NONLINEAR NORMAL MODES IN HOMOGENEOUS SYSTEM WITH TIME DELAYS

O.V.Gendelman
Faculty of Mechanical Engineering
Technion – Israel Institute of Technology
Haifa 32000 Israel
e-mail: ovgend@tx.technion.ac.il

Abstract.

Periodic synchronous regimes of motion are investigated in symmetric homogeneous system of coupled essentially nonlinear oscillators with time delays. Such regimes are similar to nonlinear normal modes (NNMs), known for corresponding conservative system without delays, and can be found analytically. Unlikely the conservative counterpart, the system possesses "oval" modes with constant phase shift between the oscillators, in addition to symmetric/antisymmetric and localized regimes of motion. Numeric simulation demonstrates that the "oval" modes may be attractors of the phase flow. These attractors are particular case of phase-locked solutions, rather ubiquitous in the system under investigation.

Keywords:
Nonlinear Normal Modes, Time – Delay Systems

1. Introduction

A concept of nonlinear normal mode (NNM) as analytic continuation of well – known normal modes of linear system originates from the work of Lyapunov [Lyapunov, 1947]. The concept has been refined by Rosenberg and other researchers [Rosenberg, 1960, 1962, Kauderer, 1958]; they have primarily considered the NNMs as particular solutions of nonlinear dynamical systems, which are characterized by synchronous evolution of all variables. Recently, there was a plenty of activity in the field of the NNMs, including successful analysis of multi-DOF systems, transient motions etc [Vakakis et al, 1996, Manevitch et al, 1989, Manevitch and Manevitch, 2005, Shaw and Pierre, 1991, 1993, Nayfeh and Nayfeh, 1994, Vakakis et al, 2003, Gendelman, 2004].

Account of time delay is necessary modification of many dynamical models if one takes into account finite speed of the signal propagation, retarded waves etc. Such models are widely studied in relation with cutting and milling processes, neural networks and other systems [Sen and Rand, 2003, Wirkus and Rand, 2002, Atay, 2003, Ramana et al, 1998, Fofana and Ryba, 2004, Wahi and Chatterjee, 2004, Stepan et al, 2005, Morrison and Rand, 2007]. Usually the nonlinearities in these models are treated by means of asymptotic expansions and averaging; these methods are very powerful but it is well – known that they can miss some effects related to essential nonlinearity.

The goal of present paper is to investigate whether the concept of the NNM may be useful for investigation of essentially nonlinear dynamical system with time delays. Of course, one can no more rely on the linear normal modes – the state space has infinite dimension from the very beginning. Instead it is proposed to treat the NNMs as synchronous solutions of the system. The study of in-phase and out-of-phase motions in a system of coupled Van-der-Pol oscillators with time delays has been performed in [Wirkus and Rand, 2002]. The phase – locked solutions in the case of relaxation oscillations were studied in [Sen and Rand, 2003]. Current paper adopts different approach – the dynamical system is chosen in a form which allows exact computation of some modal shapes, with subsequent numeric verification.

2. Description of the model and analytic treatment

The system of symmetric homogeneous coupled essentially nonlinear oscillators is postulated to have a form

$$\begin{aligned} &\ddot{y}_1 + Cy_1^m + G(y_1 - y_2(t-\tau))^m = 0 \\ &\ddot{y}_2 + Cy_2^m + G(y_2 - y_1(t-\tau))^m = 0 \end{aligned} \quad (1)$$

where $y_i \equiv y_i(t), i = 1,2$, C is the nonlinear stiffness of the oscillator, G is the coupling strength, τ is the time delay and m is odd positive number. In every equation, the time delay is introduced only to coupling term related to the other oscillator; such choice of model is suggested to describe the retarded reaction of the coupling spring. For τ=0 system (1) is reduced to system (1.2.1) in book [Vakakis et al, 1996] without linear part.

After simple rescaling system (1) is reduced to dimensionless form

$$\begin{aligned} &\ddot{y}_1 + y_1^m + k(y_1 - y_2(t-1))^m = 0 \\ &\ddot{y}_2 + y_2^m + k(y_2 - y_1(t-1))^m = 0 \end{aligned} \quad (2)$$

which is the basis for further analysis.

Periodic synchronous solutions of system (2) are searched in the form

$$y_1 = cy_2(t-T) \quad (3)$$

where c is the constant ratio of amplitudes and T is the constant phase shift. Substituting (3) to (2) and shifting the first equation of system (2) by T, one obtains

$$\begin{aligned} &\ddot{y}_2 + c^{m-1}y_2^m + kc^{-1}(cy_2 - y_2(t+T-1))^m = 0 \\ &\ddot{y}_2 + y_2^m + k(y_2 - cy_2(t-1-T))^m = 0 \end{aligned} \quad (4)$$

In order to have a consistent solution, the first and the second equations of system (4) should be equivalent. Periodic solutions are searched for; let us consider that the solution of system (4) is periodic with minimal period Δ. Due to the symmetry of initial equations, one can suggest

$$y_2 = y_2(t+\Delta), y_2 = -y_2(t+\frac{\Delta}{2}) \quad (5)$$

System (4) will be consistent, if the following relationships hold:

$$T-1 = \frac{n\Delta}{2}, T+1 = \frac{l\Delta}{2}, n,l \in \mathbb{Z} \quad (6)$$

From (6) it is easy to obtain

$$\Delta = \frac{4}{l-n}, T = (l+n)\frac{\Delta}{4} = \frac{l+n}{l-n} \quad (7)$$

Possible values of period are discrete and time shift T is multiple of Δ/4. The latter conclusion allows description of the modal shapes at configuration plane y_1-y_2: for l+n even the modal shapes will be straight lines and for l+n odd – ovals. Solutions similar to the latter type of modes were described, for instance, in [Manevitch and Manevitch, 2005] for coupled conservative oscillators with weak nonlinearity. There they were referred to as "elliptic" modes. Here this term is not justified, since solutions of essentially nonlinear system (1) are very different from sine and cosine functions and therefore the modal curves are not ellipses. Therefore the term "oval modes" is used. Case l=n corresponds to trivial solution y_1=y_2=0.

Let us consider different possibilities for l and n, which will yield different consistency conditions for system (6). Due to symmetry of the system without restriction of generality it is possible to adopt l>n.

a) Both l and n are even.

In this case $\Delta = 2/q, q = 1,2,3...$ and T=0 or Δ/2. Condition of consistency for system (4) is written as

$$1 + k(1-c)^m = c^{m-1} + kc^{-1}(c-1)^m \quad (8)$$

Equation (8) always allows solutions c=±1, corresponding to symmetric and antisymmetric modes (due to conditions (3) and (5), transformation c→-c, T→T+Δ/2 keeps the system unchanged). Besides the symmetric and the antisymmetric modes, at certain critical value of k solutions of equation (8) bifurcate, giving rise to localized modes.

b) Both l and n odd;

In this case also $\Delta = 2/q, q = 1,2,3...$ and T=0 or Δ/2. Condition of consistency for system (6) is written as

$$1 + k(1+c)^m = c^{m-1} + kc^{-1}(c+1)^m \quad (9)$$

Transformation c→-c yields equation (8); so, this series of solutions is equivalent to the previous case.

c) l even, n odd;

In this case $\Delta = 4/(2q+1), q = 0,1,...$ and T=Δ/4 or 3Δ/4. In other terms, these solutions correspond to "oval modes" mentioned above. Condition of consistency for system (4) is written as

$$1 + k(1+c)^m = c^{m-1} + kc^{-1}(c-1)^m \quad (10)$$

Structure of solutions for equation (10) is very different from that of (8). First of all, values c=±1 do not satisfy the equation for every value of k. Therefore, almost for all values of the coupling coefficient the solutions will be localized. Moreover, it is possible to demonstrate that these solutions exist only for some diapason of values of k. Reshaping equation (10), one obtains

$$k = \frac{c^m - c}{c(1+c)^m + (1-c)^m} \quad (11)$$

It is easy to demonstrate that the denominator in (10) never vanishes for any odd positive m.

d) l odd, n even.

It is easy to demonstrate that the sign inversion of c brings this case to (10-11).

So, one can conclude that system (4) possesses two families of synchronous periodic solutions. The first family corresponds to the cases a) and b) and can be presented by straight lines on the configuration plane. The second one corresponds to the cases c) and d); at the configuration plane it is described by ovals.

In order to complete the computation, one should determine the amplitude of y_2 for each case. If two equations in system (4) are consistent, both of them are reduced to the form

$$\ddot{y}_2 + Qy_2^m = 0 \quad (12)$$

where Q is a constant depending on the selected mode. This equation has well – known solution (see, e.g., [Salenger et al, 1999]):

$$y_2(t) = Av\left(\sqrt{\frac{2Q}{m+1}} A^{\frac{m-1}{2}} t + \varphi_0\right) \quad (13)$$

where A and φ_0 are determined by the initial conditions and periodic function v(t) with unit amplitude is determined by the quadrature

$$dt = \frac{dv}{\sqrt{1 - v^{m+1}}} \quad (14)$$

Minimal period of solution (13) is determined as

$$T = 2A^{\frac{1-m}{2}} \sqrt{\frac{2\pi}{Q(m+1)}} \frac{\Gamma(1/(m+1))}{\Gamma(1/2 + 1/(m+1))} \quad (15)$$

Equations (8) and (10) allow presenting the coefficient Q in equation (12) as explicit function of c for every case. For "straight" modes (cases a) and b)) this function will read

$$Q_{straight} = \frac{\pm c^m + 1}{\pm c + 1} \quad (16)$$

Positive sign corresponds to a) and negative – to b)

For "oval" modes (cases c) and d)), combination of system (6) and equation (11) yields:

$$Q_{oval} = \frac{\pm c^m (1 \pm c)^m + (1 \mp c)^m}{\pm c (1 \pm c)^m + (1 \mp c)^m} \quad (17)$$

The upper sign corresponds to c) and the lower – to d).

Combination of expression for period (15) with expression (16) or (17) for relevant mode (amplitude ratio c should be selected from the solutions of equations (9) or (11) respectively with given value of coupling k) yields condition for discrete values of amplitude (q=0,1,…, q=0 impossible for (18)):

$$A_{straight} = \left(\frac{Q_{straight}(m+1)}{2\pi q^2} \frac{\Gamma^2(\frac{1}{2} + \frac{1}{m+1})}{\Gamma^2(\frac{1}{m+1})} \right)^{-\frac{1}{m-1}} \quad (18)$$

$$A_{oval} = \left(\frac{2Q_{oval}(m+1)}{\pi (2q+1)^2} \frac{\Gamma^2(\frac{1}{2} + \frac{1}{m+1})}{\Gamma^2(\frac{1}{m+1})} \right)^{-\frac{1}{m-1}} \quad (19)$$

3. Numeric investigations

All numeric simulations presented below were performed for system (3) with m=3 (unless otherwise stated explicitly), for different values of the coupling parameter k and different initial functions $y_1(t)$ and $y_2(t)$, t<0.

a) Straight modes

The first series of simulations illustrates modes which are described by straight lines at the configurational plane. Figure 1 demonstrates the displacements y_1 and y_2 for the symmetric mode (c=1, T=0).

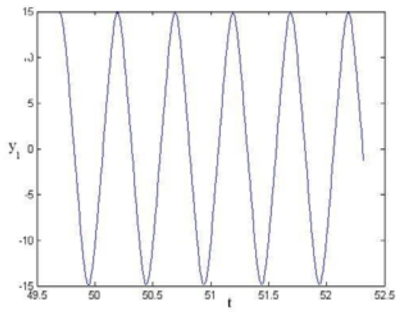
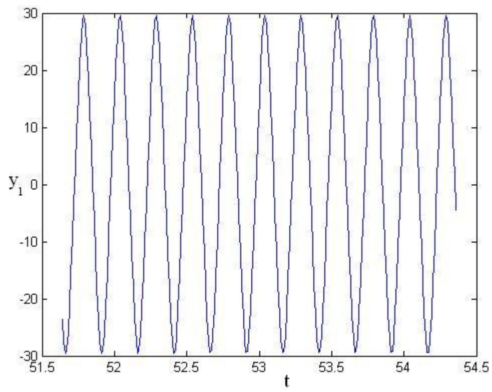


Figure 1. Displacements for the symmetric mode, k=0.1, $y_1(t)=y_2(t)=13$, $t \le 0$;

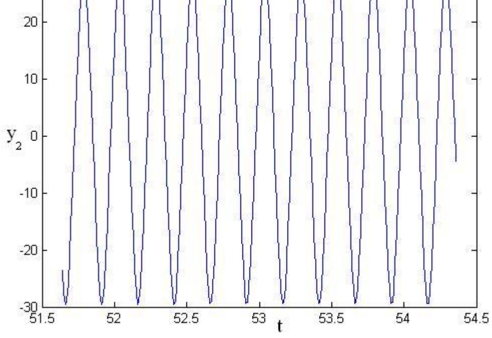
It is easy to see that the symmetric mode is realized with amplitude close to 15. Substituting c=1 and m=3 to (14) and (16), one obtains:

$$A_{straight} \approx 3.708q \quad (20)$$

Figures (1) demonstrates the solution with q=4, the period is equal to 2/q =0.5, in complete agreement with the analytic predictions above. Similarly the antisymmetric mode can be simulated. The results are presented at Figure 2, a, b.



a)



b)

Figure 2. Displacements for the antisymmetric mode, k=0.15, $y_1(t)=-y_2(t)=1$, $t\le 0$; a) $y_1(t)$; b) $y_2(t)$;

It is easy to see that this regime corresponds to q=1. In fact, here c=1 and T=Δ/2.

Stability of the symmetric and the antisymmetric solutions mentioned above is rather subtle issue. From one side, direct simulation of the exact antisymmetric response has not indicated any divergence. At least, it seems that the numeric errors do not bring about the divergence. Still, it is not sufficient to conclude about the stability of this response regime. To illustrate the point, the simulation with relatively small initial deviation from ideal antisymmetry was performed (Figure 3).

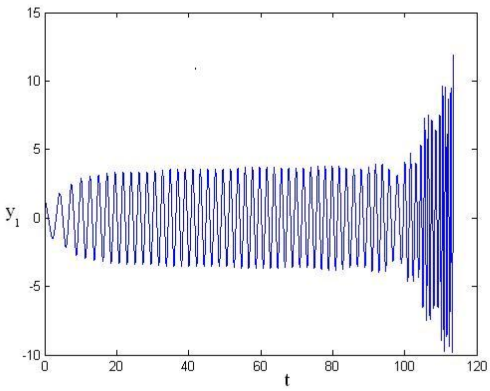


Figure 3. Displacements for the squeezed antisymmetric mode, k=0.15, $y_1(t)=1.2$, $y_2(t)=-1$, $t\le 0$;

In the beginning of the process, the responses are antisymmetric, but finally the flow leaves the antisymmetric mode and diverges – the segment of the plot for t>100 may be problematic with respect to numeric accuracy but a trend towards divergence is obvious. Simulations with smaller deviation from the exact antisymmetry also yielded the approximate capture to the antisymmetric mode with subsequent divergence, but it took much more time. Behavior of the symmetric mode is rather similar.

It is possible to explain this phenomenon by perturbation analysis of system (2) by investigation of the vicinity of the exact solution with c=1. For the sake of simplicity, the analysis will be performed only for symmetric mode with $y_1(t)=y_2(t-1)$:

$$\begin{aligned} y_1(t) &= Y_0(t)+\delta_1(t) \\ y_2(t) &= Y_0(t)+\delta_2(t) \qquad (21) \\ Y_0(t) &= Y_0(t-1), \delta_j << 1 \end{aligned}$$

Substituting (21) to (2), one obtains:

$$\begin{aligned} \ddot{\delta}_1+\{mY_0^{m-1}\delta_1+...\}+k(\delta_1-\delta_2(t-1))^m=0 \\ \ddot{\delta}_2+\{mY_0^{m-1}\delta_2+...\}+k(\delta_2-\delta_1(t-1))^m=0 \end{aligned} \qquad (22)$$

The terms in figured parentheses stem from the uncoupled system and cannot bring about the divergence; the only reason for the instability can be related with the last term. It means, first of all, that the stability of the symmetric and the antisymmetric modes cannot be analyzed in linear approximation. Then, if the deviation is extremely small (like in the case of numeric errors), it will not affect the stability of the numeric solution – the corrections yielded by these deviations will not be taken into account by numeric scheme, since it has finite accuracy. So, the situation here is a bit paradoxical – the regime in fact is unstable but the instability cannot be captured in the linear approximation and does not reveal itself via numeric errors. This case requires complete nonlinear stability analysis by analytic means.

The situation with the other modes where the linear coupling term do not disappear due to the symmetry (e.g. "straight" localized modes), is quite different – it seems that they are unstable and cannot be revealed by direct numeric simulation. Quite surprisingly, the only stable attractors revealed for the system under consideration, are those synchronous solutions which do not exist at all for the system with zero delay– the "oval" modes.

b) "Oval" modes and phase – locked solutions

Typical example of the "oval" mode is presented at Figure 4.

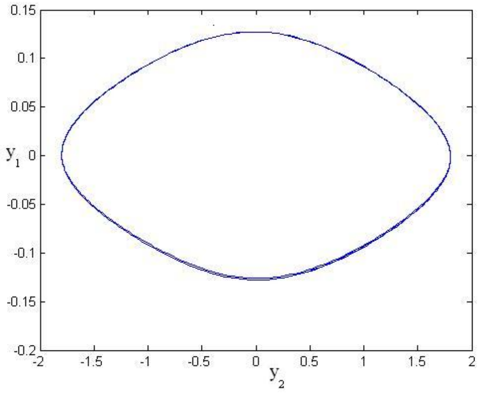
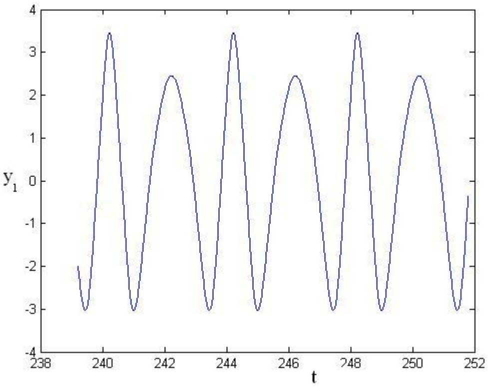


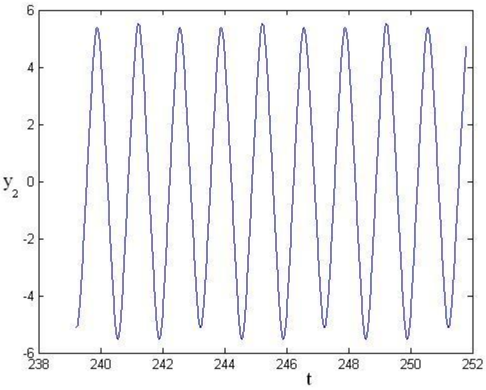
Figure 4. Displacements for the "oval" mode, k=0.06, $y_1(t)=2$, $y_2(t)=0$, $t \le 0$, configuration plane: y_1 versus y_2.

One has to mention that although the initial conditions are rather "far" from the mode obtained; it demonstrates remarkable stability, unlikely the "straight" modes. Stable "oval" modes are ubiquitous in the system under consideration for relatively small values of the coupling coefficient; for higher values of k they cease to exist.

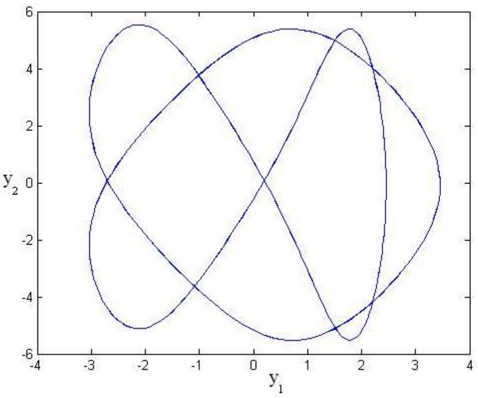
Another interesting phenomenon may be observed for the same value of the coupling but for somewhat higher amplitude of the initial function (Figure 5, a-c).



a)



b)



c)

Figure 5. Displacements for the phase – locked mode, k=0.06, $y_1(t)=3$, $y_2(t)=0$, $t \le 0$; a) $y_1(t)$; b) $y_2(t)$; c) configuration plane: y_1 versus y_2

One obtains stable phase-locked solution with period ratio 2:3. Such phase – locked solutions are also rather ubiquitous for the system; of course, the synchronous solutions described above are particular case of the phase locking with period ratio 1:1.

4. Discussion and concluding remarks

The investigation presented above demonstrates that the concept of nonlinear normal mode can make sense for the essentially nonlinear system with time delay. For such system the NNM cannot be treated as analytic continuation of the linear normal mode; instead, one should refer to the definition of "synchronous motion". In this case the NNM turns to be a particular case of the phase locking. The NNMs determined in this way may serve as stable attractors of the dynamical flow, which is an impossible in common conservative system.

Special structure of system (1) allowed investigating the synchronous solutions exactly. For other systems it will not be the case. Vast literature devoted to the NNMs describes many approximate methods of their computation. It would be desirable to develop the methods to treat the NNMs in the systems with time delays. It would allow one to find the solutions which are not available by standard averaging or asymptotic schemes.

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