A MATHEMATICAL MODEL OF HUMAN COCHLEA

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Abstract. We suggest a two-chamber model of human cochlea. The motion of the fluid is described by equations of hydrodynamics, which are supplemented by the equation of oscillations of the membrane. The equations are linearized and their solution is represented as Fourier harmonics with a given frequency. The harmonics satisfy a system of boundary value problems for ordinary differential equations with variable coefficients. Numerical solution of this system with a finite-difference approximation is hardly possible due to a big parameter in the problem and a closeness of the problem to a singular one. We suggest a new numerical method without saturation, which allows to solve the problem in a wide range of frequencies with an arbitrary and controlled precision. Computations confirmed Bekesy's theory. The low sound frequencies cause the deflection of the membrane at the upper part of the cochlea, whereas high sound frequencies cause the deflection of the main volute of the cochlea.

Key words: human cochlea, Bekesy's theory, oscillations, boundary value problem, numerical method without saturation.

A brief description of human cochlea. Human hear consists of three chambers (Fig. 1): the external, the middle, and the internal ones. Sound waves go into the external chamber (1), reach the cochlear partition (2), and cause its oscillations. By means of the three bones: malleus, anvil, and stirrup, which are situated in the middle chamber (3), the oscillations are fed into the threshold window (4) and cause pulse-like shifts of the perilymph of the threshold in the region (5) of the cochlea.

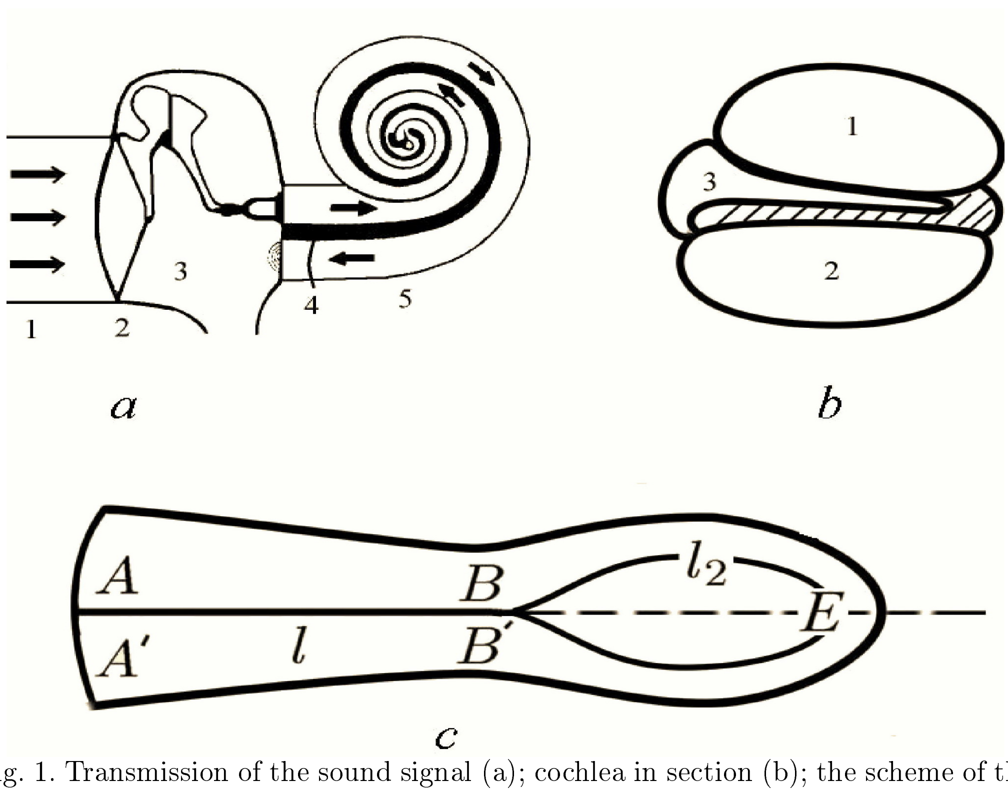

Fig. 1. Transmission of the sound signal (a); cochlea in section (b); the scheme of the spiral channel.

Human cochlea represents a spiral bone channel of about two and a half convolutions and has the length of about 32 mm. An unrolled cochlea looks like a flattened cone with the base width of about 9 mm and length of 5 mm.

The oscillations of the perilymph are transformed into mechanical oscillations and then into electrical nerve impulses. Details can be found in books on otorhinolaryngology.

Helmholz and Bekesy's theories are two different explanations of processes taking place in the inner human hear [1, 2]. Helmholz's theory is considered now as obsolete, so we concentrate on Bekesy's theory of *traveling wave*.

According to Bekesy's theory, the sound of a certain frequency generates a traveling wave on the basilar membrane. The place of the wave crest with biggest membrane shift depends on the frequency of the sound signal. The low sound frequencies cause the deflection of the membrane at the upper part of the cochlea, whereas high sound frequencies cause the deflection of the main volute of the cochlea. The basilar membrane in its turn causes the deformation of hair cells of the spiral organ at the place of the crest of the traveling wave.

System of equations. Theoretical description of hydrodynamical effects in the human cochlea is based on the approximation of the thin layer of the viscous incompressible fluid:

$$\begin{gathered} \rho\frac{\partial v}{\partial t} - \mu\frac{\partial^2 v}{\partial z^2} = -\frac{\partial p}{\partial x} + \frac{\mu}{B}\frac{\partial^2 q}{\partial x^2} \\ \frac{\partial B}{\partial t} + \frac{\partial q}{\partial x} = 0, \quad \frac{\partial p}{\partial z} = 0. \end{gathered} \tag{1}$$

where x and z are coordinates along and across the thin layer; $v(t,x,z)$, $p(t,x,z)$ and $\rho(t,x,z)$ are, respectively, longitudinal velocity, pressure, and density in the fluid; μ is the coefficient of dynamical viscosity; B is the thickness of layer; $q(t,x)$ is the volumetric flow rate through the section x.

A model of two-layer flow in the cochlea. A spread out spiral bone channel is shown in Fig. 1 (c). The basilar membrane of the length l separates the upper and the lower chambers of the channel, which communicate through the helicotrema. The helicotrema is modeled as an opening BEB' of the length l_2 and of the constant thickness b_1.

The basilar membrane is non-uniform along its length. Its Young modulus $E(x)$ depends on the longitudinal coordinate. According to different sources, the value of $E(x)$ decreases from $x = 0$ to $x = l$ by factor of 10^3 - 10^4 [3]. We assume exponential low of decreasing of $E(x)$: $E(x) = E_0 \times 10^{-4x/l}$.

The basilar membrane is attached to the side bone boundaries with variable tension $\sigma_y(x)$. The cross tension $\sigma_y(x)$ of the membrane causes tensile stress $\sigma(x)$ through the Poisson coefficient ν. Using equations of the linear theory of elasticity, and assuming ν to be constant, we obtain $\nu\sigma_y(x) = (1-\nu)\sigma(x)$, from where we derive the exponential low of decreasing $\sigma_y(x) = \sigma_y^0 \times 10^{-4x/l}$. Further we assume the Poisson coefficient ν to be close to unit, and so the cross tension is negligible in comparison to the tensile stress $\sigma(x)$.

Perilymph is modeled as a viscous fluid of the density ρ with the dynamical viscosity coefficient μ. The basilar membrane is deformed under the influence of the pressure differences of the perilymph in the upper (p_1) and lower (p_2) chambers. We assume that the membrane does not resist to the bending. The membrane is filled with endolymph, which is a Newton fluid with great viscosity. The pressure difference $p_1 - p_2$ causes membrane bending of the magnitude h (which is small) in the perpendicular direction to the membrane plane, so we neglect terms of higher order in h and assume that the tension of the membrane does not depend on h.

The Newton low for the membrane with variable tension takes the form

$$m(x)\frac{\partial^2 h}{\partial t^2} = p_2 - p_1 + \sigma(x)\frac{\partial^2 h}{\partial x^2} - k\frac{\partial h}{\partial t}, \qquad (2)$$

where $m(x)$ is the mass of the endolymph per unit of length, k is the coefficient of friction.

In the upper and lower chambers AB and $A'B'$ and in the opening BEB' of the cross-section $b_1 = b(l)$ (Fig. 1 (c) we take the system of equations (1).

The unperturbed length of the layers b_1 and b_2 is the same and equal to $b(x)$. In linear approximation with respect to the variables h, v, p, we take $b_i = b(x)$.

Boundary values. All functions are assumed to be continuous. The velocity satisfies no-slip conditions at the boundaries. The periodic signal at the entrance $x = 0$ has the frequency ω and

$$q(t, 0) = q_0 e^{i\omega t}.$$

The ends of the membrane are fixed:

$$h(t, 0) = 0, \quad h(t, l) = 0.$$

The solution of the above equations has the form

$$v = \tilde{v}(x, z)e^{i\omega t}, \quad h = \tilde{h}(x)e^{i\omega t}, \quad q = \tilde{q}(x)e^{i\omega t}, \quad p = \tilde{p}(x)e^{i\omega t}.$$

Then, after some transformations, we obtain boundary value problems for the functions $\tilde{h}(x), \tilde{q}(x), \tilde{p}(x)$.

We have three ODEs

$$\begin{gathered} \rho\omega\lambda\tilde{q} + (\lambda b - 2)i\left(-\frac{d\tilde{p}}{dx} + \frac{\mu}{b}\frac{d^2\tilde{q}}{dx^2}\right) = 0, \\ i\omega\tilde{h} - \frac{d\tilde{q}}{dx} = 0, \quad \sigma(x)\frac{d^2\tilde{h}}{dx^2} + (\omega^2 m(x) - ik\omega)\tilde{h} = 2\tilde{p}, \end{gathered} \qquad (3)$$

with the boundary conditions

$$\tilde{q}(0) = q_0, \quad \tilde{h}(0) = \tilde{h}(l) = 0, \quad \rho\omega\lambda\tilde{q}(l) + (\lambda b_1 - 2)i\left(\frac{2\tilde{p}(l)}{l_2}\right) = 0, \qquad (4)$$

The non-dimensional values. The values of the parameters are taken as follows: $\rho_0 \sim 1g/cm^3$, $b_0 = b(0) \sim 0.2cm$, $b_1 = b(l) \sim 0.1cm$, $\mu \sim 0.01g/(cm/, sec)$, $k = 10^3 g\, cm/sec$, $m \sim \rho b_0$, $\sigma(x)$ changes from $\sigma_0 = \sigma(0) = 10^7 Dyne/cm$ to $\sigma_1 = \sigma(l) = 10^3 Dyne/cm$, $l = 3$, $l_2 = 4cm$. Note that the force $1Dyne = 10^{-3}G$, hence the tension changes from 10 Kg/cm to 1 G/cм. This is confirmed by the research on the coefficient of elasticity of the membrane [3].

The function $\Sigma(X) = \sigma(x)/\sigma(0)$ decreases monotonously from $\Sigma(0) = 1$ to $\Sigma(1) \sim 10^{-4}$. The functions $B(X)$ and $M(X)$ are almost constant. The function $B(X)$ decreases monotonously from $B(0) = 1$ to $B(1) = 1/2$, and $M(X) = 1$.

The non-dimensional frequency $\Omega = \omega/\omega_0$ is expressed through the characteristic frequency

$$\omega_0 = \sqrt{\frac{\sigma_0 b_0}{\rho_0 l^4}} \approx 157 radian/sec.$$

The non-dimensional frequency Ω changes significantly and serves as a parameter of the problem.

Further, we make substitutions

$$x = lX,\ \tilde{h}(x) = b_0 H(X),\ \tilde{q}(x) = i\omega l b_0 Q(X),$$
$$\tilde{P}(x) = \sigma_0 b_0 P(X)/l^2,\ \omega = \omega_0 \Omega,\ \omega_0 = \sqrt{\sigma_0 b_0/\rho}/l^2,\ \sigma(x) = \sigma_0 \Sigma(X),$$
$$m(x) = \rho b_0 M(X),\ b(x) = b_0 B(X),\ b_0 = b(0),\ b_1 = b(l),$$
$$\mu_0 = \frac{\mu}{\sqrt{\rho \sigma_0 b_0}},\quad k_0 = \frac{k\omega_0 l^2}{\sigma_0},$$

into the equations (3), (4) and use method without saturation [4] for solution of the non-dimensional equations.

The results of computation. Fig. 2 shows the results of computation for 10 functions $|H(X)|$ corresponding to the frequencies $\Omega = 2^n$, $n = 0,\ 1,\ 2, \ldots, 9$. These functions are the amplitude envelopes of Bekesy's traveling waves. As it is seen on the Fig. 2, the maximum of $|H(X)|$ shifts to the point $X = 0$ as Ω increases.

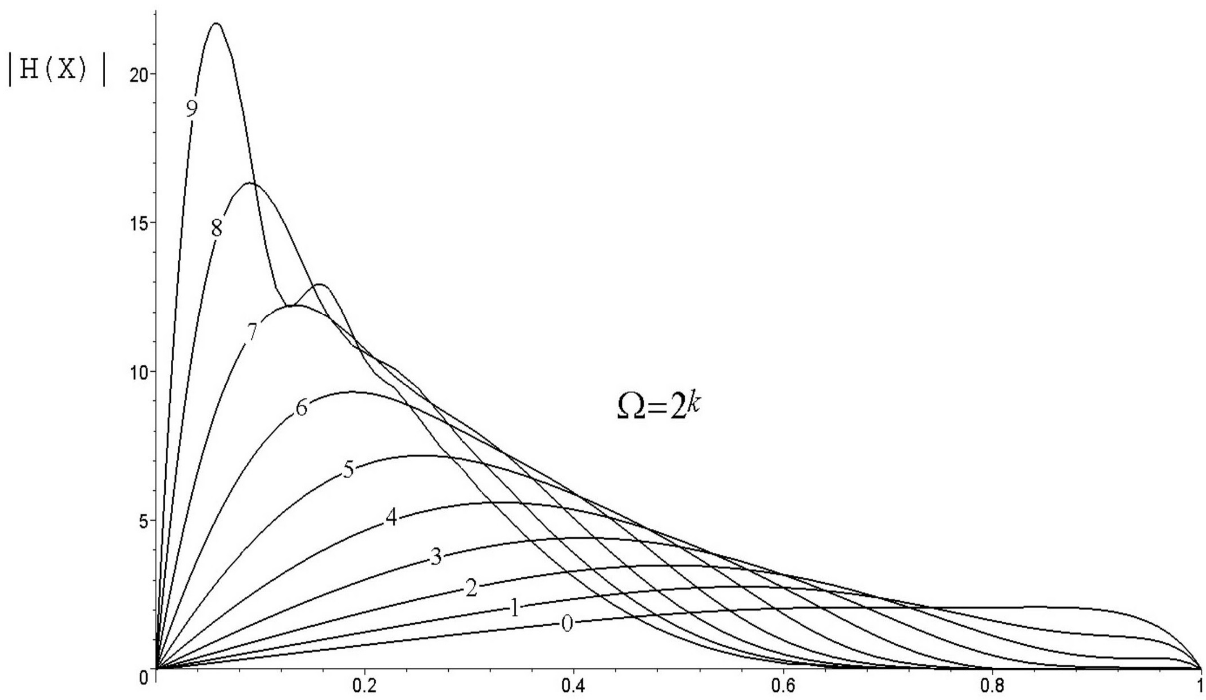

Fig. 2. Envelopes of Bekesy's traveling waves.

Fig. 3 shows various phases of the traveling waves for $\Omega = 1,\ 64$, that corresponds to the frequencies 25 and 1600 Hz. These results are in agreement with the Bekesy's theory, which was never obtained before in a mathematical model.

We note that the computational problems here are on such a scale that ordinary numerical methods were not able to obtain a solution. It is due to the problem being almost singular, and a very big parameter in the equations, that causes solutions to be very unstable. Numerical methods without saturation [4] are not subject to these difficulties.

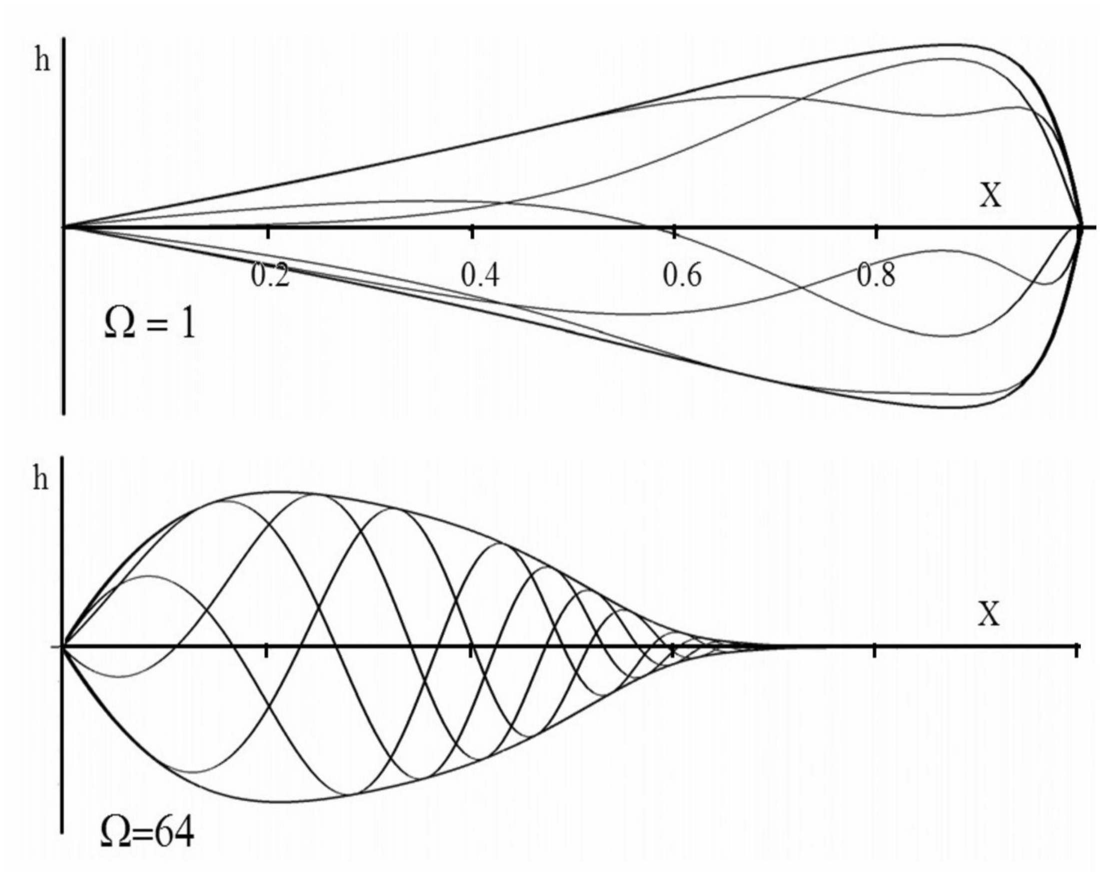

Fig. 3. phases of the traveling waves for $\Omega = 1$, 64.

This work was supported by RFBF grants 08-01-00251 and 08-01-00082.

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