STRUCTURAL STABILITY AND BIFURCATIONS IN ANALITICAL DIFFERENTIAL SYSTEMS

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Abstract
Analytical differential systems depending on parameters are investigated. A notation of space of generalized solutions is introduced and its topology is defined by fundamental solutions. The possibility of structure determination of differential system used systems of fundamental solutions is shown. Integrating of the considered systems is equivalent to existence of generalized solution of the complete uniform system; it means the solution space is analytical manifold. Symmetric functions of the operator eigenvalues of differential system gives bifurcation varieties on analytical manifold of the system parameters.

Key words
Bifurcation, fiber bundle, differential, differential form, generalized function, eigenvalue, functional, manifold, polyvector field.

1 Introduction

Mathematical models of real complex systems are described by finite system of the equations which frequently represents system of the differential equations. This system is called a dynamical or differential system. Research of differential systems allows to apply the qualitative approach which to counterbalance exact solutions of system determines their global behavior.

Global structures of orbits in dynamical system possess such features which do not depend on a choice of coordinate system, but depend on parameters. It is possible to enter natural relations of equivalence between phase portraits of the systems, connected with various classes of coordinate transformation, and to interpret a problem of the description for orbit structures as a problem of classification in conformity with these relations of equivalence.

Last years effective methods of construction of critical surfaces of symmetric bifurcations in differential systems with use of algebraic transformations, numerical integration and differential methods of the solution of the nonlinear equations are developed [Bratus' and Khalin, 2002; Khalin, 2000; Khalin, 2002a-c; Khalin, 2003a; Kuznesov, 1995].

2 Differential System

Let's consider a class of the analytical differential systems represented by autonomous system of the ordinary differential equations with an **R**-analytical right-hand side. Let **R**-analytical manifold P of finite dimension $n+m$, **R**-analytical manifold of parameters M with dimension n and an autonomous system of the differential equations in each point of manifold P dependent locally from vector variable x and vector parameter a is given:

$$\frac{dx}{dt} = f(x,a) \qquad (1)$$

where $a = (a_1,...,a_n)$ – a set of local parameters in some vicinity U of **R**-analytical manifold M, $y = (x_1,...,x_m,a_1,...,a_n)$ – a set of local coordinates in some vicinity W on manifold P, $f(x,a)$ – regular **R**-analytical vector function in vicinity W on the manifold P.

There is the regular **R**-analytical mapping $p : P \to M$ specifying a locally trivial fiber bundle $\eta = (P, p, M)$ for the manifolds P and M, in more details, there exists an **R**-analytical diffeomorfism of vicinity $W = p^{-1}(U)$ in any point of the manifold P to direct product of vicinities $U \times V$ for any vicinity U, where V – a vicinity with local coordinates $x = (x_1,..., x_m)$.

Points with coordinates of a vector variable x form phase space for the fixed value of parameter $a = (a_1,...,a_n)$, the space of fiber bundle η will be named a total phase space of differential system (1).

Let's define analytical differential system in accordance of the introduced signs.

***Definition 1**:* There exists a differential system D with a rank q on the manifold P if for each point $y \in P$ there is a vicinity in the corresponding point y of tangent bundle, in which there exists submanifold $D(y) \subset T_y(P)$ with dimension q, where $T(P)-$ tangent bundle of considered manifold P.

The differential system (1) given on tangent bundle determines analytical distribution of any dimension smaller or equal $m+n$. The system of the differential equations (1) determines on P **R**-analytical polyvector fields, which set k-dimensional analytical distributions [Grauert and Remmert, 1977; Postnikov, 1998], $k \le n+m$.

Analytical manifold P is integrated manifold of system of the differential equations (1). However, **R**-analytical stratification generally does not exist for any dimension. It is connected to existence of singular points of system (1). For simple points the theorem of existence and uniqueness of the solution of differential system takes place. An integrated surfaces of any order, less or equal $m+n$, are **R**-analytical manifold for such points.

3 Integrability of Differential System

Let's consider a layer of fiber bundle $\eta = (P, p, M)$ above manifold M, due to M is the R-analytical manifold and according to the made assumptions there exists tangent bundle. The differential system (1) is set on the tangent bundle and defines analytical distribution of any dimension smaller or equal $m+n$. The system of the differential equations (1) defines on P **R**-analytical polyvector fields which set k-dimensional analytical distributions [Khalin, 2003a; Khalin, 2004a-b], $k \le n+m$.

If $\{y_i\}$ are local coordinates in the vicinity W of a point $y \in P$, then elements $\left\{\frac{\partial}{\partial x_i}\right\}$ form basis in $T_y(P)$. Such system of vectors forms a nonzero k-vector for any point y at the fixed value of parameter and determines a nonzero k-vector field for P. For each point on P k-vector can be presented as $X_y^k = \sum_{i_1<\ldots<i_k} f^{i_1\ldots i_k} \frac{\partial}{\partial x_{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x_{i_k}}$, $k \le m$.

The polyvector is determined to within a positive constant and forms nonzero orientable **R**-analytical distribution. Let's designate through E linear space of polyvectors.

***Definition 2**:* Submanifold $Q \subset P$ will be named integrated manifold of system D (1) if for any point $y \in Q$ inclusion $T(Q) \subset D(y)$ takes place.

***Definition 3**:* Distribution is integrated if q-dimensional integrated manifold Q passes through any point of the manifold P, which is tangential to distribution in each point [Postnikov, 1998].

The space of differential forms Ω is the conjugate space to a layer of the tangent bundle. Dual fiber bundle $\eta^\bullet = (P^\bullet, \tau, TP^\bullet)$ to the tangent bundle TP of manifold P we shall define as analytical fiber bundle of a coordinate ring of analytical functions $P^\bullet$ of manifold P above base, a coordinate ring of functions of the tangent bundle, which is ring of differential forms.

For fiber bundle $\eta^\bullet$ there is always a local section in regular points of manifold P, since any **R**-analytical function set on analytical manifold is locally integrated. The system (1) sets vector basis of base of the dual fiber bundle.

Analytical manifold P is integrated manifold of system of the differential equations (1). However **R**-analytical слоения generally does not exist for any dimension. It is connected to existence of special points of the system (1). For regular points the theorem of existence and uniqueness of the solution of differential system is fair.

The vector space of polyvector fields and the space of differential forms conjugated it aren't complete in all points of the fiber bundle $\eta = (P, p, M)$ for the system (1). For global resolvability of system on all the fiber bundle η it is necessary to define the solution of system for any point of manifold P, that is to close the space of the **R**-analytical differential forms set by the system (1).

By definition a series of elements of the solution space of the system (1) normally converges, or absolutely converges, if a series of the norms being numerical series of positive members converges. The normalized vector space is complete in only case when any absolutely converging a series converges [Hutson and Pym, 1983].

***Definition 4**:* A normal vicinity of space of differential forms is corresponded to the solutions of system having normally converging series in some normal vicinity of zero section of the fiber bundle η. Such solutions of system is named fundamental.

4 Space of the Generalized Solutions

Property of integration for distribution of the differential system (1) is local, there is enough to consider it for a vicinity W for any point on the **R**-analytical manifold P.

The space $\mathbf{\Omega}$ locally defines on linear space of polyvectors E space of analytical Lebesque's measures. The measure of any dimension is determined to within a positive constant.

***Definition 5**:* Distribution of dimension q is integrated in a point if the one belongs locally finite measurable submanifold of the manifold P with dimension q.

In order to define the solution of system for any point of the manifold P it is necessary to expand or complete space of the analytical polyvector fields. It is necessary to add the differential equations for parameters to the system (1) which determine a zero sections fiber bundle. Due to equality to zero of differentials from parameters the manifold of parameters is determined by system (1) as a manifold of zero Lebesque's measure. The system can be completed by the equations with parameter derivatives, which are identically equal to zero, and such system will be called complete.

Thus, due to integration of the system (1) there exists a space of linear functionals on space of analytical differential forms, being conjugated for space of differential forms. It is a complete finite space of dimension $n+m$. That space will be named the generalized **R**-analytical functions, where the basic functions are **R**-analytical functions on compact vicinities of zero on the fiber bundle E [Fuchs, 1962]. Then integration of the system is equivalent to existence finite **R**-analytical functional in any point of the fiber bundle P.

Definition 6*:* Subset U of locally compact space TP is integrated concerning the measure μ on E if characteristic function φ of the subset U is integrated. The finite number $\mu(U) = \int \varphi \, d\mu$ is called a measure of the subset U [Kolmogorov and Fomin, 1989].

In order that a measure μ on locally compact space $K \subset E$ has been limited, it is necessary and enough, that K was integrated set concerning μ, or this is equivalent, that any constant function has been integrated, then

$$\mu(K) = \int d\mu$$

is a linear mapping of analytical functions on compact sets to subspace of the limited functionals in the Banach space of functionals. The subspace of analytical functions on the compact sets is dense all over the space of analytical functions on analytical manifold. Limited integration by Lebesque can be continued analytically to all space due to Riemann's theorem [Bourbaki, 1952].

The space of such functionals will be named one of the generalized **R**-analytical functions. Thus the space of the basic functions is made with locally constant functions. Set of all generalized functions with the given basic space coincides with the conjugate space to space of **R**-analytical differential forms.

5 The Completeness of the Generalized Solutions Space of Differential System

Integrability of the differential system (1) is supposed. It is equivalent to existence of finite value of a functional $F(y) = \tau \circ f \in P^{\bullet}$ in any point, and regularity of a point is equivalent to integrability of the systems (1). In order to that functional $F(y)$, being simple and analytical in complete space, was regular, it is necessary and enough that there was vicinity W of a point y_0 in which function is limited by norm: $\lim_{y \to y_0} F(y) = M < \infty$ [Fuchs, 1962]. The norm of solutions space $\max_{y \in W} |F(y)| = L$ is supposed, where W - compact closing of a normal vicinity of the point y_0.

According to an Weierstrass's indicator of even convergence, if function $F(y)$ is analytical in W and continuous on its closing, so convergence in the whole vicinity is following by convergence on its bondary Γ [Efimov, 1970].

Any function $F(y)$ being holomorphic in a ring area $W = \{r < |y - y_0| < R\}$ is presented as the sum of converging series $F(y) = \sum_{n=-\infty}^{\infty} c_n (y - y_0)^n$, which coefficients are defined by the formula

$$c_n = \frac{1}{2\pi i} \int_{|y-y_0|=\rho} \frac{F(\tau) d\tau}{(\tau - y_0)^{n+1}}, \; n = \pm 1, \pm 2, \ldots,$$

where $r < \rho < R$ [Fuchs, 1962].

The generalized solution of system (1) is simple defined by locally integrated function $f(y)$ on W in only case, when considered generalized function is equal to the generalized derivative $\left(\int_W f(y) dy \right)'$, $y \in W$ [Antosik, Mikusinski and Sikorski, 1976].

According to Lowville's theorem function $F(y)$ is identically equal to a constant if it is analytical and limited on space $\boldsymbol{C}^{n+m}$. The sequence of constant functions converges in the generalized sense in only case, when it converges in usual sense [Bourbaki, 1952].

According to the theorem of existence of the solution of system of the ordinary differential equations the system of the equations (1) defines system of local one-parametrical groups of local operators $\{G_i\}, i = 1, \ldots, n$, in some vicinity of any point of a layer of fiber bundle η generating corresponding system of analytical polyvector fields [Narasimhan, 1968].

The fiber bundle with structural group G_i defines associated fiber subbundle ξ_i where the layer of tangent bundle of a layer of fiber subbundle η is vector G-space. System of integrated distributions by system (1) defines a class of locally isomorphic fiber G-bundle above manifold M and the corresponding class of locally homotopic G-layers of fiber G-bundle of manifold P [Khalin, 2003b; Kuznesov, 1995].

Sections of locally trivial fiber bundle ξ_i are in natural isomorphic conformity with local continuous solutions of system (1) [Postnikov, 1998]. For section $s : M \to P$ of fiber bundle η satisfies to the equation $p \circ s = id$. Thus, local resolvability of system is equivalent to existence of local sections s_i of fiber G-bundle of ξ_i. The sequence $\{G_i\}$ is growing sequence of the self-adjoint operators. Then, if it is $\sup_i \|G_i\| = M < \infty$ there is linear operator G such, that for any $y \in P$: $Gy = \lim_{i \to \infty} G_i y$, thus $\|G\| \le M$ [Kantorovich and Akilov, 1977].

The converse statement follows from Cauchy's inequalities for derivatives. For analytical operator $G(y)$ all its derivatives satisfy to an inequalities $\left\|G^{(n)}(x_0)\right\| \le \frac{Mn!}{R^n}$, $n = 1, 2, ...$ in a closed circle $|y - y_0| \le R$, where M is norm of the operator $G(\xi)$ on a circle $|\xi - y_0| = R$ [Euderman, 2002].

6 One-Connectivity of Space of Solutions of the Complete Differential System

According to Hilbert - Schmidt's theorem for the compact self-adjoint operator G_i and its eigenvectors $\{g_i\}$ it is possible to make orthonormal basis on manifold P [Bourbaki, 1952], that is decomposition

$$Gf = \sum \lambda_k (f, g_k) g_k \ , f \in P^{\bullet}$$

is fair. Resolventa $R(\lambda; G_i)$ is an analytical function from λ in resolventa set $\rho(G_i)$ of the operator G_i, for $\lambda \in \rho(G_i)$ and any differential form $g \in P^{\bullet}$ the solution of the equation $(\lambda I - G_i) f = g$ can be presented as

$$f = R_i(\lambda; G_i) f = \sum (\lambda - \lambda_k)^{-1} (f, g_k) g_k \ .$$

$R(\lambda; G_i)$ is an operator analytical function from λ with simple poles in the eigenvalues.

The spectrum of the operators G_i is set of singular points of the analytical functions $R_i(\lambda; G_i)$. Resolventa has a finite set of singular points, and all of them are poles. Let $\{\lambda_k\}$ are the closed set of isolated points of open set D of space **C**, $R_i(\lambda; G_i)$ is holomorphic in $D / \{\lambda_k\}$. Let D is the closed area containing all set $\{\lambda_k\}$, and its boundary Γ does not contain any of the points $\{\lambda_k\}$. Then the formula of deductions

$$\int_{\Gamma} R(\lambda) dy = 2\pi i \sum_k \mathrm{Re}\, s_{k_i} f = 2\pi i \sum_k \lambda_{k_i} \ .$$

is fair. Thus, the generalized solution is determined for any point of manifold P.

Set of all eigenvalues are named a spectrum of operator G_i. Regular points form open set, the spectrum forms the closed set. By virtue of compactness of the operator the spectrum of the operator coincides with set of its eigenvalues. The generalized solutions of the system are the holomorphic functions, therefore according to the global theorem of existence of an antiderivative for the one-connectivity area [Efimov, 1980]. The complete analytical differential system is a weak one-connective space, and space of its solutions converges in the generalized sense.

***Statement 1**:* The space of solutions of the system (1) is complete in only case when there are fundamental solutions of the system which represents a class of equivalence of normally converging generalized Loran's series for the generalized solutions of system.

7 Structural Stability and Bifurcations of Differential System

The topology of even convergence in space of the generalized **R**-analytical functions is a weak topology on the Banach space of differential forms. Finiteness of the spaces of considered linear functionals and the differential forms gives equivalence between closing of space of differential forms in weak topology and one in strong topology which is locally convex topology in the space.

***Statement 2**:* Closed submanifolds in space of **R**-analytical differential forms are **R**-analytical sets of space of the generalized **R**-analytical functions.

The space of the introduced generalized functions is linear subspace sections of the fiber bundle *P*. The section manifold of the fiber bundle is closed, if it is closed in zero that is on parameter submanifold in total space of the fiber bundle η.

***Definition 7**:* A set of critical points or bifurcation points on the fiber bundle η will be called a set with a zero measure concerning measure determined by differential system (1). Therefore, this set is the closed set in space of the generalized **R**-analytical functions.

***Definition 8**:* The differential system (1) will be called a transversal system or structurally stable in a point of parameter manifold if this point is not a bifurcation point.

***Definition 9**:* Function g determined on *TP* will be called insignificant concerning Lebesque's measure μ if $\mu(g) = 0$ on *TP*. That linear function g determined on *TP* was insignificant, it is necessary and enough, that $g(y) = 0$ almost everywhere [Bourbaki, 1952].

***Definition 10**:* Two sections of the fiber bundle E will be called equivalent ones concerning μ if they are equal almost everywhere on E.

Integrated functions on compact vicinities of zero in E is locally constant functions, finite almost everywhere. Such function is integrated, and its integral is **R**-analytical functional. A zero measure sets on space E are carriers of insignificant functions. The corresponding generalized functions on set of a zero measure is equal to zero.

***Statement 3**:* The set of bifurcation points of the differential system (1) is an **R**-analytical variety.

For topological vector space of the generalized functions there is a filter of the closed vicinities forming a fundamental vicinity system of zero. On this space it is possible to enter a normal vicinity of zero section of fiber bundle η [Bourbaki, 1952]. The critical point of linear functional is determined on tangent bundle by equality to zero of a determinant of one from the main minors of derivative of the operator (1) or symmetric function of eigenvalues of the derivative.

The complete space of polyvector fields on P is isomorphic to the conjugate space of polylinear analytical forms which local basis is the set of eigenvalues. Thus the normal vicinity of zero section for P is an image exponentional mapping, so analytical, from corresponding vicinity in the fiber bundle Ω [Postnikov, 1998]. On tangent bundle the set of points with degenerate eigenvalues forms variety with codimension not less one, therefore they do not influence on analyticity of exponentional function according to Riemann theorem of continuation of analytical functions [Grauert and Remmert, 1977].

***Statement 4**:* The generalized elementary symmetric functions of eigenvalues determine fundamental system of normal vicinities in space of the generalized **R**-analytical functions.

***Statement 5**:* The set of bifurcation points of the differential system (1) is determined by system of the **R**-analytical equations defined by the elementary symmetric functions of eigenvalues of the differential system (1).

8 Conclusion

Integrability of the **R**-analytical differential system (1) is equivalent to existence of the fundamental solution, since the common solution of homogeneous system in the space of the generalized functions always exists. Any integrated differential system has the fundamental solution and on the contrary. The complete system has the solution in a class of the generalized functions at any right-side part [Gelfand and Shilov, 2000].

A symmetric function of the system (1) gives bifurcation varieties on analytical manifold of the system parameters. The system of the analytical equations for bifurcation variety can be received by join of the equations determining stationary points and the equations determined by symmetric forms concerning eigenvalues of a derivative for the operator of system (1).

Methods of calculation of critical surfaces for symmetric bifurcations in systems of the differential equations with nonlinear right-hand side by use of algebraic transformations, numerical integration and differential methods of solution search are developed and published [Bratus' and Khalin, 2002; Khalin, 2000; Khalin, 2002a-c; Khalin, 2003a].

Acknowledgements

I extremely thank Larisa V. Tuchina for important and helpful discussions of the all main ideas of this paper and for supporting of its implementation.

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