown but bounded with given constraints and the matrix in the linear part of state velocities is also unknown but bounded.

Basing on results of ellipsoidal calculus developed earlier for some classes of uncertain systems we present the modified state estimation approach which uses the special constraints on the controls and uncertainty in the impulsive system and allows to construct the external ellipsoidal estimates of reachable sets.

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