PATTERNS OF CHAOS SYNCHRONIZATION

Wolfgang Kinzel
Institute for Theoretical Physics
University of Würzburg
Germany
kinzel@physik.uni-wuerzburg.de

Ido Kanter
Department of Physics
Bar-Ilan University
Israel

Johannes Kestler
Institute for Theoretical Physics
University of Würzburg
Germany

Evi Kopelowitz
Department of Physics
Bar-Ilan University
Israel

Abstract

We investigate small networks of chaotic units which are coupled by their time-delayed variables. In spite of the time delay, the units can synchronize isochronally, i.e. without time shift. Moreover, networks can not only synchronize completely, but can also split into different synchronized sublattices. These synchronization patterns are stable attractors of the network dynamics. In this contribution we present different networks with their associated behaviors and synchronization patterns.

Key words

synchronization, patterns, chaos, sublattice

1 Introduction

Chaos synchronization is a counter-intuitive phenomenon. On one hand, a chaotic system is unpredictable. Two chaotic systems, starting from almost identical initial states, end in completely different trajectories. On the other hand, two identical chaotic units which are coupled to each other can synchronize to a common chaotic trajectory. The system is still chaotic, but after a transient the two chaotic trajectories are locked to each other [Pikovsky *et al.*, 2001; Schuster and Just, 2005]. This phenomenon has attracted a lot of research activities, partly because chaos synchronization has the potential to be applied for novel secure communication systems [Pecora and Carroll, 1990; Cuomo and Oppenheim, 1993]. In fact, synchronization and bit exchange with chaotic semiconductor networks has recently been demonstrated over a distance of 120 km in a public fiber-optic communication network [Argyris *et al.*, 2005]. In this case, the coupling between the chaotic lasers was uni-directional, the sender was driving the receiver. For bi-directional couplings, when two chaotic units are interacting, additional interesting applications have been suggested. In this case, secret information may be transmitted over a public channel. Although the algorithm as well as all the parameters are public, any attacker is not able to decipher the secret message [Klein *et al.*, 2006*a*, 2005].

Typically, the coupling between chaotic units has a time delay due to the transmission of the exchanged signal. Nevertheless, chaotic units can synchronize without time shift, isochronically, although the delay time may be extremely long compared to the time scales of the chaotic units. This – again counter-intuitive – phenomenon has recently been demonstrated with chaotic semiconductor lasers [Klein *et al.*, 2006*b*; Fischer *et al.*, 2006; Sivaprakasam *et al.*, 2003; Lee *et al.*, 2006], and it is discussed in the context of corresponding measurements on correlated neural activity [Cho, 2006; Engel *et al.*, 1991; Campbell and Wang, 1998].

Several chaotic units may be coupled to a network with delayed interactions. Such a network can synchronize completely to a single chaotic trajectory, or it may end in a state of several clusters, depending on the topology of the network or the distribution of delay times [Atay *et al.*, 2004; Matskiv *et al.*, 2004; Masoller and Martí, 2005; Topaj *et al.*, 2001]. Recently another phenomenon has been reported for chaotic networks: Sublattice synchronization. If a small network can be decomposed into two sublattices, then the units in each sublattice can synchronize to a common chaotic trajectory although they are not directly coupled. The coupling of one sublattice is relayed by the chaotic trajectory of a different sublattice. The trajectories of different sublattices are only weakly correlated, but not synchronized [Kestler *et al.*, 2007].

In this talk we want to investigate patterns of chaos synchronization for several lattices with uni- and bi-directional couplings with time delay. There exists a mathematical theory to classify possible solutions

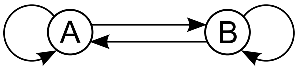


Figure 1. Two mutually coupled units.

of nonlinear differential or difference equations for a given lattice [Golubitsky and Stewart, 2006]. However, this theory does not determine the stability of these solutions. But in order to describe physical or biological dynamic networks, we are interested in stable patterns of chaotic networks. The patterns which are discussed in this presentation are attractors in phase space, any perturbation of the system will relax to these patterns which move chaotically on some high dimensional synchronization manifold.

Our results are demonstrated for iterated maps, for the sake of simplicity and since we can calculate the stability of these networks analytically. But we found these patterns for other systems, as well, for example for the Lang-Kobayashi rate equations describing semiconductor lasers. Hence we think that our results are generic.

2 Two interacting units

We start with the simplest network: Two units with delayed couplings and delayed self-feedback, as sketched in Fig. 1.

For iterated maps, this network is described by the following equations:

$$\begin{aligned} a_t &= (1-\varepsilon)f(a_{t-1}) + \varepsilon\kappa f(a_{t-\tau}) + \varepsilon(1-\kappa)f(b_{t-\tau}) \\ b_t &= (1-\varepsilon)f(b_{t-1}) + \varepsilon\kappa f(b_{t-\tau}) + \varepsilon(1-\kappa)f(a_{t-\tau}) \end{aligned} \tag{1}$$

where $f(x)$ is some chaotic map, for example the Bernoulli shift,

$$f(x) = \alpha\, x \mod 1 \tag{2}$$

with $\alpha > 1$. In this case, the system is chaotic for all parameters $0 < \varepsilon < 1$ and $0 < \kappa < 1$. ε measures the total strength of the delay terms and κ the strength of the self-feedback relative to the delayed coupling.

Obviously, the synchronized chaotic trajectory $a_t = b_t$ is a solution of Eq. (1). Its stability is determined by τ conditional Lyapunov exponents which describe perturbations perpendicular to the synchronization manifold. For the Bernoulli map, these Lyapunov exponents have been calculated analytically [Lepri *et al.*, 1993; Kestler *et al.*, 2007], and for infinitely long delay, $\tau \to \infty$, one obtains the phase diagram of Fig. 2.

In region I and II the two units are synchronized to an identical chaotic trajectory $a_t = b_t$. Although the two units are coupled with a long delay τ, they are completely synchronized without any time shift. For $\tau \to \infty$, this region is symmetric about the line $\kappa = \frac{1}{2}$.

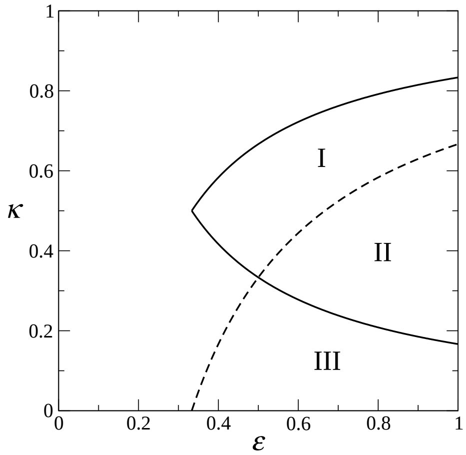


Figure 2. Phase diagram for $\alpha = 3/2$.

Complete synchronization can be understood by considering a single unit driven by some signal s_t:

$$a_t = (1-\varepsilon)f(a_{t-1}) + \varepsilon\kappa f(a_{t-\tau}) + s_t\,. \tag{3}$$

If the system is not chaotic, i.e. if its Lyapunov exponent is negative, then the trajectory a_t relaxes to a unique trajectory determined by the drive s_t. For the Bernoulli shift, this region is indicated by II + III in Fig. 2.

Now let us rewrite Eq. (1):

$$\begin{aligned} a_t = (1-\varepsilon)f(a_{t-1}) &+ \varepsilon(2\kappa-1)f(a_{t-\tau}) + \\ &+ \varepsilon(1-\kappa)f(a_{t-\tau}) + \varepsilon(1-\kappa)f(b_{t-\tau})\,, \\ b_t = (1-\varepsilon)f(b_{t-1}) &+ \varepsilon(2\kappa-1)f(b_{t-\tau}) + \\ &+ \varepsilon(1-\kappa)f(b_{t-\tau}) + \varepsilon(1-\kappa)f(a_{t-\tau}). \end{aligned} \tag{4}$$

Both systems are driven by the identical signal

$$s_t = \varepsilon(1-\kappa)\big[f(a_{t-\tau}) + f(b_{t-\tau})\big]. \tag{5}$$

Hence, for

$$\tilde{\kappa} = 2\kappa - 1 \tag{6}$$

the system is described by Eq. (3). The phase boundary of the driven system, region II + III, and the phase boundary of the interacting system, region I + II, are connected with each other: With the mapping of Eq. (6), one phase boundary can be obtained from the other. This mapping does not only hold for the Bernoulli shift but for any chaotic system, provided that the signal does not change the Lyapunov exponent of the driven system. For example, we found this relation for the Lang-Kobayashi laser equations.

Let us assume that we record the synchronized trajectory $a_t = b_t$ of two interacting chaotic units. Now

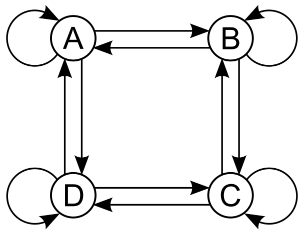


Figure 3. Ring of 4 units.

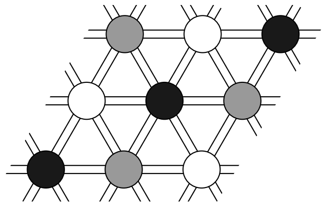


Figure 4. Sublattice synchronization in a triangular lattice with periodic boundaries. The double lines signify bi-directional couplings. The self-feedback is not drawn to simplify the illustration.

let us insert the recorded trajectory b_t into Eq. (1). How will a_t respond to this drive? We find that in regions II and III the unit A will synchronize completely to the recorded trajectory b_t, whereas in region I the unit A does not synchronize. Although the two interacting units A and B do synchronize, the unit A does not follow the recorded trajectory in region I. This shows that bi-directional interaction is different from uni-directional drive.

3 Sublattice synchronization

The response of a single chaotic unit to an external drive, Fig. 2, determines also the phase diagram of a ring of four chaotic units. Additionally, it shows a new phenomenon: sublattice synchronization. Consider the ring of four identical units of Fig. 3.

Obviously, the two units A and C receive identical input from the units B and D. Consequently, they will respond with an identical trajectory in the regions II and III of Fig. 2, since for those parameters the units have negative Lyapunov exponents. The same argument holds for the two units B and D. That leads to sublattice synchronization in region III of Fig. 2: A and C have an identical chaotic trajectory and B and D have a different one. Although there is a delay of arbitrary long time of the transmitted signal, synchronization is complete, without any time shift. The synchronization of A and C is mediated by the chaotic trajectory of B and D. But the two trajectories have only weak correlations, they are not synchronized. Numerical calculations of the Bernoulli system with small values of τ show that there is no generalized synchronization, either.

Sublattice synchronization has been shown for other lattices, as well. For example, the lattice of Fig. 4 can be decomposed into three sublattices. For some parameters of the Bernoulli system we find sublattice synchronization with three chaotic trajectories. Again, the synchronized units are not directly coupled, but they are indirectly connected via the trajectories of the other sublattices.

4 Spreading chaotic motifs

The response of a chaotic unit to an external drive, Fig. 2, points to another interesting phenomenon. Consider a triangle of chaotic units with bi-directional couplings as sketched in Fig. 5(a).

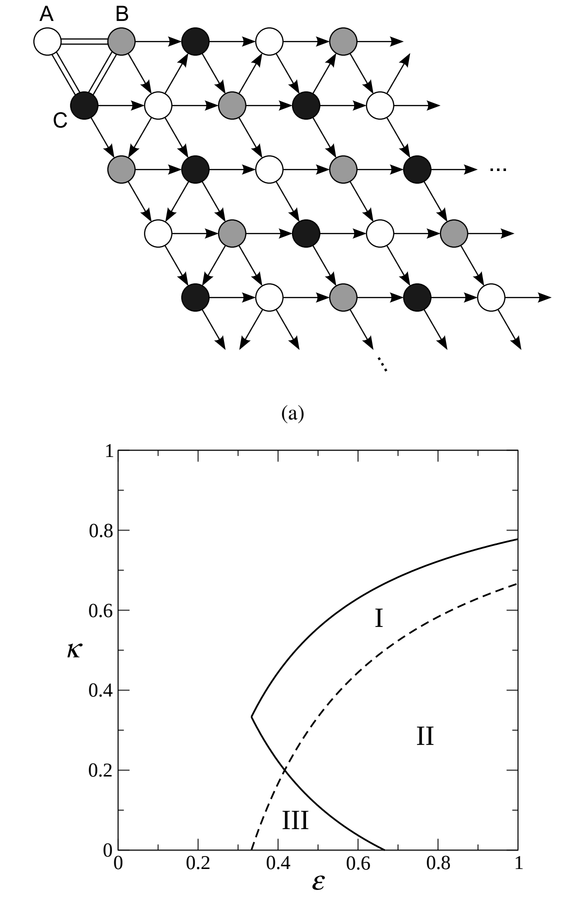
Choose the parameters such that the triangle is completely disordered, but each unit has negative Lyapunov exponents when it is separated from the two others. (Both conditions are fulfilled in region III of Fig. 5(b), which shows analytical results for the Bernoulli system.) When we record the three time series a_t, b_t and c_t we find three different weakly correlated chaotic trajectories. Now feed the two trajectories b_t and c_t into an infinitely large lattice of identical units with uni-directional couplings as shown in Fig. 5(a). Each unit receives two input signals from two other units. But since all Lyapunov exponents are negative, the system responds with the three chaotic trajectories a_t, b_t and c_t. Although the units of the initial triangle are not synchronized, their pattern of chaos is transmitted to the infinite lattice, after some transient time. All units of the same sublattice are completely synchronized without time shift, although the coupling has a long delay time τ.

For some parameters κ and ε, namely in regions I and II of Fig. 5(b), the three units of the triangle are completely synchronized. In region I, only the the three units are synchronized while the other units remain unsynchronized. In region II, the other units, too, get synchronized to the triangle (because all Lyapunov exponents are negative), so the whole lattice is completely synchronized.

5 Synchronization by restoring symmetry

In general we expect that the larger the network is, the smaller the region in the parameter space is where the network synchronizes. For example, a ring of $N = 6$ units has a smaller region of synchronization than the region II and III of Fig. 2 for $N = 4$. With increasing N synchronization finally disappears completely. However, we found a counterexample where adding a unit restores synchronization. Consider the chain of 5 units shown in Fig. 6.

The coupling to the two outer units has a longer delay time than the internal couplings. There is no self-feedback, $\kappa = 0$. Now remove unit E and rescale the coupling to unit D. In this case, a synchronized solution does not exist. Numerical simulations of the Bernoulli system and the laser equations show high correlations between units A and C with time shift $\Delta = \tau_1 - \tau_2$, and between B and D with zero time shift, but the correla-



(a)

(b)

Figure 5. Triangle (three bi-directionally coupled units) with a uni-directionally attached infinitely large lattice. (a) Double lines signify bi-directional couplings whereas arrows show uni-directional couplings. The self-feedback is not drawn to simplify the illustration. The colors indicate the synchronization pattern (sublattice synchronization) of region III. (b) Phase diagram for Bernoulli system, $\alpha = 3/2$, analytical result.

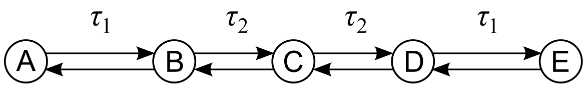


Figure 6. Chain of five units with two different time delays and without self-feedback.

tion coefficient does not achieve the value one. On the other side, if we add unit E we restore the symmetry of the chain. In this case we find sublattice synchronization with time shift between the outer units and the central one:

$$a_t = e_t = c_{t-\Delta}; \quad b_t = d_t \tag{7}$$

If τ_1 is greater than τ_2, the central unit is earlier than the chaotic trajectory of the outer ones, it leads, whereas for the opposite case it lags behind.

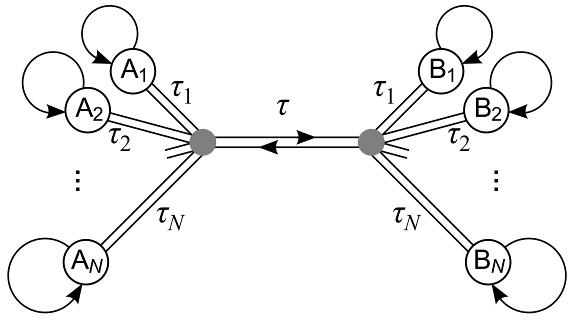


Figure 7. Each unit on one side is coupled to all units of the other side. The delay times are pairwise identical.

6 Cooperative pairwise synchronization

Is it possible to synchronize two sets of chaotic units with a single coupling channel? In fact, we found an example where two sets of chaotic units are bi-directionally connected by the sum of their units, as indicated in Fig. 7.

All units are identical, but the delay times of their couplings are different. The units have pairwise identical delay times, i.e. A_k and B_k have a coupling delay time $2\tau_k + \tau$ which is enforced by a self-feedback with delay time $\tau_s = 2\tau_k + \tau$. Hence, for one pair, $N = 1$, we obtain the phase diagram of Fig. 2, where the two units are completely synchronized in regions I and II. For a large number N of units, each A unit receives the signal

$$s_t = \varepsilon(1-\kappa)\frac{1}{N}\sum_{k=1}^{N} f(b_{t-(2\tau_k+\tau)}) \tag{8}$$

and vice versa. Hence the unit A_k receives only a weak signal of the order $1/N$ from its counterpart B_k. Nevertheless, we find that the network synchronizes to a state of pairwise identical chaotic trajectories, $a_{k,t} = b_{k,t}; k = 1, \ldots, N$. For the Bernoulli system, the region of pairwise synchronization is similar to region II of Fig. 2. There is no synchronization among units of the same side. Each unit receives the sum of all chaotic trajectories, but it responds only to the tiny part which belongs to its counterpart. The synchronization is a cooperative effect. As soon as a single unit is detuned, the whole network loses synchronization.

7 Summary

Small networks of chaotic units with time-delayed couplings show interesting patterns of chaos synchrony. These patterns are stable attractors of the network dynamics.

Two interacting units with self-feedback can synchronize completely, without time shift, even if the delay time is extremely large.

Sublattice synchronization is found for lattices which can be decomposed into a few sublattices. Each sublattice is completely synchronized, but different sublattices are only weakly correlated. Synchronization is relayed by different chaotic trajectories.

Synchronization may depend on the symmetry of the network. When the symmetry of a disordered chain is restored, sublattice or complete synchronization is restored, too.

Finally, a bi-partite network, where the two parts are coupled by a single mutual signal, shows pairwise complete synchronization, whereas the units of each part do not synchronize. Each unit responds to the weak contribution of its partner in the other part of the network. Pairwise synchronization is a cooperative effect: Detuning a single unit destroys the complete synchronization of the whole network.

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