SPEED-GRADIENT CONTROL OF AN INVARIANT FOR MULTISPECIES POPULATIONS

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Abstract. The problem of control of an invariant for the model of the multispecies Lotka-Volterra ecosystem model is examined. The algorithm for control of oscillatory behavior of the ecosystem based on the speed-gradient method is designed. The conditions of achievement of the control goal are proposed. The results of numerical experiments for control of four-species model by two parameters (birth rates) are presented.

Keywords: Lotka-Volterra ecosystem, ecological stability, speed-gradient algorithm

1. Introduction

Stability analysis of ecosystems is important both for theory and for practice. Only stable ecosystems are able to exist for a long time and their stability limits define those maximum loadings which excess can lead to ecocatastrophes. The stability problem is connected with questions of an operation of natural populations, estimations of pollution limits of an environment, the forecast of farming.

The classical Lotka-Volterra model of the population dynamics ("predator-prey" model) and its generalization to the case of N species are well recognized in mathematical ecology and biophysics [1,2]. The Lotka-Volterra models were considered in detail in [2] where a special attention was paid to stability analysis of the mathematical model of an ecosystem. In [3] these models as the thermodynamic systems were investigated, and generalized expressions of entropy-production for the systems and the study of their role in the analysis of ecological stability were derived.

The objective of the present work is to develop the algorithm based on the speed-gradient method for control of oscillatory behavior of the generalized Lotka-Volterra model in order to improve its stability. Analogous results for the case of N=2 are obtained in [4].

2. Mathematical Model of an Ecosystem

In a class of the ordinary differential equations a generalized Lotka-Volterra model represents the system [1]:

$$\frac{dx_i}{dt} = x_i(t)\cdot\left(k_i + \beta_i^{-1}\sum_{j=1}^{N} a_{ij}\cdot x_j(t)\right), \tag{1}$$

Where $i = 1,2,..N$, k_i is the speed of the natural increase or death rate of the *i*th kind in the absence of all other species: $k_i < 0$, if the *i*th species lives at the expense of others and $k_i > 0$ else.

The parameter β_i reflects the fact that the appearance of a predator is usually connected with vanishing of one or more preys. Quantities $a_{ij}, i \neq j$ evaluate the type and intensity of the interaction between *i*-th and *j*-th species and form an antisymmetric matrix. The stability of the ecosystem can be interpreted as the special behavior of solutions of (1) when all species stay alive, that is their numbers are always more than zero. Obviously for the stability of the system (1) its solutions must not approach the border of the positive ortant.

3. Controlled Model

In this paper we consider the controlled version of the model (1). Suppose the birth rate of the species $x_l, l = M+1,..N$ can be controlled. Then the interaction between the species is described by the differential system:

$$\begin{cases} \frac{dx_i}{dt} = x_i(t) \cdot \left(k_i + \beta_i^{-1} \sum_{j=1}^{N} a_{ij} \cdot x_j(t) \right), i = 1,2,..M, \\ \frac{dx_l}{dt} = x_l(t) \cdot \left(k_l + \beta_l^{-1} \sum_{j=1}^{N} a_{lj} \cdot x_j(t) + u(t) \right), l = M+1,..N. \end{cases} \quad (2)$$

Assume that there exists at least one positive equilibrium of (1) for some values of the system parameters:

$$n = (n_1, n_2, .., n_N), n_i > 0, \ i = 1,..N, \quad (3)$$

and consider an auxiliary function W:

$$W(x) = \sum_{i=1}^{N} \beta_i n_i \left(\frac{x_i}{n_i} - \log \frac{x_i}{n_i} \right). \quad (4)$$

In [1] it is shown if the condition (3) is satisfies, that W will be an invariant of (1). Since $W(x)$ can measure the amplitude of oscillations, we can use it to achieve the desired amplitude of oscillations. Therefore the control goal can be stated in terms of achieving the desired level of the quantity W:

$$W \to W^*, t \to \infty. \quad (5)$$

Apply the speed gradient (SG) method [5] for solution of the problem. To this end introduce the function Q:

$$Q(x,u) = \frac{1}{2}\left(W(x) - W^*\right)^2. \quad (6)$$

In order to achieve the goal (5), it is necessary and sufficient that Q converges to zero. According to the SG method one needs to evaluate A) derivative (speed of changing) of Q with respect to the system (2) and B) the gradient of $\dot{Q}$ with respect to u.

Calculation of time derivative of Q with respect to the system (2) yields:

$$\dot{Q}(x,u) = \left(W(x) - W^*\right) \sum_{l=M+1}^{N} (x_l(t) - n_l) u_l. \quad (7)$$

Partial derivatives on u_l are as follows:

$$\frac{\partial}{\partial u_l} \dot{Q}(x,u) = (W(x) - W^*)(x_l(t) - n_l), \quad (8)$$

where $l = M+1,..N$. According to the speed gradient algorithm control action is chosen as follows:

$$u_l(t) = -\gamma_l (W(x) - W^*)(x_l(t) - n_l), \gamma_l > 0, \quad (9)$$

where $l = M+1,..N$. The main result of this paper is the following proposition.

Theorem: Assume that there exist an equilibrium in the system (1) such that (3) holds. Then either the algorithm (9) provides the goal (5), or the quantities of the controlled species tend to their equilibrium values.

Proof: Consider the time derivative of the goal function Q (6):

$$\frac{d}{dt} Q(x,u) = -2\gamma Q \sum_{l=M+1}^{N} (x_l - n_l)^2 \leq 0. \quad (10)$$

Since Q doesn't increase, there exists a limit of $Q(t)$ as t→∞. Denote it as $\overline{Q}$. Suppose the goal (5) doesn't hold. Then $\overline{Q} > 0$. Apparently $Q(t) \geq \overline{Q}$ for all $t \geq 0$ and

$$\frac{d}{dt}Q(x,u) = -2\gamma\overline{Q}\sum_{l=M+1}^{N}(x_l - n_l)^2 \leq 0. \quad (11)$$

Integration of (11) yields

$$0 \leq Q(x(t),u(t)) \leq Q(x(0),u(0)) -$$
$$-2\gamma\overline{Q}\sum_{l=M+1}^{N}\int_0^t (x_l(s) - n_l(s))^2 ds \leq 0. \quad (12)$$

Therefore

$$\sum_{l=M+1}^{N}\int_0^t (x_l(s) - n_l(s))^2 ds < \infty. \quad (13)$$

The integrand function converges to zero according to Barbalat Lemma [6], that is

$$x_l(t) \to n_l, t \to \infty, \ \ l = M+1,..N. \quad (14)$$

Thus either the algorithm (9) provides the control goal (5), or the number of the controlled species $x_l(t)$ converges to its equilibrium n_l ■

Remark. In Theorem 1 it has been proposed that the system (1) has at least one positive equilibrium for some its parameters. For a nonsingular matrix composed of a_{ij} we always can choose values of the birth rate k_i such that (3) holds [3]. For a nonsingular matrix composed of a_{ij} positivity conditions depending only on a_{ij} were found in [2]. Finally, for both nonsingular and singular cases positivity conditions were given in [7].

4. Numerical experiments

We present the results of numerical experiments demonstrating the dynamics of the system controlled by the algorithm (9). Below the behavior of uncontrolled system (1) (Fig.1) and the behavior of the system (2) for the case of controlled the third and forth species (Fig.2, Fig.3, Fig.4) for $N=4$ are shown. We take initial numbers of the species $x_{01} = [4;7;6;5]$; $x_{02} = [2;3;5;3]$ and the system parameters

$$\beta_1 = 4;\ \beta_2 = 2; \beta_3 = 3;\ \beta_4 = 6;$$
$$k_1 = -9; k_2 = -8; k_3 = 7; k_4 = 6;$$
$$a_{12} = 2;\ a_{13} = 3;\ a_{14} = 5;$$
$$a_{23} = 4;\ a_{24} = 3.5;\ a_{34} = 2.$$

Three versions of the desired levels of W are considered: $W_1{*} = 52; W_2{*} = 62; W_3{*} = 40.$

The coefficients in the control algorithm are taken as $\gamma_3 = \gamma_4 = 0.08$. The equilibrium of the system (1) for these parameters is $n_1 = 3;\ n_2 = 5;\ n_3 = 2;\ n_4 = 4$, the equilibrium value of the quantity W is $W^e \cong 51.2.$

In Fig.2 and Fig.3 initial numbers of the species are picked such that $W^e < W^0 < W*$, in this case value of W have converged to its desired level rather fast. For the initial numbers of the species in Fig.4 $W* < W^e < W^0$ and the desired level of W has not been achieved, but the numbers of the controlled species have converged to their equilibrium value.

5. Conclusion

In this work we have demonstrated the application of the speed-gradient method for solving non-traditional control problems of nonlinear network models, a special case of which is a Volterra model of the dynamics of the N species.

The simulation results have shown at the smaller value of *W* the oscillation variations in the number of the species are lower. Thus, to improve ecosystem stability it is necessary to reduce the value of *W*. The algorithm based on the speed-gradient method can do it with small control signal, which is important in controlling real ecosystems where control action should be sufficiently small.

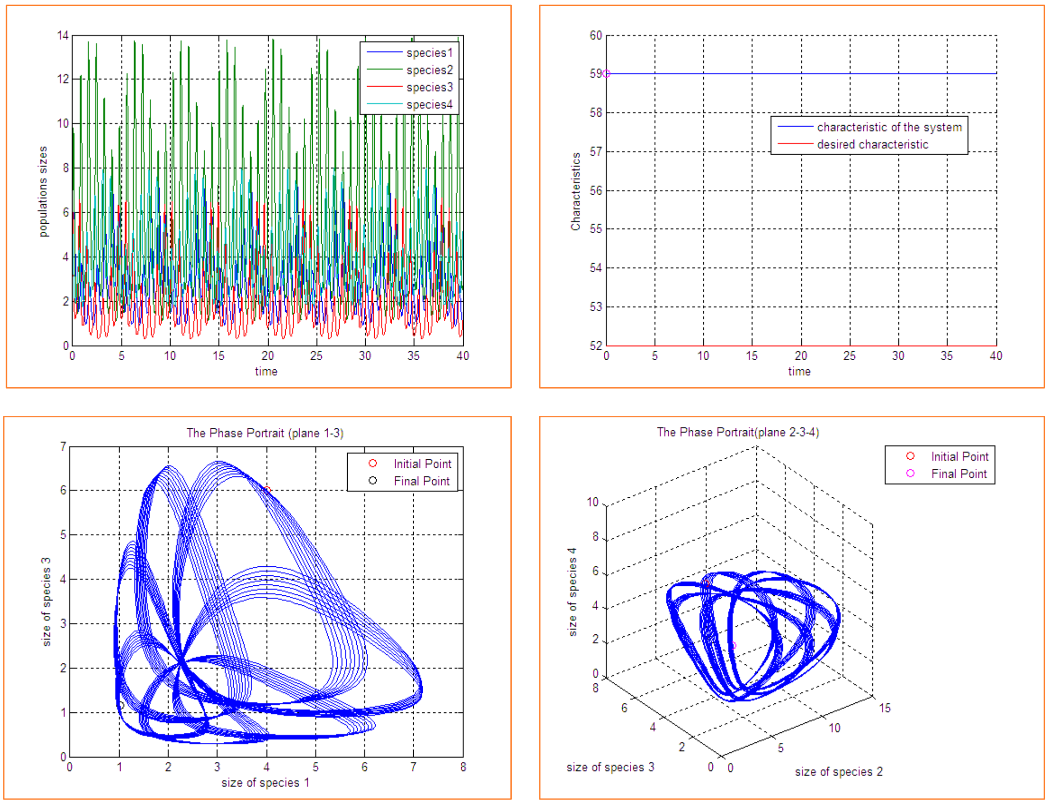

Fig.1. Plots of the numbers of the species versus time (top left) and W versus time (top right) and the phase portraits (bottom) of the uncontrolled system (1) for N=4 and initial numbers of the species $x_{01} = [4;7;6;5]$

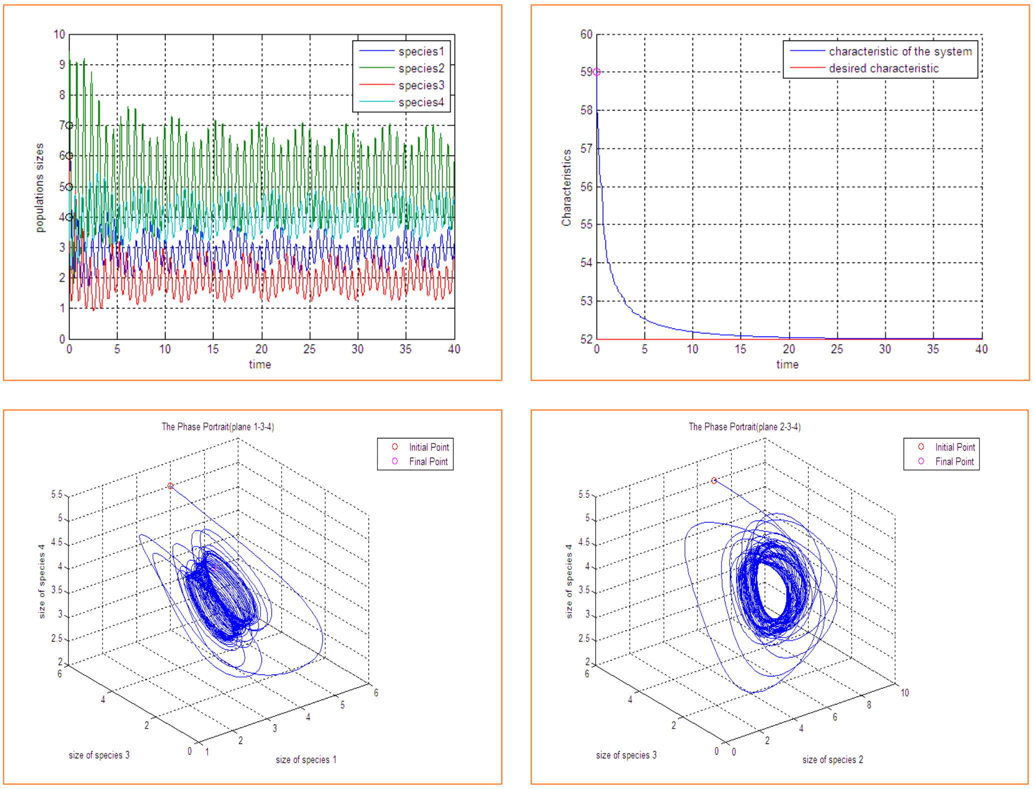

Fig.2. Plots of the numbers of the species versus time (top left) and W versus time (top right) and the phase portraits (bottom) of system (2) when controlling the numbers of the third and forth species, for N=4, $x_{01} = [4;7;6;5]$ *and desired value W*=52.*

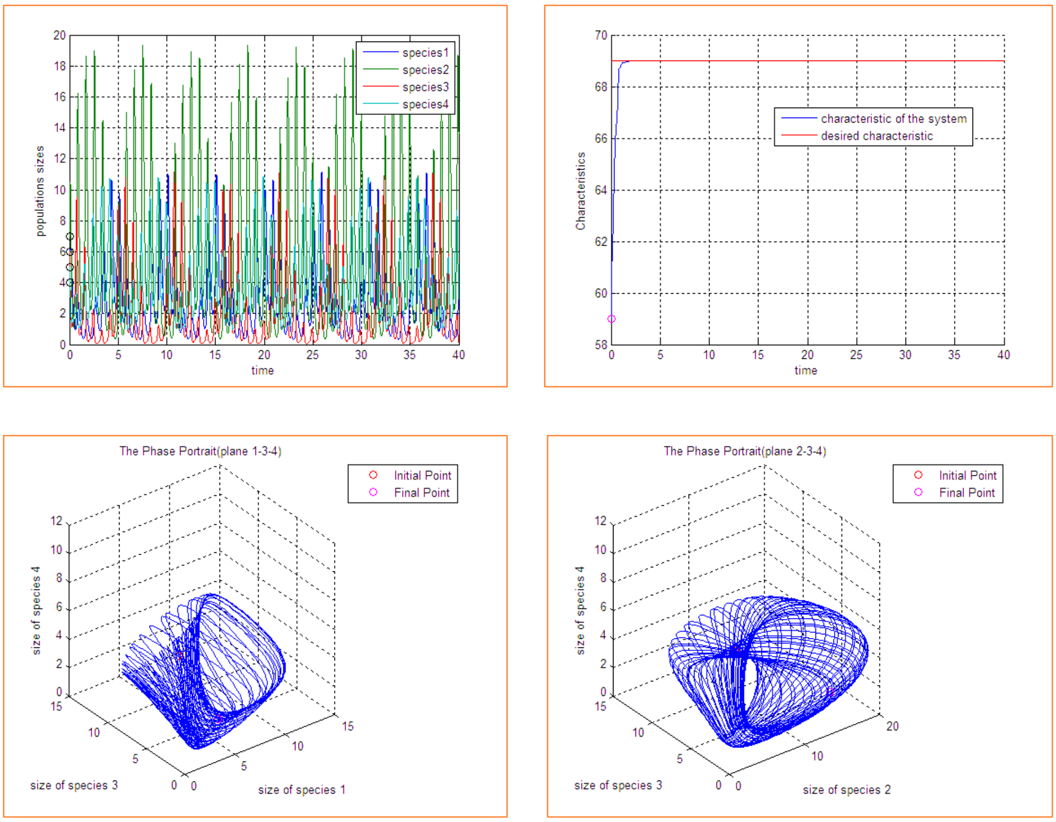

Fig.3. Plots of the numbers of the species versus time (top left) and W versus time (top right) and the phase portraits (bottom) of the controlled system (2) when controlling the numbers of the third and forth species, for N=4, $x_{01} = [4;7;6;5]$ *and W*=69.*

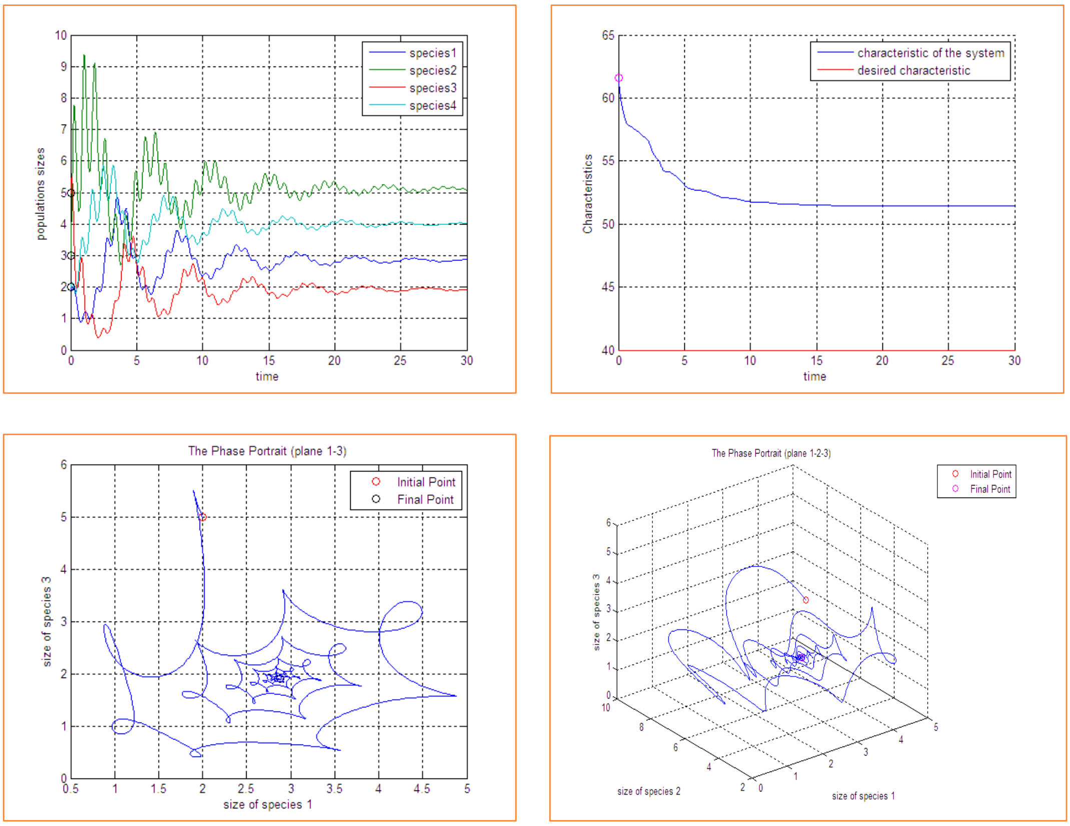

Fig.4. Plots of the numbers of the species versus time (top left) and W versus time (top right) and the phase portraits (bottom) of the controlled system (2) when controlling the numbers of the third and forth species, for N=4, initial numbers of the species $x_{02} = [2;3;5;3]$ *and desired value W*=40.*

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