Phase and Frequency Control for Resonance Oscillations in a System with Bounded Perturbations

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Locally optimal control is designed to minimize the phase and frequency deviations from resonance in a nonlinear system with bounded perturbation. The control strategy is independent of the shape of perturbations and the structure of the conservative subsystem. As an example, the phase and frequency control of forced oscillations in a system of weakly coupled oscillators is constructed.

I. Introduction

This paper considers the problem of keeping the system in a stable near-resonance state as a control problem. For a wide range of oscillatory models the analysis of weakly perturbed motion in the neighbourhood of resonance is reduced to the analysis of "an equivalent pendulum" [1]. The phase and frequency deviations from the resonance surface correspond, respectively, to the phase and the frequency of the pendulum oscillations. The phase plane of the pendulum is divided into the domains of libration and rotation separated by the separatrix of resonance. The domain of libration Σ is interpreted as an admissible domain; the passage through the separatrix is associated with failure of resonance. The control task is thus to keep the system within Σ on the maximum time interval.

The pendulum-like model allows making use of the well-developed asymptotic methods of control of oscillations [2 - 4]. However, the solution of resonance control problems for multidimensional systems is tedious, and the closed-form asymptotics is unavailable.

Locally optimal control strategy is considered as an alternative approach to motion control. As known [5], locally optimal control closely corresponds to the solution of the maximum residence time problem. However, the locally optimal control design is much simple than the direct solution of the residence time problem. Then, as shown in the paper, this approach does not require the solution of the maximum principle or dynamic programming equations. This makes locally optimal control a proper tool for controlling systems with uncertain perturbations. The paper demonstrates an application of the locally optimal control approach to the problem of near-resonance control in a nonlinear system with unknown perturbations. We write the equations of motion in the near-resonance domain, introduce the local criterion of optimality, and construct the solution of the control problem. Then we specify the phase and frequency criteria of local optimality and define the phase and frequency control associated with these criteria. We show that the phase and frequency control is asymptotically equivalent to locally optimal control. As an example, we construct the phase and frequency control for a nonlinear system of coupled oscillators.

II. Basic Methodology

2.1. The equations of motion

For brevity, we consider a two-frequency system with a scalar slow variable, scalar perturbation and scalar control. The MIMO system may be studied in a similar way.

The equations of motion are reduced to the standard form [1]

$$\dot{x} = \varepsilon f(x,\theta_1,\theta_2) + \varepsilon^n F(x,\theta_1,\theta_2)u + \varepsilon\Delta(x,\theta_1,\theta_2,\xi(t)), \tag{1}$$

$$\dot{\theta}_i = \omega_i(x) + \varepsilon f_i(x,\theta_1,\theta_2) + \varepsilon^n G_i(x,\theta_1,\theta_2)u + \varepsilon\Delta_i(x,\theta_1,\theta_2,\xi(t)),$$

$i = 1, 2.$

Here $0 < \varepsilon << 1$, the scalar variable $x \in X$ describes the slow evolution of the system, $\theta_i(\mathrm{mod}\ 2\pi)$, i = are the fast scalar phases, control $u \in U \subset R^1$. The coefficient ε^n shall be so chosen that control would remain weak but counteracting the external perturbation.

Following [1], we specify the resonance relationships between the system frequencies. Consider the subsystem

$$\dot{x} = \varepsilon f(x,\theta_1,\theta_2),\ \dot{\theta}_i = \omega_i(x). \tag{2}$$

Define the time average of the function $f(x,\theta_1(t),\ \theta_2(t))$ as the function of the slow variable x and the frequencies ω_1, ω_2

$$\langle f\rangle = \Phi(x,\ \omega_1,\ \omega_2) = \lim_{T\to\infty}\frac{1}{T}\int_0^T f(x,\ \omega_1 t,\ \omega_2 t)\mathrm{d}t.$$

The function $\Phi(x,\ \omega_1,\ \omega_2)$ is assumed to be continuous uniformly in all variables except for a set $(\omega_1,\ \omega_2)$ of the solutions of the equation $m_1\omega_1 + m_2\omega_2 = 0$, where m_1 and m_2 are some integers, $m_1^2 + m_2^2 \neq 0$. Define the function

$$\rho(x) = m_1\omega_1(x) + m_2\omega_2(x) = 0 \tag{3}$$

Formula (3) determines the resonance frequencies of system (2). Let Eq. (3) have a unique solution x^* such that

$$\rho(x^*) = 0,\quad \mathrm{d}\rho(x^*)/\mathrm{d}x = r \neq 0. \tag{4}$$

We presume that

1°. The right-hand sides of system (1) are 2π – periodic in θ_1, θ_2, and smooth enough in all variables;

2°. The perturbation $\xi(t)$ is uniformly bounded, $|\xi(t)| \le \xi_0$, $-\infty \le t \le \infty$.

3°. In the neighbourhood of $x = x^*$ the perturbation $\xi(t)$ does not yield a new resonance relationship similar to (3).

It follows from Assumptions 1°-3° that in the neighbourhood of resonance the solution of system (1) exists and the requisite transformations remain valid for any admissible control $u \in U$.

Let Eqs. (3), (4) determine the stable resonance state of system (1). Perturbation may result in escaping from the near-resonance domain. The control task is to keep the frequencies in the near-resonance domain in the presence of perturbation. We shall describe this requirement as a control problem.

Following the standard approach [1], we introduce the variables v and φ characterizing the frequency and phase deviations in the near-resonance domain. As known [1], the frequency deviations in the near-resonance domain are of order $\mu = \varepsilon^{1/2}$. Thus we write

$$\mu v(x) = \rho(x) = m_1\omega_1(x) + m_2\omega_2(x), \quad \varphi = m_1\theta_1 + m_2\theta_2. \tag{5}$$

Conditions (4), (5) allows representing the variables in the near-resonance domain in the form

$$x = X(\mu v) = x^* + \mu x_1 + \mu^2, \;\; x_1 = r^{-1}v,$$

$$\theta_1 = \theta,\; \theta_2 = m_2^{-1}(\varphi - m_1\theta). \tag{6}$$

Inserting (5), (6) into system (1), we obtain the equations of motion in the near-resonance domain in the form

$$\dot v = \mu[\beta(\varphi) + b(\varphi, \theta) + \Delta^*(\varphi, \theta, \xi(t))] + \mu^{2n-1}F^*(\varphi, \theta)u +$$

$$\mu^2 R_1(v, \varphi, \theta, \xi(t), u, \mu),$$

$$\dot\varphi = \mu v + \mu^{2n}G^*(\varphi, \theta)u + \mu^2 R_2(v, \varphi, \theta, \xi(t), u, \mu)], \tag{7}$$

$$\dot\theta = \omega^* + \mu\,\kappa v + \mu^{2n}G_1^*(\varphi, \theta)u + \mu^2 R_3(v, \varphi, \theta, \xi(t), u, \mu)],$$

where the residual terms R_i $i = 1,2,3$, vanish as $\mu \to 0$, and

$$\theta = \theta_1,\; \omega^* = \omega_1(x^*),\; \kappa = \omega_{1x}(x^*),$$

$$\beta(\varphi) = \langle f^*(\varphi, \theta)\rangle,\; b(\varphi, \theta) = f^*(\varphi, \theta) - \beta(\varphi),$$

$$\Phi^*(\varphi, \theta) = r^{-1}\Phi(x^*, \theta, \theta_2(\varphi, \theta)). \tag{8}$$

Here $\langle f^*(\varphi, \theta)\rangle$ denotes the averaging in θ, Φ and Φ^* are the vectors with the components $(f, \Delta, F, G_{1,2})$ and $(f^*, \Delta^*, F^*, G_{1,2}^*)$, and

$$G^*(\varphi, \theta) = m_1G_1^*(\varphi, \theta) + m_2G_2^*(\varphi, \theta) \tag{9}$$

We now define the admissible domain of motion. Consider the slow subsystem of (3)

$$\dot v = \mu\beta(\varphi), \;\; \dot\varphi = \mu v. \tag{10}$$

Equations (10) describe motion of the pendulum-like system with energy $E = \mu H(\varphi, v)$, where

$$H(\varphi, v) = \Pi(\varphi) + v^2/2. \tag{11}$$

The *cos*-like potential $\Pi(\varphi)$ is defined by the equation $\Pi_\varphi(\varphi) = -\beta(\varphi)$. For clarity we take $\Pi(0) = 0$. For the typical pendulum-like model we have $\Pi(\varphi) = \Pi(-\varphi)$, $\beta(-\varphi) = -\beta(\varphi)$ [1]. Let $\varphi = 0$ and $\varphi = \pm\,\varphi^s$ be the solutions of the equation $\Pi_\varphi(\varphi) = -\beta(\varphi) = 0$ corresponding to the minimum and the symmetric maxima of the function $\Pi(\varphi)$, respectively. It follows from the minimum condition that $\Pi_{\varphi\varphi}(0) > 0$, $\beta_\varphi(0) < 0$. Definition (5) and inequality $\beta_\varphi(0) < 0$ imply that the point O: $\{\varphi = 0, v = 0\}$ is the stable steady-state solution of system (10) associated with the stable resonance in the unperturbed system.

The phase plane of the pendulum-like system (10) is divided into the domains of libration and rotation separated by the separatrix; the passage from libration to rotation is associated with failure of resonance [1]. The domain of libration Σ can thus be treated as the reference domain. The control task is not so much to minimize deviations from the reference point O but to prevent the perturbed system from leaving the domain Σ.

2.2. The control problems

We consider system (7) as the equations of the perturbed motion of the pendulum (10). Introduce a measure of deviations from the steady-state point O as

$$h = v^2/2 + k\varphi^2/2, \tag{12}$$

where $k > 0$ is a weight coefficient. Define the domain $\Sigma^h \in \operatorname{int}\Sigma$ such that $(\varphi, v) \in \Sigma^h \Rightarrow h \in [0, h^*)$. The control task is to keep the function $h(t)$ within the interval $[0, h^*)$ on the maximum time interval.

The associated locally optimal control problem is reduced to minimization of the derivative

$$J(u) = \dot h(t) \tag{13}$$

at each moment t [5]. The control constraint is taken in the form $|u| \le U_0$. The locally optimal control is thus defined as

$$u_{\rm opt} = \arg\min_{|u|\le U_0} J(u). \tag{14}$$

The physical meaning of this approach is quite obvious. If $\dot h > 0$, the function h increases, and the control task is to minimize the velocity of the outward motion from the core of the domain Σ^h. If $\dot h < 0$, the function h decreases, and the control task is to maintain motion toward the core of the domain Σ^h.

Calculating (13) by virtue of Eqs (7), we find

$$\dot h = v\dot v + k\varphi\dot\varphi = [\mu^{2n-1}F^*(\varphi, \theta)v + \mu^{2n}k\varphi G^*(\varphi, \theta)]u +$$

$$\mu R(\varphi, v, \theta, \xi(t), \mu, \mu u), \tag{15}$$

where R is the residual term insubstantial for the further analysis.

It follows from (14), (15) that

$$u_{\rm opt} = -U_0\,\mathrm{sgn}[\mu^{2n-1}F^*(\varphi, \theta)v + \mu^{2n}k\varphi G^*(\varphi, \theta) + O(\mu^2)]. \tag{16}$$

Introduction of the parameter μ allows constructing of a relatively simple near-optimal control u^* such that $u^* \to u_{\rm opt}$ as $\mu \to 0$. We consider near-optimal control for two models of the controlled systems.

2.2.1. The frequency control. Let $F(x, \theta_1, \theta_2) \neq 0$, $n = 1$. In this case the control term in the first equation of system (7) is of the leading order, the control terms in the other equations are negligibly small. This yields

$$u^* = -U_0 \operatorname{sgn} F^*(\varphi, \theta) \operatorname{sgn} v \tag{17}$$

as $\mu \to 0$. Near-optimality of control (17) can be proved in the standard way.

Under the assumptions accepted in this item, control u counteracts the frequency deviations v but it is negligible in the last equations of system (7). This allows exclusion of the phase dependence from the cost criterion. Introduce the function

$$h^v = v^2/2 \tag{18}$$

as a measure of the frequency deviations from resonance. Let control u^v minimize the derivative $\dot{h}^v(t)$, that is

$$J^v(u) = \dot{h}^v(t),\ u^v = \arg\min_{|u|\le U_0} J^v(u). \tag{19}$$

In the very same way as before we find

$$u^v = u^* = -U_0 \operatorname{sgn} F^*(\varphi, \theta) \operatorname{sgn} v. \tag{20}$$

Equality (20) implies that criterion (18) may be used instead of (13). The associated control laws (17) or (20) can thus be interpreted as the frequency control.

2.2.2. The phase control. If $F(x, \theta_1, \theta_2) = 0$, we take $n = ½$. In this case control u is not involved in the main terms of the first equation in (7) but becomes substantial in the second equation.

It follows from conditions (14), (15) that

$$u^* = -U_0 \operatorname{sgn} G^*(\varphi,\theta) \operatorname{sgn} \varphi \tag{21}$$

as $\mu \to 0$. This implies that control (21) directly counteracts the phase deviations. Now we introduce the function

$$h^\varphi = \varphi^2/2 \tag{22}$$

as a measure of the phase deviation and find control u^φ minimizing the derivative $\dot{h}^\varphi(t)$. Calculating $\dot{h}^\varphi(t)$ by virtue of Eqs (7) ($n = ½$) and omitting the higher order terms, we obtain $\dot{h}^\varphi = \dot{\varphi}\,\varphi = \mu[v + G^*(\varphi, \theta)u]\varphi$, as $\mu \to 0$, and, therefore,

$$u^\varphi = u^* = -U_0 \operatorname{sgn} G^*(\varphi, \theta) \operatorname{sgn} \varphi. \tag{23}$$

Hence, in case $F = 0$ the phase criterion (22) may be used instead of (13). The associated control law (23) can be interpreted as the phase controls.

2.3. *Locally optimal control as the solution of the maximum residence time problem*

We demonstrate that locally optimal control (14) corresponds to the solution of the maximum residence time problem. Let (φ^u, v^u) be an orbit of system (7) governed by control u, h^u be function (12) calculated along the orbit (φ^u, v^u) starting at the point O at $t = 0$. Let T^u be the first moment the function h^u reaches the upper admissible value h^*. In case $u = u_{opt}$, we denote $h^u = h^0$, $T^u = T^0$. We show that

$$T^0 = \sup_{|u|\le U_0} T^u$$

To this end, we write

$$h^* = h^u(T^u) = \int_0^{T^u} \dot{h}^u(t)\,dt\,.$$

By definitions (13), (14), $h^0(T^u) < h^u(T^u) = h^*$ for any T^u. This means that the locally optimal system does not reach the boundary of the admissible domain by the moment T^u, and that is $T^0 = \sup_{|u|\le U_0} T^u$.

III. The frequency control of coupled oscillators

Equalities (20) and (23) imply that criteria (18) or (21) can replace the general criterion (13). We illustrate this approach by an example.

Consider a linear resonance circuit weakly connected with a nonlinear system. The linear circuit enhances a weak periodic signal of frequency Ω; then the transformed enhanced signal is fed to the input of the nonlinear system. In the absence of perturbations the nonlinear system generates oscillations of a resonance frequency correlated to Ω. The control task is to sustain the resonance mode of nonlinear oscillations in the presence of perturbation.

Next we investigate different control strategies.

1. Let the equations of motion have the form

$$\ddot{\psi} + \varepsilon b\dot{\psi} + \Omega^2\psi + \varepsilon\delta_1(\psi, \xi_1(t)) = \varepsilon a \sin \Omega t + \varepsilon s(x, \dot{x}, \psi, \dot{\psi}),$$

$$\ddot{x} + \varepsilon n\dot{x} + \phi(x) + \varepsilon\delta_2(x, \xi_2(t)) = \varepsilon q(x, \dot{x}, \psi, \dot{\psi}) + \varepsilon u\,. \tag{24}$$

Here $\phi(x) = d\Pi(x)/dx$, $\Pi(x)$ is the potential of the conservative counterpart of the nonlinear system. The perturbations $\xi_{1,2}(t)$ satisfy the assumptions of Section 1. The terms $q(\psi, \dot{\psi})$ and $s(x, \dot{x})$ describe the interaction of the subsystems. Control u is designed by the criteria of Section II.

Using the standard transformations [1], we reduce (24) to the standard form (1). Following [1], we introduce the change of variables $\psi, \dot{\psi} \to R, \theta_1$, and $x, \dot{x} \to y, \theta_2$. The slow variables R and y are defined as the amplitude of oscillations of the linear subsystem and the partial energy of the nonlinear subsystem, respectively, that is

$$R^2 = (\Omega^2\psi^2 + \dot{\psi}^2), \quad y = \tfrac{1}{2}\dot{x}^2 + \Pi(x), \tag{25}$$

θ_1 and θ_2 are the associated fast phases with the corresponding angular frequencies Ω and $\omega(y)$. Substituting the obtained functions $\psi(R,\theta_1)$, $x(y,\theta_2)$, etc., in Eqs (24) and reproducing the transformations of [1], we obtain the system

$$\dot{R} = -\frac{\varepsilon}{\Omega}[\Psi(R, y, \theta_1, \theta_2, \theta_3) + \Delta_1(R, \theta_1, \xi_1(t))]\sin\theta_1,$$

$$\dot{y} = \varepsilon\{f(y,\theta_2) + [Q(R,y,\theta_1,\theta_2)+ u]\,\dot{x}(y,\theta_2) + \Delta_2(y,\theta_2, \xi_2(t))\},$$

$$\dot{\theta}_1 = \Omega - \frac{\varepsilon}{\Omega R}[\Psi(R, y,\theta_1, \theta_2, \theta_3) + \Delta_1(R,\theta_1, \xi_1(t))]\cos\theta_1,$$

$$\dot{\theta}_2 = \omega(y) + \varepsilon\frac{\partial\omega}{\partial y}\{f(y, \theta_2) + [Q(R, y, \theta_1, \theta_2) + u]\,\dot{x}(y,\theta_2) +$$

$$\Delta_2(y, \theta_2, \xi_2(t))\},$$

$$\dot{\theta}_3 = \Omega, \qquad (26)$$

As shown in Section II, the form of the coefficients, independent of control u, is insubstantial. It is useful to mention that, despite the complicated structure of system (26), the resulting control strategy is independent of the transformations performed and has a simple physical meaning.

We investigate the main resonance, at which

$$\rho(y^*) = \omega(y^*) - \Omega = 0,\ \mathrm{d}\rho(y^*)/\mathrm{d}y = \mathrm{d}\omega(y^*)/\mathrm{d}y = r \neq 0. \qquad (27)$$

As in (5), we introduce the variables

$$\varphi = \theta_2 - \theta_3,\ \varphi_1 = \theta_1 - \theta_3,\ \theta_3 = \theta,$$

$$\mu v = \rho(y) = \omega(y) - \Omega,\ \ \mu = \varepsilon^{1/2}. \qquad (28)$$

Substituting (27), (28) into (26) and ignoring the insubstantial higher-order terms, we obtain the system

$$\dot{v} = \mu[F^*(\varphi, \theta)u + V(R, v, \varphi, \theta, \xi_2(t))],\ \ \dot{\varphi} = \mu v,$$

$$\dot{R} = \mu^2 P_1(v, R, \varphi, \varphi_1, \theta, \xi_1(t)),$$

$$\dot{\varphi}_1 = \mu^2 P_2(v, R, \varphi, \varphi_1, \theta, \xi_1(t)), \qquad (29)$$

$$\dot{\theta} = \Omega,$$

where

$$F^*(\varphi, \theta) = r^{-1}\,\dot{x}(y^*, \theta + \varphi), \qquad (30)$$

the other coefficients are unimportant.

System (29) does not allow separation of a conservative subsystem similar to (10). However, as seen from (29), control u affects directly the frequency deviation v. Hence, control can be chosen by criterion (19). By the same arguments as above, we obtain the frequency control

$$u^v = -U_0\,\mathrm{sgn}\,F^*(\varphi, \theta)\,\mathrm{sgn}\,v \qquad (31)$$

coinciding with (20). Using the representation (30) we obtain the associated feedback control in the form

$$u^v = -U_0\mathrm{sgn}(r^{-1}\dot{x})\mathrm{sgn}[\omega(y) - \Omega]. \qquad (32)$$

The only parameter requisite for the control design is $\mathrm{sgn}r = \mathrm{sgn}\omega_y(y^*)$. This parameter can be found without calculating the frequency $\omega(y)$, namely, $r > 0$ if the system is "hard", and $r < 0$ if the system is "soft" in the neighbourhood of the point y^*. The physical meaning of solution (31) is obvious. Let $r > 0$, that is an increase of the nonlinear subsystem energy y entails an increase of the frequency ω. Let $\omega(y) > \Omega$ at some moment t. In this case control (31) takes the form $u^v = -U_0\mathrm{sgn}(\dot{x})$. Control of this type slows down the motion, diminishes the system energy and, as a result, diminishes the frequency $\omega(y)$. If $\omega(y) < \Omega$ at some moment t, control $u^v = U_0\mathrm{sgn}(\dot{x})$ acts in the opposite direction.

2. Let control u acts upon the excitation frequency. The equations of the controlled motion take the form

$$\ddot{\psi} + \varepsilon b\dot{\psi} + \Omega^2\psi + \varepsilon\delta_1(x,\xi_1(t)) = \varepsilon a\sin\theta_3 + \varepsilon s(x,\dot{x},\psi,\dot{\psi}),$$

$$\ddot{x} + \varepsilon n\dot{x} + \phi(x) + \varepsilon\delta_2(x,\xi_2(t)) = \varepsilon q(x,\dot{x},\psi,\dot{\psi}), \qquad (33)$$

$$\dot{\theta}_3 = \Omega + \varepsilon^{1/2}u.$$

The right-hand sides of Eqs (33) use the same notation as Eq. (24). In the same way as above we obtain the equations of motion in the near-resonance domain

$$\dot{v} = \mu V,\quad \dot{\varphi} = \mu v - \mu u,$$

$$\dot{R} = \mu^2 P_1,\ \dot{\varphi}_1 = -\mu u + \mu^2 P_2, \qquad (34)$$

$$\dot{\theta} = \Omega + \mu u.$$

where $P_{1,2}$ and V are defined as in system (29). Control u is involved in three equations of system (33). In case the task is to sustain the resonance oscillations of the nonlinear subsystem regardless the dynamics of the linear circuit, we can consider the subsystem for the variables (φ, v) and construct a control minimizing criterion (22). Arguing as above, using formula (21), and considering $G^* = -1$, we obtain

$$u^\varphi = U_0\mathrm{sgn}\varphi. \qquad (35)$$

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References

[1] V.I. Arnold, V. V. Kozlov, and A.I. Neistadt, *Mathematical Aspects of Classical and Celestial Mechanics.* Springer, Berlin, 1987

[2] A.S. Kovaleva, Asymptotic solution of the optimal control problem for nonlinear oscillations in the neighbourhood of resonance. *Appl. Math. Mechs.,* 1999, **62**, 6, 913-922.

[3] A.S. Kovaleva, Control of resonance oscillations in stochastic systems, in: G. Deodatis, ed. *Computational Stochastic Mechanics.* Rotterdam: Millpress, 2003. P. 335-341.

[4] A. S. Kovaleva, Near-resonance frequency control in the presence of random perturbations of parameters. *Appl. Math. Mechs.,* 2004, **68**, 2, 294-305.

[5] N.N. Moiseev, *Numerical Methods in the Theory of Optimal Systems.* Nauka, Moscow, 1971.