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# AVERAGE CONSENSUS IN SYMMETRIC NONLINEAR MULTI-AGENT NETWORKS WITH NON-HOMOGENEOUS DELAYS

**Anton Proskurnikov**
Department of Mathematics and Mechanics
Saint Petersburg State University and Institute for Problems of Mechanical Engineering RAS
Russia
avp1982@gmail.com

**Abstract**
We study nonlinear continuous-time decentralized consensus algorithms with delayed couplings for networks of single-order agents. The network topology is undirected and may switch and the couplings may be nonlinear and uncertain, but supposed to satisfy conventional sector conditions. Using the absolute stability methods, we obtain effective condition for convergence of such consensus protocols, given that the network graph is uniformly connected.

**Key words**
Consensus, synchronization, networked systems, delayed systems.

## 1 Introduction

The problems of decentralized cooperative control in networked multi-agent systems are attracting enormous attention of the different research communities. These problems deal with complex dynamical systems constituted by autonomous simpler units, or *agents*, and are focuse mainly on achieving some desired collective behavior of the agents by means of local interactions. Examples of such a behavior are rendezvouz (gathering at one point), bypassing an area, moving in a swarm, target following etc. The term ”local interaction” presumes that agent does not use any information on the network in whole but interacts (via communication, sensoring or otherwise) to a restricted circle of other agents (*neighbors*). The relation of neighborship (in general, non-symmetric) determines the interaction graph, or *topology* of the network. This topology may be unknown and time-varying. Below we address the *consensus* or synchronization problem which concerns the design of algorithms, enabling the agents to get their states (or some output variables) synchronized.

Consensus algorithms take their origin in applied statistics, probability theory and theory of positive matrices [DeGroot, 1974], [Seneta, 1981] on the one hand, and in the computer science on the other hand [J.N. Tsitsiklis and Athans, 1986]. The idea of consensus (or synchronism) achieved via local interactions lies at the heart of numerous natural and social phenomena. Among them are synchronization of oscillator networks [Kuramoto, 1984], [Strogatz, 2000], [Yeung and Strogatz, 1999], [Earl and Strogatz, 2003], alignment in the flows of self-driven particles [Vicsek *et al.*, 1995], opinion dynamics in social networks [Krause, 2000], synchronization in complex networks [Wu, 2007]. One of the most important applications of the consensus algorithms is the formation control and analysis of the regular collective behavior and intelligence of complex biological populations, such as swarms of insects, schools of fish, flocks of birds, herds of mammals etc. In the pioneering paper [Reynolds, 1987] the three empirical rules of flocking were proposed: to avoid collisions with nearby flockmates, to match velocity with nearby flockmates, to stay close to nearby flockmates. The second of Reynolds' rules is nothing but the synchronization (alignment) of the agents velocities. The history and profound results on convergence of the consensus algorithms, as well as further applications, can be found in e.g. in [Jadbabaie *et al.*, 2003], [Olfati-Saber and Murray, 2004], [Blondel *et al.*, 2005], [Moreau, 2005], [Ren and Beard, 2005], [Olfati-Saber *et al.*, 2007], [Lin *et al.*, 2007] just to mention a few.

Despite that the consensus protocols were seriously investigated, a number of questions still remain open even for the networks of simplest dynamical agents modeled by the single integrator dynamics. One of such problems is delay robustness of consensus protocols. This problem is of high importance since inevitable small delays in actuators, sensors and communication channels may potentially cause instability of the closed-loop system. Despite the considerable efforts towards finding effective criteria for consensus establishing in the presence of delays (see e. g. [Münz, 2010] for the review of recent results), such conditions are known for a few situations only.

The most exhaustively investigated are the consensus algorithms with so-called "communication" delays that do not affect the own state of the agent, see [Moreau, 2004], [Chopra and Spong, 2006], [Papachristodoulou *et al.*, 2010], [Münz, 2010] among the other. In other words, each agent directly measures its state without any delay but the data from neighboring agents may be retarded. The Lyapunov-based techniques developed for non-delayed protocols [Moreau, 2004], [Moreau, 2005], [Lin *et al.*, 2007] may be directly extended to consensus algorithms with bounded communication delays. The common idea of those methods is to consider the convex hull of the agents state vectors over sufficiently long interval as a vector-valued analogue of Lyapunov function (that shrinks as time progresses). The diameter (calculated in appropriate norm) of the convex hull appears to be a scalar Lyapunov function for the system. Using such an approach, it may be shown that bounded delays (with arbitrarily large upper bound) do not violate the consensus between the agents.

The protocols with self-delays, arising e. g. if the agents have retarded inputs [Tian and Liu, 2008] or use delayed relative measurements (deviations between its own and neigbors states) are tolerable to sufficiently small delays only and can not be investigated by the contraction arguments. Such algorithms were investigated mainly for the case of fixed topology and linear stationary couplings [Olfati-Saber and Murray, 2004], [Bliman and Ferrari-Trecate, 2008], [Michiels *et al.*, 2009], [Münz, 2010], [Tian and Liu, 2008], [Lestas and Vinnicombe, 2010] by using frequency-domain methods. The known results for switching topology also deal with linear couplings, are not analytic and lead to high-dimensional systems of LMIs [Qin *et al.*, 2009], [Lin and Jia, 2011]. Below we consider the case of *nonlinear* consensus algorithms with switching topology and give easily verifiable sufficient analytic conditions for convergence of such protocols. Unlike the previous paper [Proskurnikov, 2010], the delays in communication links may be different and the graph is not assumed to be constantly connected.

## 2 Problem Set Up

Throughout the paper $\mathbb{G}_N$ stands for the set of all undirected graphs (possibly disconnected) with common set of vertices $V_N = \{1, 2, \dots, N\}$, having no loops (arcs with coincident ends). For any $G \in \mathbb{G}_N$ and $j \in V_N$ let $N_j(G)$ stand for the set of all *neighbors* (adjacent vertices) of the node $j$ in $G$. By definition of undirected graph, $k \in N_j(G) \iff j \in N_k(G)$.

We consider a team of $N$ independent *agents* modeled by the first order equations

$$\dot{x}_j(t) = u_j(t) \in \mathbb{R}^d, \quad j = 1, 2, \dots, N. \qquad (1)$$

Here $x_j, u_j \in \mathbb{R}^d$ stand respectively for the state vector and the control input of $j$-th agent. Let the topology of the network be defined by a graph-valued function $G(\cdot) : [0; +\infty] \to \mathbb{G}_N$. That is, the interaction (by means of data transmission, mechanical links, etc.) between the agent $j, k$ is possible at time $t \geq 0$ if and only if $k \in N_j(G(t))$ (and thus $j \in N_k(G(t))$). Throughout the paper we assume the function $G(\cdot)$ to be the Lebesgue measurable, i.e. for any $\Gamma \in \mathbb{G}_N$ the set $G^{-1}(\Gamma) = \{t : G(t) = \Gamma\}$ is Lebesgue measurable.

Below we investigate distributed control policies or *protocols* as follows

$$u_j(t) = \sum_{k \in N_j(G(t))} \varphi_{jk}(t, z_{jk}(t - \tau_{jk}(t)), \qquad (2)$$

where by definition

$$z_{jk}(t) = x_k(t) - x_j(t), \quad 1 \leq j, k \leq N. \qquad (3)$$

Here $\{\varphi_{jk}(t, y)\}_{j \neq k}$ is a family of functions (with arguments $t \geq 0$, $y \in \mathbb{R}^d$) referred to as *couplings* and describing the interaction strength between the agents. The delays $\tau_{jk} = \tau_{kj} \geq 0$ are assumed to be constant. Such control strategies are typical for decentralized coordination and synchronization problems without global reference frame in presence of communication and measurement delays. Each input $u_j$ is a function of delayed measurements of neighbors state vectors, made in the $j$-th agent reference frame.

To provide the unique solvability of the closed loop system (1), (2) one has to specify initial data:

$$x_j(t) = \alpha_j(t), t < 0, \quad \lim_{t \downarrow 0} x_j(t) = \alpha_j^0 \qquad (4)$$

We assume that $\alpha_j \in L_2([-\max_k \tau_{jk}; 0] \to \mathbb{R}^d)$ but do not suppose the solutions to be continuous at $t = 0$, so $\alpha_j^0$ may be chose independently of $\alpha_j$.

We say that the protocol (2) establishes the *asymptotic consensus* if for any $i, j$, $1 \leq i, j \leq N$ and arbitrary initial data set $\{\alpha_j(\cdot)\}$, $\{\alpha_j^0\}$ one has

$$\lim_{t \to +\infty} |x_i(t) - x_j(t)| = 0. \qquad (5)$$

If additionally the states $x_j$ have common limit

$$\lim_{t \to +\infty} x_j(t) = \frac{1}{N} \sum_{k=1}^{N} \alpha_j^0, \, \forall j = 1, 2, \dots, N, \qquad (6)$$

the protocol (2) is said to provide *average consensus*.

Under the Assumption 1 below (symmetry of the protocol) the asymptotic consensus and average consensus conditions are equivalent since $\sum_{j=1}^{N} u_j = 0$ and $\sum_{j=1}^{N} x_j(t) = const.$

The aim of the paper is to disclose easily verifiable conditions for average consensus (6) for the wide class of control algorithms (2) with the couplings $\varphi_{jk}$ being nonlinear and uncertain, but assumed to satisfy conditions as follows.

*Assumption 1.* (Symmetry of delays and couplings) For any pair $1 \le k, j \le N$, $i \ne j$ and any $t \ge 0$, $y \in \mathbb{R}^d$ one has $\varphi_{jk}(t, y) = -\varphi_{kj}(t, -y)$ and $\tau_{jk} = \tau_{kj}$.□

*Assumption 2. (Sector condition).* A constant $\gamma > 0$ exists such that $\varphi_{ij}(t, x)^T x \ge \gamma^{-1}|\varphi_{ij}(t, x)|^2$ for any $i \ne j$.□

For the scalar case ($x_j(t) \in \mathbb{R}$) the sector condition means that the graph of the function $\varphi_{ij}(\cdot)$, i.e. the set $\{(x, y) : y = \varphi_{ij}(x)\}$ lies between the lines $y = 0$ and $y = \gamma x$.

*Assumption 3.* If $x \in \mathbb{R}^d$ is separated from 0, the same is true for $\varphi_{ij}(t, x)$: for any $\varepsilon > 0$ one has

$$\eta_{ij}(\varepsilon) = \inf\{\varphi_{ij}(t, x) : t \ge 0, |x| \ge \varepsilon\} > 0. \quad (7)$$

Our last assumption concerns the interaction graph $G(t)$. This supposition is analogous to the uniform connectivity assumptions [Moreau, 2004], [Moreau, 2005], [Lin *et al.*, 2007] and prevents decomposition of the network into separate clusters. Let $E(t)$, $t \ge 0$ stands for the set of arcs of the graph $G(t)$. For any unordered pair of vertices $e = \{i, j\}$ (with $i \ne j$) and interval $\Delta = [t_1; t_2] \subset [0; +\infty)$ let $l_e(\Delta) = mes\{t \in \Delta : e \in E(t)\}$ be the total time (over the interval $\Delta$) when the arc $e$ exists in graph $G(t)$, here $mes$ denotes the Lebesgue measure. Consider the set $S_\varepsilon(\Delta) = \{e : l_e(\Delta) > \varepsilon\}$ of all possible edges with "lifetime" on $\Delta$ greater than $\varepsilon > 0$. We say the graph $(V_N, S_\varepsilon(\Delta)) \in \mathbb{G}_N$ to be *$\varepsilon$-skeleton* of the graph $G(\cdot)$ on the interval $\Delta$.

*Assumption 4.* There exist $\varepsilon > 0, T > 0$ such that the $\varepsilon$-skeleton of $G(t)$ on any interval $[t; t + T]$, $t \ge 0$, is connected.

*Remark.* Often the switching topology is piecewise-constant with the *dwell time* (infimum of time lags between consequent switchings) positive. For this case Assumption 4 means that for some $T > 0$ all of the graphs $(V_N, \bigcup_{t=t_0}^{t_0+T} E(t))$ are connected. The latter condition is a "UQSC (uniform strong quasi connectedness) property" proposed in [Lin *et al.*, 2007], [Moreau, 2004]. Therefore Assumption 4 may be treated as a generalization of UQSC property for the case of arbitrary Lebesgue measurable underlying graph $G(\cdot)$ . Notice that, as shown in [Lin *et al.*, 2007], Theorem 3.8 for the case of positive dwell-time and undelayed protocols, the UQSC property is almost necessary for achieving consensus (and becomes necessary if one requires the uniform convergence).

It should be noticed that the case of fixed topology and linear time-invariant couplings $\varphi_{ij}(t, y) = w_{ij} y$ ($w_{ij} = w_{ji} > 0$ for $i, j$ being neighbors in $G_0$ and $w_{ij} = 0$ otherwise) was exhaustively studied in [Olfati-Saber and Murray, 2004], [Bliman and Ferrari-Trecate, 2008], [Münz, 2010]. Let $L_0$ be the Laplacian matrix of the obtained weighted graph:

$$L_0 = \begin{bmatrix} \sum_{j=1}^N w_{1j} & -w_{12} & \dots & -w_{1N} \\ -w_{21} & \sum_{j=1}^N w_{2j} & \dots & -w_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -w_{N1} & -w_{N2} & \dots & \sum_{j=1}^N w_{Nj} \end{bmatrix} \ge 0,$$

and $\lambda_N = \lambda_{max}(L_0)$ be its maximal eigenvalue. The brilliant result by [Bliman and Ferrari-Trecate, 2008] states that consensus is established if $\tau_{jk} < \bar{\tau} := \frac{\pi}{2\lambda_N}$ for any $i, j$. This estimate can not be improved, moreover, for homogeneous delays $\tau_{ij} = \tau$ the inequality $\tau < \bar{\tau}$ is necessary and sufficient for consensus [Olfati-Saber and Murray, 2004]. More general analytic results for discrete-time case were obtained in [Tian and Liu, 2008]. In the papers [Lestas and Vinnicombe, 2010], [Münz, 2010] the case of fixed topology was analyzed by means of extension of the Nyquist criterion.

However, the methods of mentioned papers use frequency-domain analysis that is not applicable for time-varying graphs and nonlinear protocols (2).

## 3 Main Result

The aim of this section is to give a simply verifiable consensus conditions (expressed in terms of the topology $G(\cdot)$, sector bound $\gamma > 0$ and maximal delay magnitude) for the protocol (2) with the couplings satisfying Assumptions 1-4.

We define the *adjacency matrix* of the graph $G \in \mathbb{G}_N$ by $a_{ij}(G) := 1$ if the nodes $i, j$ are adjacent in $G$ and $a_{ij}(G) := 0$ otherwise.

Let $\xi_{ij}(t) = a_{ij}(G(t))\varphi_{ij}(t, x_j(t-\tau_{ij}) - x_i(t-\tau_{ij}))$, where $a_{ij}(G)$ stands for the adjacency matrix of the graph $G \in \mathbb{G}_N$. In particular, for any $j$ one has

$$\dot{x}_j(t) = u_j(t) = \sum_{k=1}^N \xi_{jk}(t) \quad (8)$$

(by definition we have $\xi_{jj}(t) = 0$). Notice that all functions $\xi_{ij}$ are Lebesgue measurable due to the measurability of the graph-valued function $G(\cdot)$.

Our main result is based on the following lemma stating that for sufficiently small delays all of the functions $\xi_{ij}$ are uniformly bounded in $L_2$-norm.

**Lemma 1.** *Suppose that the couplings $\varphi_{ij}$ and delays $\tau_{ij}$ satisfy Assumptions 1,2 and the claim is valid:*

$$2\gamma(N - 1)\tau < 1, \quad \tau := \max \tau_{jk}. \quad (9)$$

*Then a constant $C > 0$ exists depending on $\gamma$ and the delays $(\tau_{jk})$ only such that*

$$\sum_{j,k=1}^N \int_0^{+\infty} |\xi_{jk}(t)|^2 dt < C \sum_{j=1}^N \left( |\alpha_j^0|^2 + \int_{-\tau}^0 |\alpha_j(t)|^2 dt \right). \quad (10)$$

*Here $\alpha_j(\cdot)$,$\alpha_j^0$ are initial data from (4) and we take by definition $\alpha_j(t) = 0$ for $t < -\tau$.*

The proof of Lemma 1 is based upon the absolute stability theory techniques extending the Popov method [Popov, 1973], [Yakubovich, 2000] and can be found in the Appendix.

It should be noticed that assumptions of Lemma 1 do not concern the network topology, so the functions $\xi_{jk}$ and control inputs $u_j$ are $L^2$-bounded under Assumptions 1,2 even if the protocol (2) can not establish consensus (e.g. if the graph is totally disconnected).

Now we present the main result of the paper.

**Theorem 1.** *Suppose that Assumptions 1-4 and (9) hold. Then the protocol (2) provides consensus (and thus average consensus).*

*Proof.* Suppose on the contrary that the consensus (5) is not achieved, so a number $\delta > 0$, indices $i, j$ and a sequence $t_n \uparrow +\infty$ exist such that $|x_i(t_n) - x_j(t_n)| \geq \delta$. Bounding ourselves with an appropriate subsequence $\{t_{n_k}\}$, we may suppose that $t_{n+1} - t_n > T$, thus intervals $\Delta_n = [t_n - T/2; t_n + T/2]$ are disjoint (here $T$ is the number from Assumption 4). Since $\varepsilon$-skeleton of $G(\cdot)$ on $\Delta_N$ is a connected graph, for some indices $i_n, j_n$ one has: 1) the existence time of the arc arc $e_n = (i_n, j_n)$ on $\Delta_n$ is greater then $\varepsilon$; 2) $|x_{i_n}(t_n) - x_{j_n}(t_n)| > \delta' = \delta/(N-1)$. From Lemma 1 one may conclude that $u_j \in L_2$, thus $\int_{t_n}^{+\infty} |u_j(t)|^2 dt \to 0$ as $n \to +\infty$. This implies that for sufficiently large $n$ the inequality $|x_{i_n}(t) - x_{j_n}(t)| > \delta'$ holds for any $t \in [t_n - T/2 - \tau; t_n + T/2]$, where $\tau = \max_{j,k} \tau_{jk}$. Using Assumption 3, one obtains that $\varphi_{i_n j_n}(t, x_{i_n}(t - \tau_{i_n j_n}) - x_{j_n}(t - \tau_{i_n j_n})) \geq \eta_0 = \eta(\delta')$ whenever $t \in \delta_n$. Therefore for $n$ sufficiently large one has

$$\sum_{j,k=1}^{N} \int_{\Delta_n} |\xi_{jk}(t)|^2 dt \geq \int_{\Delta_n} |\xi_{i_n j_n}(t)|^2 dt \geq \varepsilon\eta_0,$$

which obviously contradicts to (10).■

**Appendix A Proof of Lemma 1.**

The proof is divided on two stages: the first stage (summarized by Lemma 2) is to prove uniform bound (10) (with common constant $C > 0$) for the case of "stable" (in $L_2$ sense) solutions, and the second one is to prove that every solution is stable.

We start with the first part of the proof which is based on the important result of V.A. Yakubovich on quadratic functionals semiboundedness.

Let $Z$ and $\Xi$ be two complex Hilbert spaces of finite dimension. Consider a linear stabilizable system

$$\dot{z}(t) = Az(t) + B\xi(t), t \geq 0. \tag{11}$$

Here $A : Z \to Z$, $B : Z \to \Xi$ are linear operators. For any $a \in Z$ denote by $\mathfrak{M}_a \subset L_2([0; +\infty) \to Z \times \Xi)$ the set of all pairs $w(\cdot) = [z(\cdot), \xi(\cdot)]$ such that $|w(\cdot)| \in L_2[0; +\infty]$, (11) is satisfied and $z(0) = a$.

Consider a Hermitian functional $J_0(w) = \int_{-\infty}^{+\infty} \hat{w}(i\omega)^* P(i\omega)\hat{w}(i\omega)d\omega$, where $\hat{w}$ stands for the Fourier transform of $w$, $P(\cdot)$ bounded and analytic at any point $\omega \in \mathbb{R}$ operator-valued function such that $P(i\omega) = P(i\omega)^* : Z \times \Xi \to Z \times \Xi$. Let

$$J(w) = J_0(w) + 2\, Re \int_{-\infty}^{+\infty} s(i\omega)^* \hat{w}(i\omega)d\omega \tag{12}$$

where $s(\cdot) \in L_2(i\mathbb{R} \to Z)$. We also introduce an auxiliary operator-valued function $\Pi(i\omega)$ defined for $\omega \in \mathbb{R}$ such that $\det(i\omega I_n - A) \neq 0$:

$$\Pi(i\omega) = W(i\omega)^* P(i\omega) W(i\omega),$$
$$\text{where} \quad W(i\omega) = \begin{bmatrix} (i\omega I_n - A)^{-1}B \\ I_m \end{bmatrix}$$

The straightforward computation shows that for $w = [z, \xi] \in \mathfrak{M}_0$ one has $\hat{w}(i\omega) = W(i\omega)\hat{\xi}(i\omega)$ and thus $J_0(w) = \int_{-\infty}^{+\infty} \hat{\xi}(i\omega)^* \Pi(i\omega)\hat{\xi}(i\omega)d\omega$.

Our goal is to find conditions which guarantee the quadratic function $J$ to be bounded from above on the set $\mathfrak{M}_a$ for any $a$. It is evident that necessary condition is non-strict negative definiteness of $J_0$ on the correspondent linear space $\mathfrak{M}_0$ which is easily rewritten as $\Pi(i\omega) \leq 0$ for any $\omega \in \mathbb{R}$. This condition appears to be sufficient under certain additional assumption.

**Theorem 2.** $\Pi(i\omega) \leq 0$ *for any $\omega$ if and only if then $J_0(w) \leq 0$ for any $w \in \mathfrak{M}_0$. If $\Pi(i\omega) \leq 0$ and a matrix $\Pi_\infty > 0$ exists such that $\Pi(i\omega) \leq -\Pi_\infty < 0$ for sufficiently large $|\omega|$, then a constant $C > 0$ exists depending on $P, Q, R, A, B$ only, such that*

$$\sup_{w \in \mathfrak{M}_a} J(w) \leq C(|a|^2 + \|s\|_{L_2}^2).$$

Proof. In the case of Hurwitz matrix $A$ the Theorem 2 directly follows from [Arov and Yakubovich, 1981], Theorem 2, see analogous reasoning in the proof of [Likhtarnikov and Yakubovich, 1983], Theorem 2. The non-stable case reduces to the case of stable system by the change of variables $\xi = \xi' + Kz$ such that the matrix $A + BK$ is Hurwitz.■

Consider the space $Z$ of all matrices $z = (z_{jk})$, $1 \leq j, k \leq N$ and the space $\Xi \subset Z$ consisting of all skew-symmetric matrices. Taking $z_{jk} = x_k - x_j$, $z = (z_{jk})$, $\xi = (\xi_{jk})$, where $x_j$ is a solution of (1), (2) and $\xi_{jk}$ are the same as in Lemma 1, the system (8) is easily rewritten as (11) for appropriate $A$, $B$. Now we take an *integral quadratic constraint* into account which follows follows from Assumption 2. Consider a quadratic function $J(z(\cdot), \xi(\cdot))$ defined by

$$J = \int_0^{+\infty} \sum_{j,k=1}^{N} \xi_{jk}(t)^* z_{jk}(t-\tau_{jk})dt - \\ -(\gamma^{-1}-\varepsilon) \int_0^{+\infty} \sum_{j,k=1}^{N} |\xi_{jk}(t)|^2 dt.$$

Here $\varepsilon > 0$ is a sufficiently small number to be detailed below. Assumption 2 implies that $\xi_{jk}^*(t)z_{jk}(t-\tau_{jk}) \geq \gamma|\xi_{jk}|^2$ and thus $J(z,\xi) \geq \varepsilon \sum_{j,k=1}^{N} \|\xi_{jk}\|_{L_2}^2$.

Using the Plancherel theorem, the functional $J$ is easily seen to have the form (12) with $P(i\omega)$ bounded and analytic. A straightforward computation shows that the operator-valued function $\Pi(i\omega)$ is defined by

$$\hat{\xi}^* \Pi(i\omega)\hat{\xi} = 2 \sum_{j,k=1}^{N} Re \frac{\hat{\xi}_{jk}^* \hat{u}_j e^{-i\omega\tau_{jk}}}{i\omega} - \\ - \left(\frac{1}{\gamma} - \varepsilon\right) \sum_{j,k=1}^{N} |\hat{\xi}_{jk}|^2.$$

where $\hat{\xi} \in \Xi$, $\hat{u}_j = \sum_{k=1}^{N} \hat{\xi}_{jk}$. Since $e^{i\omega\tau_{jk}} = 1 + \beta_{jk}$ where $|\beta_{jk}| \leq |\omega|\tau$, $\tau = \max \tau_{jk}$, one has

$$\sum_{k=1}^{N} Re \frac{\hat{\xi}_{jk}^* \hat{u}_j e^{-i\omega\tau_{jk}}}{i\omega} \leq \tau|\hat{u}_j| \sqrt{(N-1)\sum_{k=1}^{N} |\hat{\xi}_{jk}|^2}$$

(the multiplier $(N-1)$ appears here instead of $N$ since $\hat{\xi}_{jj} = 0$). At the same time $|\hat{u}_j|^2 \leq (N-1)\sum_{k=1}^{N} |\hat{\xi}_{jk}|^2$. Therefore

$$\hat{\xi}^* \Pi(i\omega)\hat{\xi} \leq \sum_{j,k=1}^{N} |\hat{\xi}_{jk}|^2 (2\tau(N-1) - \gamma^{-1} + \varepsilon)$$

Since $2\tau(N-1)\gamma < 1$, for sufficiently small $\varepsilon > 0$ one has $\Pi(i\omega) \leq -\delta I$, where $\delta > 0$ is some small constant. Due to Theorem 2, one obtains the result as follows:

**Lemma 2.** *The conclusion of Lemma 1 is valid whenever the solution of (1), (2) satisfies $\xi_{jk} \in L_2[0; +\infty]$, $x_k - x_j \in L_2[0; +\infty]$ for any $j, k$. The constant $C > 0$ in (10) is independent on partial solution and is determined by the delays $\tau_{ij}$ and constant $\gamma$ only.*

To accomplish the proof of Lemma 1, we need now a result proving absence of "unstable" in $L_2$-sense solutions. This will be done by the following standard trick from absolute stability theory, used for proving "minimal stability" conditions [Yakubovich, 2000], [Yakubovich, 2002]. Consider arbitrary protocol (2) satisfying Assumptions 1,2 with the topology function $G(t)$. For some $T > 0$ and $\mu \in (0; \gamma)$ consider the new coupling functions

$$\tilde{\varphi}_{jk}(t,y) = \begin{cases} \varphi_{jk}(t,y), & t \leq T \\ \mu y, & t > T \end{cases}$$

and the new underlying graph function $\tilde{G}(t)$ which coincides with $G(t)$ for $t < T$ and is the complete graph for $t > T$. The protocol

$$u_j(t) = \sum_{k \in N_j(\tilde{G}(t))} \tilde{\varphi}_{jk}(t, x_k(t-\tau_{jk}(t)) - x_j(t-\tau_{jk}(t)),$$

is known to provide consensus with exponential rate of convergence due to the inequality $\mu < \gamma < \frac{1}{2(N-1)} < \frac{\pi}{2N}$ (see Theorem 1). At the same time, the solution of the new closed-loop system coincides with the solutions of (1),(2) for $t < T$. Accordingly to Lemma 2, one has

$$\sum_{j,k=1}^{N} \int_0^T |\xi_{jk}(t)|^2 dt < C \sum_{j=1}^{N} \left( |\alpha_j^0|^2 + \int_{-\tau}^{0} |\alpha_j(t)|^2 dt \right).$$

with $C$ independent of $T$ and initial data. taking limit as $T \to +\infty$, one proves Lemma 1.

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