STATE VARIABLES SCALING TO SOLVE THE MALKIN'S PROBLEM ON PERIODIC OSCILLATIONS IN PERTURBED AUTONOMOUS SYSTEMS

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Abstract

By means of a version of the implicit function theorem for directionally continuous functions we establish the existence, uniqueness and the asymptotic stability of periodic solutions of a T-periodically perturbed autonomous system bifurcating from a T-periodic limit cycle of the autonomous unperturbed system (Malkin's problem). The main point of this method is the scaling of the state variables in a suitably defined map whose zeros are T-periodic solutions of the perturbed system. In order to define this map we introduce a projector by means of a convenient change of the state variables of the unperturbed system. Finally, by applying to this map the implicit function theorem mentioned before we solve the Malkin's problem without any reduction of the dimension of the state space as it is done in the literature.

Key words

Periodic solutions, asymptotic stability, Lyapunov-Schmidt reduction, directionally continuous functions.

1 Introduction

Consider the perturbed autonomous system

$$\dot{x} = f(x) + \varepsilon g(t, x, \varepsilon)$$

where $\varepsilon > 0$, g is T-periodic with respect to the first variable and assume that the unperturbed autonomous system

$$\dot{x} = f(x)$$

has a T-periodic limit cycle x_0. Let $\mathcal{P}_\varepsilon$ be the Poincaré map associated to the perturbed system, it is easy to see that the derivative of $\mathcal{P}_0(x_0(\theta)) - I$ is singular, for any $\theta \in [0, T]$, this fact does not allow the employ of the implicit function theorem for studying existence, uniqueness and stability of the fixed points of $\mathcal{P}_\varepsilon$ bifurcating from $x_0([0, T])$. On the other hand, $\mathcal{P}_0'(x_0(\theta)) - I$ normally is a nonzero matrix and so the classical averaging method, i.e. the Bogolubov's second theorem, does not allow to carry out this kind of analysis either. Existence, uniqueness and stability of fixed points of $\mathcal{P}_\varepsilon$ bifurcating from $x_0([0, T])$ has been studied in classical papers [Malkin, 1949], [Loud, 1959] and [Blekhman, 1971, p. 186-202] by means of the Lyapunov-Schmidt reduction which permits to apply implicit function theorem to reduce the system under consideration to a one-dimensional equation for which the averaging method can be applied. In this paper, by a suitable scaling of the state variables, we introduce a map $\widehat{\mathcal{P}}_\varepsilon(v)$[1] having the same fixed points as $\mathcal{P}_\varepsilon(v)$ and whose derivative is (v, ε)-directionally continuous with respect to a suitable cone at the points of $x_0([0, T])$. The directional derivative $\widehat{\mathcal{P}}_0'(x_0(\theta)) - I$ turns out to be not necessarily singular and it is related to the derivative of the perturbation term. For such a map we propose an implicit function theorem which permits to study existence, uniqueness and stability of fixed points of $\widehat{\mathcal{P}}_\varepsilon$ bifurcating from $x_0([0, T])$ without any reduction of the dimension of the state variables. This method simplifies considerably the work done in the papers by Malkin, Loud and Blekhman. (The talk will also present a comparison of our results with those of the previous cited authors). The paper is organized as follows: in Section 2 after some preliminaries we precise the problem that we want to tackle and the results that solve it. Finally, the appendix collects the proofs.

[1] Indeed we introduce the map $F(v, \varepsilon) = \widehat{\mathcal{P}}_\varepsilon(v) - v$.

2 Results

Consider the function

$$\Phi(v,\varepsilon) = P(v) + \varepsilon Q(v,\varepsilon), \qquad (1)$$

where $P \in C^2(\mathbb{R}^n, \mathbb{R}^n)$, $Q \in C^1(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$ and $\varepsilon > 0$ is a small parameter. Assume the following conditions.

(P1) $P\left(\begin{pmatrix}\theta \\ 0_{n-1\times 1}\end{pmatrix}\right) = 0$ for any $\theta \in [-\delta, \delta]$, where $\delta > 0$ is sufficiently small and $0_{n-1\times 1}$ is the $n-1$-dimensional zero vector,

(P2) $\begin{pmatrix} 0 \\ \mathbb{R}^{n-1}\end{pmatrix}$ is an invariant space of $P'\left(\begin{pmatrix}\theta \\ \xi\end{pmatrix}\right)$ for any $\theta \in [-\delta,\delta]$, $\xi \in B_\delta^{n-1}(0)$.

Here and in the following $B_r^p(x_0)$ denotes the ball in $\mathbb{R}^p$ centered at x_0 of radius r.
Functions P with this properties appear in a natural way when we consider the system of ordinary differential equations

$$\dot{x} = f(x) \qquad (2)$$

with a T-periodic limit cycle x_0. In fact, let $\mathcal{P} : \mathbb{R}^n \to \mathbb{R}^n$ be the Poincaré map associated to (1) over the period $T > 0$, then it is known from the paper [Guckenheimer, 1975] that there exists a family of so-called isochronous surfaces $S(\theta, \cdot) \in C^1(\mathbb{R}^{n-1}, \mathbb{R}^n)$ which transversally intersect the cycle having the following properties.

1) $\bigcup_{\theta \in [-T/2, T/2]} S(\theta, \mathbb{R}^{n-1}) \supset V_\delta(x_0([-T/2, T/2]))$ for $\delta > 0$ sufficiently small,

2) for any $v \in S(\theta, \mathbb{R}^{n-1}) \bigcap V_\delta(x_0([-T/2,T/2]))$ we have that $\mathcal{P}(v) \in S(\theta, \mathbb{R}^{n-1})$.

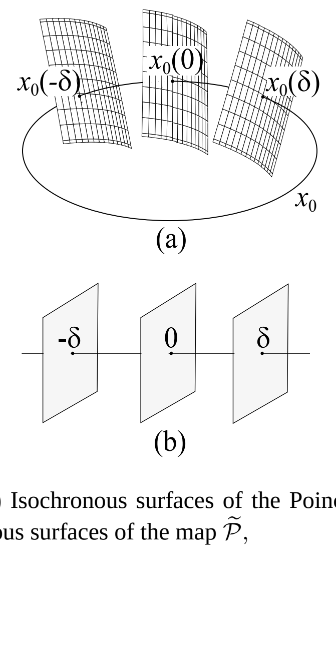

Figure 1. (a) Isochronous surfaces of the Poincaré map $\mathcal{P}$, (b) isochronous surfaces of the map $\widetilde{\mathcal{P}}$,

Therefore, taking

$$\widetilde{\mathcal{P}}\left(\begin{pmatrix}\theta \\ \xi\end{pmatrix}\right) = S^{-1}\left(\mathcal{P}\left(S\left(\begin{pmatrix}\theta \\ \xi\end{pmatrix}\right)\right)\right),$$

for any $\theta \in [-\delta, \delta]$ and any $\xi \in \mathbb{R}^{n-1}$ such that $(\theta, \xi) \in V_\delta(x_0([-T/2, T/2]))$, see Fig. 1, we have that the function $P(v) = \widetilde{\mathcal{P}}(v) - v$ satisfies properties (P1)-(P2). Indeed, (P1) is obvious. To prove (P2) we observe that

$$\widetilde{\mathcal{P}}'\left(\begin{pmatrix}\theta \\ \xi\end{pmatrix}\right)\begin{pmatrix}0 \\ \zeta\end{pmatrix} + o(\zeta) = \widetilde{\mathcal{P}}\begin{pmatrix}\theta \\ \xi+\zeta\end{pmatrix} - \widetilde{\mathcal{P}}\begin{pmatrix}\theta \\ \xi\end{pmatrix},$$

for any $\xi,\ \zeta \in B_\delta(0)$. Since the first component of $\widetilde{\mathcal{P}}\begin{pmatrix}\theta \\ \xi+\zeta\end{pmatrix} - \widetilde{\mathcal{P}}\begin{pmatrix}\theta \\ \xi\end{pmatrix}$ is zero by the definition of $\widetilde{\mathcal{P}}$ then the first component of

$$\widetilde{\mathcal{P}}'\left(\begin{pmatrix}\theta \\ \xi\end{pmatrix}\right)\begin{pmatrix}0 \\ \zeta/\|\zeta\|\end{pmatrix} + \frac{o(\zeta)}{\|\zeta\|} \qquad (3)$$

is zero as well. On the other hand for any $h \in \mathbb{R}^{n-1}$, $\|h\| = 1$, there exists $\zeta_k/\|\zeta_k\| \to h$ as $k \to \infty$ hence from (3) we obtain that the first component of $\widetilde{\mathcal{P}}'\left(\begin{pmatrix}\theta \\ \xi\end{pmatrix}\right)\begin{pmatrix}0 \\ h\end{pmatrix}$ is zero for any $h \in \mathbb{R}^{n-1}$, $\|h\| = 1$. Therefore, for any $\xi \in B_\delta^{n-1}(0)$ and $\theta \in [-\delta, \delta]$ the map $\widetilde{\mathcal{P}}'\left(\begin{pmatrix}\theta \\ \xi\end{pmatrix}\right)$ maps $\begin{pmatrix}0 \\ \mathbb{R}^{n-1}\end{pmatrix}$ into itself. Therefore, we have $\pi P(v)(I-\pi) = \pi\widetilde{\mathcal{P}}(v)(I-\pi) = 0$ for any $v \in B_\delta^n(0)$, and so (P2), where π is a projector defined as follows

$$\pi h = \begin{pmatrix} h_1 \\ 0_{n-1\times 1}\end{pmatrix}, \qquad (4)$$

Function Q appears in (1) when we perturb (2) as follows

$$\dot{x} = f(x) + \varepsilon g(t, x, \varepsilon), \qquad (5)$$

where g is T-periodic with respect to the first variable. Therefore, the problem of the existence and uniqueness of zeros v_ε of Φ near the set $\begin{pmatrix}[-\delta,\delta] \\ 0_{n-1\times 1}\end{pmatrix}$ is equivalent to the problem of the existence and uniqueness of T-periodic solutions x_ε of (5) near $x_0(t)$. Moreover, observe that if the real parts of eigenvalues of $\Phi'_v(v_\varepsilon, \varepsilon)$ are negative then the associated T-periodic solution x_ε is asymptotically stable.

This paper aims at obtaining sufficient conditions ensuring that

(Φ1) there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ there exists $v_\varepsilon \in \mathbb{R}^n$ such that $\Phi(v_\varepsilon, \varepsilon) = 0$ and $v_\varepsilon \to 0$ as $\varepsilon \to 0$,

(Φ2) the real parts of eigenvalues of $\Phi'_v(v_\varepsilon, \varepsilon)$ are negative for $\varepsilon \in (0, \varepsilon_0)$.

The classical method to solve this problem under assumptions (P1)-(P2) is based on the Lyapunov-Schmidt reduction (to a one-dimensional equation) and it was completely developed in [Malkin, 1949], [Loud, 1959] and [Blekhman, 1971, p. 186-202].
In this paper we propose an alternative method based on the scaling of the state variables of a suitably defined map, whose zeros are the T-periodic solutions of the perturbed systems. This metod permits to avoid the reduction of the dimension of the state space in order to solve the problem. The proposed method considerably simplifies the procedure required in the Lyapunov-Schmidt reduction approach. A first result in this direction has been obtained by the authors in [Kamenskii, Makarenkov and Nistri, 2008] by using the topological degree theory, but this tool did not allow us to obtain (Φ2). In this paper the approach is instead based on a new implicit function theorem. The main idea is to introduce the following auxiliary function

$$F(v, \varepsilon) = \Phi(v, \varepsilon) - \pi\Phi(v, \varepsilon) + \frac{1}{\varepsilon}\pi\Phi(v, \varepsilon),$$

where π is the projector defined in (4), hence it possesses the following properties

$$\pi P'\left(\begin{pmatrix} \theta \\ 0_{n-1\times 1} \end{pmatrix}\right) = 0, \quad \text{for any } \theta \in [-\delta, \delta], \quad (6)$$

$$\pi P'(v)(I - \pi) = 0, \quad \text{for any } v \in B_\delta^n(0). \quad (7)$$

Following [Bressan, 1988] we give the following definition.

Definition 1. *Given $M > 0$, consider the cone*

$$K_M = \{(v, \varepsilon) \in \mathbb{R}^n \times (0, \infty) : \|v\| \leq \varepsilon M\}. \quad (8)$$

We say that a map $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is directionally continuous with respect to K_M at a point (v_0, ε_0) if and only if $f(v_k, \varepsilon_k) \to f(v_0, \varepsilon_0)$ for every sequence $(v_k, \varepsilon_k) \to (v_0, \varepsilon_0)$ with $(v_k - v_0, \varepsilon_k - \varepsilon_0) \in K_M, k \in \mathbb{N}$.

For an application of directionally continuous selections of multivalued maps to the control theory we refer to [Gorniewicz and Nistri, 1994].

Functions Φ and F are equivalent in the sense that

$$\Phi(v, \varepsilon) = 0 \quad \text{if and only if} \quad F(v, \varepsilon) = 0.$$

The properties of the function F can be now summarized as follows.

Lemma 1. *For any $M > 0$ the functions F and F'_v are directionally continuous with respect to the cone K_M at $(0, 0)$, provided that $F(0, 0)$ and $F'_v(0, 0)$ are defined as follows*

$$F(0, 0) = (I - \pi)P(0) + \pi G(0, 0),$$
$$F'_v(0, 0) = (I - \pi)P'(0) + \pi G'_v(0, 0).$$

Lemma 2. *Let $\{v_\varepsilon\}_{\varepsilon > 0}$ be a sequence of zeros of Φ such that $(v_\varepsilon, \varepsilon) \in K_M$ for some fixed $M > 0$ and any $\varepsilon > 0$ sufficiently small. If the real parts of eigenvalues of the matrix $F'_v(0, 0)$ are negative then the real parts of the eigenvalues of the matrix $\Phi'_v(v_\varepsilon, \varepsilon)$ are also negative for any $\varepsilon > 0$ sufficiently small.*

To obtain (Φ1)-(Φ2) it is now sufficient to show that the conclusion of Lemma 1 implies that the assumptions of Lemma 2 are satisfied. In turn Lemma 2 follows from the following result, due to the authors, that is a modified version of the classical implicit function theorem (see [Kolmogorov and Fomin, 1976, p. 492]).

Theorem 1. *(Implicit function theorem for directionally continuous functions). Assume that*

1. *There exists $\Delta_0 > 0$, $L > 0$ such that $\|F'_\varepsilon(0, \varepsilon) - F'_\varepsilon(0, 0)\| \leq L\,\varepsilon$ for any $0 < \varepsilon \leq \Delta_0$.*
2. *$F(0, 0) = 0$,*
3. *The functions F and F'_v are directionally continuous at $(0, 0)$ with respect to the cone K_M defined in (8), whenever $M > 0$, and the matrix $F'_v(0, 0)$ is nonsingular.*

Then there exist $\Delta \in (0, \Delta_0)$ and $M > 0$ such that for any $0 < \varepsilon \leq \Delta$ the equation

$$F(v, \varepsilon) = 0$$

has a unique solution $v = V(\varepsilon)$ in the εM-neighborhood of $0 \in \mathbb{R}^n$. Moreover, the function V is continuous at 0.

3 Conclusion

We have proposed a new approach to study the bifurcation of asymptotically stable periodic solutions from an isolated cycle. This approach turns out to be simpler with respect to the classical Lyapunov-Schmidt reduction, it can be also applied to study the existence of bifurcation of asymptotically stable periodic solutions from families of periodic solutions and nondegenerate cycles of Hamiltonian systems. However, it does not provide new physical understanding, but it simplifies the mathematical analysis of those physical problems for which the Lyapunov-Schmidt reduction is employed as, for instance, the synchronization problems (see [Blekhman, 1971]).

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Appendix A Proofs

Proof of Lemma 1. To prove that F is directionally continuous at $(0,0)$ with respect to the cone K_M it is enough to show that $\frac{1}{\varepsilon}\pi P(v_\varepsilon) \to 0$ as $\varepsilon \to 0$. Indeed, taking into account (6), we have that $\frac{1}{\varepsilon}\pi P(v_\varepsilon) = \frac{1}{\varepsilon}\pi P(v_\varepsilon) - \frac{1}{\varepsilon}\pi P(0) = \frac{1}{\varepsilon}\pi P'(0)v_\varepsilon + \frac{o(v_\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0$.

Analogously, to prove that F'_v is directionally continuous at $(0,0)$ with respect to the cone K_M it is enough to show that $\frac{1}{\varepsilon}\pi P'(v_\varepsilon) \to 0$ as $\varepsilon \to 0$. For this, using (7) we have that $\frac{1}{\varepsilon}\pi P'(v_\varepsilon) = \frac{1}{\varepsilon}\pi P'(v_\varepsilon)\pi = \frac{1}{\varepsilon}\pi(P'(v_\varepsilon) - P'(0))\pi = \pi P''(0)\pi\frac{v_\varepsilon}{\varepsilon} + \frac{o(v_\varepsilon)}{\varepsilon}$. But (6) implies that $\pi P''(0)\pi = 0$ and thus the proof is complete. □

The proof of Lemma 2 is contained in an implicit way in Malkin [Malkin, 1949], Loud [Loud, 1959] and Blechman [Blekhman, 1971, p. 186-202]. For the reader convenience we provide in the sequel an explicit proof of Lemma 2.

Proof of Lemma 2. Let $\Gamma \subset \mathbb{C}$ be a circumference centered at 0 and containing none of the eigenvalues of $P'(v_0)$ different from 0. Since $\Phi'_v(v_\varepsilon,\varepsilon) \to P'(v_0)$ then for any $\varepsilon > 0$ sufficiently small the real parts of all the $n-1$ eigenvalues of $\Phi'_v(v_\varepsilon,\varepsilon)$ belonging to the exterior of Γ are less than zero. Thus, it remains to determine the sign of that eigenvalue of $\Phi'_v(v_\varepsilon,\varepsilon)$ which belongs, for $\varepsilon > 0$ sufficiently small, to the interior of Γ. Denote this eigenvalue by λ_ε and let l_ε be the associated eigenvector of unitary length, hence

$$\Phi'_v(v_\varepsilon,\varepsilon)l_\varepsilon = \lambda_\varepsilon l_\varepsilon. \tag{9}$$

Clearly, $\lambda_\varepsilon \to 0$ and $l_\varepsilon \to \begin{pmatrix} +1 \\ 0_{n-1\times 1}\end{pmatrix}$ or $l_\varepsilon \to \begin{pmatrix} -1 \\ 0_{n-1\times 1}\end{pmatrix}$ as $\varepsilon \to 0$. Without loss of generality assume that and $l_\varepsilon \to \begin{pmatrix} +1 \\ 0_{n-1\times 1}\end{pmatrix}$.

Now observe that

$$F'_v(v_\varepsilon,\varepsilon) = \Phi'_v(v_\varepsilon,\varepsilon) - \pi\Phi'_v(v_\varepsilon,\varepsilon) + \frac{1}{\varepsilon}\pi\Phi'_v(v_\varepsilon,\varepsilon)$$

and using (9) we get the following identity

$$\pi F'_v(v_\varepsilon,\varepsilon)l_\varepsilon = \frac{1}{\varepsilon}\lambda_\varepsilon \pi l_\varepsilon, \tag{10}$$

for any $\varepsilon > 0$ sufficiently small. Since $\pi l_\varepsilon \to \begin{pmatrix} +1 \\ 0_{n-1\times 1}\end{pmatrix}$, then from (10) we have that $\frac{\lambda_\varepsilon}{\varepsilon} \to a \in \mathbb{R}$, as $\varepsilon \to 0$ such that

$$\pi G'_v(0,0)\begin{pmatrix} \pm 1 \\ 0_{n-1\times 1}\end{pmatrix} = a\begin{pmatrix} +1 \\ 0_{n-1\times 1}\end{pmatrix}.$$

Our assumptions imply that $a < 0$ and taking into account that ε is positive, we obtain that $\lambda_\varepsilon < 0$ for $\varepsilon > 0$ sufficiently small. □

Proof of Theorem 1. Let

$$A_\varepsilon(v) = v - \left[F'_v(0,0)\right]^{-1} F(v,\varepsilon).$$

The equation $A_\varepsilon(v) = v$ is equivalent to the equation $F(v,\varepsilon) = 0$. We claim that for any $\Delta \in (0,\Delta_0)$ sufficiently small the map A_ε with $0 < \varepsilon \le \Delta$ maps the ball $\|v\| \le \varepsilon M$ into itself.

For this, we evaluate $\|A_\varepsilon(0)\|$. We have

$$\begin{aligned}\|A_\varepsilon(0)\| \leq \left\|\left[F'_v(0,0)\right]^{-1}\right\| \|F(0,\varepsilon)\| = \\ = \left\|\left[F'_v(0,0)\right]^{-1}\right\| \|F(0,\varepsilon) - F(0,0)\| \leq \\ \leq \left\|\left[F'_v(0,0)\right]^{-1}\right\| L\varepsilon,\end{aligned}$$

for any $0 < \varepsilon \leq \Delta_0$.

Put $M = 2\left\|\left[F'_v(0,0)\right]^{-1}\right\| \dot{L}$ and evaluate the norm of the map $(v,\varepsilon) \mapsto (A_\varepsilon)'(v)$ on K_M. To this end, consider

$$\begin{aligned}(A_\varepsilon)'(v) &= I - \left[F'_v(0,0)\right]^{-1} F'_v(v,\varepsilon) \\ &= \left[F'_v(0,0)\right]^{-1} \left[F'_v(0,0) - F'_v(v,\varepsilon)\right].\end{aligned}$$

Since of $(v,\varepsilon) \mapsto F'_v(v,\varepsilon)$ is directionally continuous at $(0,0)$ with respect to K_M we can find $\Delta \in (0,\Delta_0)$ such that

$$\|(A_\varepsilon)'(v)\| \leq \frac{1}{2},$$

for any v satisfying $\|v\| \leq \varepsilon M$, and $0 < \varepsilon \leq \Delta$. Therefore,

$$\begin{aligned}\|A_\varepsilon(v)\| &\leq \|A_\varepsilon(0)\| + \|A_\varepsilon(v) - A_\varepsilon(0)\| \leq \\ &\leq \frac{1}{2}M\varepsilon + \sup_{0\leq\theta\leq 1} \|(A_\varepsilon)'(\theta v)\| \|v\| \leq \\ &\leq \frac{1}{2}M\varepsilon + \frac{1}{2}M\varepsilon = M\varepsilon.\end{aligned}$$

Summarizing, we have that for any $0 < \varepsilon \leq \Delta$ the map A_ε maps the closed ball $\|v\| \leq \varepsilon M$ into itself and it is a contraction on this ball. Thus, there exists a unique fixed point $v^* = V(\varepsilon)$ of A_ε in this ball, that is

$$v^* = v^* - \left[F'_v(0,0)\right]^{-1} F(v^*,\varepsilon),$$

or, equivalently, $F(v^*,\varepsilon) = 0$.

Since for any $0 < \varepsilon \leq \Delta$ we have that $\|V(\varepsilon)\| \leq \varepsilon M$ then $V(\varepsilon) \to 0$ as $\varepsilon \to 0$. $\square$