STRONG CONVERGENCE ON A STOCHASTIC CONTROLLED LOTKA-VOLTERRA 3-SPECIES MODEL WITH LÉVY JUMPS

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1 Abstract

In this paper we study two properties of the numerical solutions of a controlled stochastic Lotka-Volterra one-predator-two-prey model, namely the boundedness in the mean of the numerical solutions and the strong convergence of these solutions. We also establish and solve, by means of the Stochastic Maximum Principle, the corresponding optimal control problem in a population modeled by a Lotka-Volterra system with two types of stochastic environmental fluctuations: white noise and Lévy jumps. Our study shows, assuming standard linear growth and Lipschitz conditions on the drift and diffusion coefficients, that the boundedness of the numerical solutions and the strong convergence of the scheme are preserved in this stochastic model.

Key words

Stochastic Optimal Control, Pontryagin's maximum stochastic principle, Lotka-Volterra model, Euler-Maruyama Scheme, Jumping noise.

2 Introduction

The Lotka-Volterra equations represent an important and useful model of interaction between predator-prey populations and in this paper we consider a controlled stochastic approach to this model with three species, one-predator-two-prey. This model assumes that the prey population finds food all the time and that the predator's food is completely dependent on the prey population. We will assume that the environment in which the three species develop presents natural random variations, which plays an important role in any real ecosystem and can influence the dynamics of the system, [Arnold, Horsthemke, and Stucki, 1979]. We propose Wiener processes to modeled them, [Romero-Meléndez, Castillo-Fernández, and González-Santos, 2021]. In addition, we consider some sudden environmental changes or phenomena of an abrupt nature, such as climate change, storms, volcanic eruptions, accidental oil spills or radioactive catastrophes, for which we propose Lévy processes in our stochastic model.

In this work we will address the study of the numerical solutions of the model, given the impossibility of obtaining its analytical solutions, although it is possible to have existence and uniqueness of the solutions with the conditions at most linear growth over difussionn and Lévy jumps and Lipschitz continuity over deterministic, difussion and Lévy jumps terms. We are interested in the boundedness and the strong convergence of numerical solutions of that Optimal Control problem, assuming standard linear growth and Lipschitz conditions on the drift and diffusion terms, [Oksendal and Sulem, 2005], [Situ, 2005]. Convergence rate and strong convergence rate for numerical solutions of Stochastic Differential Equations with driven jumps was studied in [Bao, *et al.*, 2011], [Higham and Kloeden, 2007] and [Higham and Kloeden, 2005], respectively, but there are not control functions in the processes. A characteristic of our model is that it contains control functions in the deterministic terms and in the terms corresponding to the process of Lévy. We will establish an Optimal Control problem and using the Pontryagin's maximum stochastic principle, we obtain the optimal controls in terms of Lévy process and the state and adjoint variables.

Lévy processes are stochastic processes with stationary and independent increments, like sub-martingales or Markov processes, that is, they are processes $Z(t)$ such that $Z(t+s) - Z(s)$ and $Z(r)$ are independent distributions with the same probability, for $s, t \geq 0$ and

$0 \leq r \leq t$. For the sake of versatility, we consider the Lévy jumps in our model to be driven by a random Poisson measure $N(t, \Omega)$, with characteristic measure $\nu(\Omega)$.

Our model consists in a non-linear stochastic ordinary differential equations system.

The stochastic differential system is of the general type

$$dx = f(t, x(t), u(t))dt + g(t, x(t), u(t))dW(t) + x(t)u(t)\int_{\mathbb{R}^n} \gamma(t, x(t-), z)\tilde{N}(dt, dz), \quad (1)$$

where $f(t, x, u) = (f^1(t, x, u), \ldots, f^n(t, x, u))^\top$ is a measurable function defined for $(t, x, u) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^m$ and $\mathbb{R}^m$-valued, known as the drift, $u(t) = (u_1(t), \ldots, u_m(t))$, $u : \mathbb{R} \to \mathbb{R}^m$ is a measurable and bounded function called the control which belongs to a compact space U, $g(t, x, u) = (g^1(t, x, u), \ldots, g^m(t, x, u))$, with $g^j(t, x, u) = (g^{1j}(t, x, u), \ldots, g^{nj}(t, x, u))^\top, 1 \leq j \leq m$, is a measurable function defined also on $[0, T] \times \mathbb{R}^n \times \mathbb{R}^m$ and $\mathbb{R}^{n\times m}$-valued ($n \times m$ - real matrix), called the diffusion coefficient and for the compensated Poisson random measure $\tilde{\mathrm{N}}(dt, dz)$, we write, according to Lévy decomposition theorem [Oksendal and Sulem, 2005], $\tilde{\mathrm{N}}(dt, dz) = (\tilde{\mathrm{N}}_1(dt, dz), \ldots, \tilde{\mathrm{N}}_m(dt, dz))$ and $\tilde{\mathrm{N}}_j(dt, dz) = N_j(dt, dz) - \nu_j(dz_j)dt, \quad 1 \leq j \leq m$, with $N_j(dt, dz)$ Poisson counting measure. Denoting by $x_1(t)$, $x_2(t)$ and $x_3(t)$ the differentiable functions meaning the density of the population of two preys and predator, respectively, our model is given by

$$\begin{aligned}
dx_1(t) &= (\eta x_1(t) - \beta x_1(t)x_2(t)) - \delta x_1(t)x_3(t) \\
&\quad -Ax_1(t)u_1(t))dt + \alpha_1\, dW_1(t) \\
&\quad +x_1(t)u_1(t)\int_{\mathbb{R}^n} \gamma(t, x_1(t-), z)N(dt, dz) \\
dx_2(t) &= (\omega x_2(t) - \beta x_2(t)x_1(t) - \epsilon x_2(t)x_3(t) \\
&\quad -Bx_2(t)u_2(t))dt + \alpha_2\, dW_2(t) \\
&\quad +x_2(t)u_2(t)\int_{\mathbb{R}^n} \gamma(t, x_2(t-), z)N(dt, dz) \\
dx_3(t) &= (-\kappa x_3(t) + \delta x_3(t)x_1(t) + \epsilon x_3(t)x_2(t) \\
&\quad -Cx_3(t)u_3(t))dt + \alpha_3\, dW_3(t) + \\
&\quad x_3(t)u_3(t)\int_{\mathbb{R}^n} \gamma(t, x_3(t-), z)N(dt, dz) \quad (2)
\end{aligned}$$

where $x_i(t-)$ denotes the left hand limit of x at time t, and initial and final conditions:

$$\begin{aligned}
x_1(0) = x_{10}, \quad x_2(0) = x_{20}, \quad x_3(0) = x_{30} \\
x_1(T) = x_{11}, \quad x_2(T) = x_{21}, \quad x_3(T) = x_{31} \quad (3)
\end{aligned}$$

where η, ω, κ are positive constants in $(0, 1]$, being the intrinsic growth rate of two preys and predator population, respectively, β, δ, η and ϵ in $(0, 1]$, are positive constants, meaning the contact rates per unit of time between prey-prey, predator-first prey and predator-second prey, respectively, $u_1(t), u_2(t), u_3(t)$ are the controls, $W_1(t), W_2(t), W_3(t)$ are standard independents Wiener processes, $N(t)$ is a Poisson process independent of $B(t)$ and $\gamma : \mathbb{R}^3 \to \mathbb{R}^3$ the jump coefficient or Poisson's process coefficient. In this stochastic controlled model we introduce controls $u_1(t), u_2(t), u_3(t)$, representing, by example, the hunting in each population, for which we have modulated their effect with constants $A, B, C \in (0, 1]$ and we take into account environmental fluctuations on the preys and the predator populations with parameters $\alpha_1, \alpha_2, \alpha_3 \in (0, 1]$, respectively, in three independent random variations for each population, $W_1(t), W_2(t), W_3(t)$, given by standard Wiener process and defined over a probability space $(\Omega, \mathcal{F}, P)$. In the above, as is conventional, P denotes a probability measure in the sample space Ω of the stochastic process $X : [0, T] \times \Omega \to [0, +\infty)$ and $\mathrm{E}[X]$ denotes the expected value with respect to the probability measure P, that is, the integral $\mathrm{E}[X_T] = \int_\Omega X_T(\omega)\, \mathrm{d}P(\omega)$ in the sense of Lebesgue integration. $\mathcal{F}_s$ denotes the σ-algebra generated by all random variables X_i with $i \leq s$; the collection of such σ-algebras forms a filter of the probability space. The class of admissible controls $\mathcal{U}$ is the set of $\mathcal{F}_s$-predictable processes with values in U.

For the sake of simplicity we have selected $\eta = \omega = \kappa = 1$ and we have placed the equilibrium point of the system at $(1, 1)$. So, according to equation (1), we set,

$$f(t, x, u) = \begin{pmatrix} x_1(t) - \beta x_1(t)x_2(t) - \delta x_1(t)x_3(t) - Ax_1(t)u_1(t) \\ x_2(t) - \beta x_2(t)x_1(t) - \epsilon x_2(t)x_3(t) - Bx_2(t)u_2(t) \\ -x_3(t) - \delta x_3(t)x_1(t) + \epsilon x_3(t)x_2(t) - Cx_3(t)u_3(t) \end{pmatrix}$$

$$g(t, x, u) = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}.$$

and we have the following stochastic optimal control problem:

To find the controls $u_1(t)$, $u_2(t)$, $u_3(t)$ and the states $x_1(t), x_2(t), x_3(t)$ of the system (1) which minimize the following expected cost functional

$$J(u_1, u_2, u_3) = \mathrm{E}\left\{\frac{1}{2}\int_0^T \sum_{i=1}^{3} \Big(x_i^2(t) + u_i^2(t)\Big)\, dt\right\}. \quad (4)$$

To solve this optimal control problem, we use the Pontryagin Maximal Stochastic Principle [Oksendal and Sulem, 2005]. We note that

setting the forward differential stochastic equations (2) and the following backward differential stochastic equations, or terminal value problem:

$$
\begin{aligned}
dp(t) &= -\Big\{ f_x(t,x,u)^\top p(t) + \sum_{j=1}^{m} g_x^j(t,x(t),u(t))^\top q_j(t) \\
&\quad - (f_0(t,x(t),u(t)))_x \Big\} dt + q(t)\, dW(t) \\
&\quad + u(t)p(t) \int_{\mathbb{R}^n} \gamma(t,x(t-),z)N(dt,dz) \\
p(T) &= (p_{11}, p_{31}, p_{31})^\top. \qquad (5)
\end{aligned}
$$

where

$$
f_0(t,x,u) = \frac{1}{2}\sum_{i=1}^{3}\Big(x_i^2(t) + u_i^2(t)\Big).
$$

The adjoint variable p, the matrix q and the matrix of processes dW are represented by

$$
p(t) = \begin{pmatrix} p_1(t) \\ p_2(t) \\ p_3(t) \end{pmatrix},\ q(t) = \begin{pmatrix} q_{11}(t) & q_{12}(t) & q_{13}(t) \\ q_{21}(t) & q_{22}(t) & q_{23}(t) \\ q_{31}(t) & q_{32}(t) & q_{33}(t) \end{pmatrix},
$$

$$
dW(t) = \begin{pmatrix} dW_1(t) \\ dW_2(t) \\ dW_3(t) \end{pmatrix}.
$$

We define the following extended Hamiltonian [Yong and Zhou, 1999]:

$$
\begin{aligned}
&H(x(t),p(t),q(t),u(t)) = \\
&< p(t), f(t,x,u)^\top > + tr[q(t)g(t,x,u)^\top] - f_0(t,x,u) \\
&\quad + \sum_{i=1}^{3} x_i(t)p_i(t)u_i(t) \int_{\mathbb{R}^n} \gamma_i(t,x_i(t-),z)N(dt,dz)
\end{aligned}
$$

That is to say

$$
\begin{aligned}
H &= x_1(t)p_1(t) - \beta p_1(t)x_1(t)x_2(t) - \delta p_1(t)x_1(t)x_3(t) \\
&\quad - Ap_1(t)x_1(t)u_1(t) + p_2(t)x_2(t) - \beta p_2(t)x_2(t)x_1(t) \\
&\quad - \epsilon p_2(t)x_2(t)x_3(t) - Bx_2(t)p_2(t)u_2(t) - p_3(t)x_3(t) \\
&\quad + \delta p_3(t)x_3(t)x_1(t) + \epsilon p_3(t)x_3(t)x_2(t) \\
&\quad - Cp_3(t)x_3(t)u_3(t) \\
&\quad - \frac{1}{2}\textstyle\sum_{i=1}^{3}\Big(x_i^2(t) + u_i^2(t) - 2\alpha_i q_{ii}\Big) \\
&\quad + \textstyle\sum_{i=1}^{3} x_i(t)p_i(t)u_i(t)\int_{\mathbb{R}^n}\gamma_i(t,x_i(t-),z))N(dt,dz) \qquad (6)
\end{aligned}
$$

Then, the adjoint equations corresponding to the processes $p(t)$, are the followings:

$$
\begin{aligned}
dp_1(t) &= \Big(x_1(t) - p_1(t) + \beta p_1(t)x_2(t) + \delta p_1(t)x_3(t) \\
&\quad + Ap_1(t)u_1(t) + \beta p_2(t)x_2(t) - \delta p_3(t)x_3(t)\Big)dt \\
&\quad + \sum_{i=1}^{3} q_{1i}dW_1(t) \\
&\quad + u_1(t)p_1(t)\int_{\mathbb{R}^n}\gamma(t,x_1(t-),z)N(dt,dz) \\
dp_2(t) &= \Big(x_2(t) - p_2(t) + \beta p_2(t)x_1(t) + \epsilon p_2(t)x_3(t) \\
&\quad + Bp_2(t)u_2(t) + \beta p_1(t)x_1(t) - \epsilon p_3(t)x_3(t)\Big)dt \\
&\quad + \sum_{i=1}^{3} q_{2i}dW_2(t) \\
&\quad + u_2(t)p_2(t)\int_{\mathbb{R}^n}\gamma(t,x_2(t-),z)N(dt,dz) \\
dp_3(t) &= \Big(x_3(t) + p_3(t) - \delta p_3(t)x_1(t) - \epsilon p_3(t)x_2(t) \\
&\quad + Cp_3(t)u_3(t) + \delta p_1(t)x_1(t) + \epsilon p_2(t)x_2(t)\Big)dt \\
&\quad + \sum_{i=1}^{3} q_{3i}dW_3(t) \\
&\quad + u_3(t)p_3(t)\int_{\mathbb{R}^n}\gamma((t,x_3(t-),z)N(dt,dz) \qquad (7)
\end{aligned}
$$

And, according to the necessary conditions of Stochastic Maximum Principle,

$$
\frac{\partial H(x,p,q,u)}{\partial u_1} = 0, \frac{\partial H(x,p,q,u)}{\partial u_2} = 0, \quad \frac{\partial H(x,p,q,u)}{\partial u_3} = 0, \qquad (8)
$$

we find:

$$
\begin{aligned}
u_1(t) &= p_1(t)x_1(t)\Big(-A + \int_{\mathbb{R}^n}\gamma(t,x_1(t-),z)N(dt,dz)\Big) \\
u_2(t) &= p_2(t)x_2(t)\Big(-B + \int_{\mathbb{R}^n}\gamma(t,x_2(t-),z)N(dt,dz)\Big) \\
u_3(t) &= p_3(t)x_3(t)\Big(-C + \int_{\mathbb{R}^n}\gamma(t,x_3(t-),z)N(dt,dz)\Big) \qquad (9)
\end{aligned}
$$

We note that the following condition:

(H) $\forall\ (t,x,p,q) \in [0,t] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times m}$, $H(t,x,p,q,\cdot)$ achieves its minimum; i.e., there exists $u_0 \in U$ such that

$$
H(t,x,p,q) := \min_{u\in U} H(t,x,p,q,u) = H(t,x,p,q,u_0)
$$

guarantees the existence of the optimal control, see ([Chighoub and Mezerdi, 2013], [Poznyak, 2008], [Wan and Davis, 1979]). We have assumed that $u(t)$ has values in a given compact set $U \in \mathbb{R}^k$ and that $u(t)$ is predictable in such a way that condition (H) is satisfied. Systems (2), (7) and (9) can be solved by numerical methods, like the Euler-Maruyama scheme.

The Euler-Maruyama scheme, corresponding to equation (1), is the simplest effective computational method used in stochastic differential equations. The Euler-Maruyama approximation is a continuous time stochastic process x, obtained by truncating Itô's formula of the stochastic Taylor series after the first terms. From now on, to carry out the numerical calculations, we will consider (see [Higham and Kloeden, 2005])

$$\int_{\mathbb{R}^n} \gamma_i(t, x_i(t-), z) N(dt, dz) = h(t_i, x_i(t-), u_i) N(t),$$

for some function $h : \mathbb{R}^m \to \mathbb{R}^m$, $h \in \mathcal{C}^1$.

Let M be a positive integer and taking the time-step size as $\Delta = \dfrac{T}{M} \in (0, 1)$, we define, for $0 \leq t \leq T$,

$$k(t) = |\frac{t}{\Delta}|\Delta, \tag{10}$$

where $|a|$ is the integer part of a,

$$\Delta W_k = W(t_{k+1}) - W(t_k), \quad \Delta N_k = N(t_{k+1}) - N(t_k)$$

and

$$\begin{aligned} x_{k+1} &= x_k + f(t_k, x_k, u_k)\Delta t_{k+1} \\ &+ g(t_k, x_k, u_k)\Delta W_{k+1} \\ &+ h(t_k, x_k, u_k)\Delta N_{k+1} \end{aligned} \tag{11}$$

For $0 \leq t \leq T$, we define

$$\bar{x}_k(t) = \sum_{k=0}^{M-1} x(t_k) \mathbf{1}_{[t_k, t_{k+1}]}(t)$$

and

$$\begin{aligned} x_\Delta(t) = x_0 + \int_0^t f(s, \bar{x}, u)\, ds + \int_0^t g(s, \bar{x}, u)\, dW(s) \\ + \int_0^t h(s, \bar{x}, u)\, dN(s) \end{aligned} \tag{12}$$

The Euler-Maruyama scheme can be expressed in our case by the systems:

$$\begin{aligned}
\bar{x}_{k+1,1} &= \bar{x}_{k,1} + (\eta \bar{x}_{k+1,1} - \beta \bar{x}_{k+1,2} - \delta \bar{x}_{k+1,1} \bar{x}_{k+1,3} \\
&\quad - A \bar{x}_{k+1,1} u_{k+1,1})\Delta \\
&\quad + \alpha_1 \Delta W_{k+1,1} \\
&\quad + u_{k+1,1} h(t_k, x_{k+1,1}, u_{k+1,1}) \Delta N_{k+1,1}, \\
\bar{x}_{k+1,2} &= \bar{x}_{k,2} + (\omega \bar{x}_{k+1,2} - \beta \bar{x}_{k+1,2} - \epsilon \bar{x}_{k+1,2} \bar{x}_{k+1,3} \\
&\quad - B \bar{x}_{k+1,2} u_{k+1,3})\Delta \\
&\quad + \alpha_2 \Delta W_{k+1,2} \\
&\quad + u_{k+1,2} h(t_k, x_{k+1,2}, u_{k+1,2}) \Delta N_{k+1,2}, \\
\bar{x}_{k+1,3} &= \bar{x}_{k,3} + (-\kappa \bar{x}_{k+1,3} - \omega \bar{x}_{k+1,3} - \delta \bar{x}_{k+1,2} \bar{x}_{k+1,3} \\
&\quad - \epsilon \bar{x}_{k+1,3} u_{k+1,3})\Delta \\
&\quad + \alpha_3 \Delta W_{k+1,3} \\
&\quad + u_{k+1,3} h(t_k, x_{k+1,3}, u_{k+1,3}) \Delta N_{k+1,3}, \\
\bar{p}_{k+1,1} &= \bar{p}_{k,1} + (\bar{x}_{k+1,1} - \beta \bar{p}_{k+1,1} \bar{x}_{k+1,2} \\
&\quad + \delta \bar{p}_{k+1,1} \bar{x}_{k+1,3} + A \bar{p}_{k+1,1}(t) \bar{u}_{k+1,1} \\
&\quad + \beta \bar{p}_{k,+1,1} \bar{x}_{k+1,1} - \kappa \bar{p}_{k,+1,1} \bar{x}_{k+1,3})\Delta \\
&\quad + \sum_{i=1}^{3} q_{1i} \Delta W_{k+1,1} \\
&\quad + \bar{u}_{k+1,1} h(t_k, x_{k+1,1}, u_{k+1,1}) \Delta N_{k+1,1}, \\
\bar{p}_{k+1,2} &= \bar{p}_{k,2} + (\bar{x}_{k+1,2} - \bar{p}_{k+1,2} - \eta \bar{p}_{k+1,2} \bar{x}_{k+1,1} \\
&\quad + \epsilon \bar{p}_{k+1,2} \bar{x}_{k+1,3} \\
&\quad + B \bar{p}_{k+1,2}(t) \bar{u}_{k+1,2} + \beta \bar{p}_{k,+1,1} \bar{x}_{k+1,1} \\
&\quad - \kappa \bar{p}_{k,+1,3} \bar{x}_{k+1,3})\Delta \\
&\quad + \sum_{i=1}^{3} q_{2i} \Delta W_{k+1,2} \\
&\quad + \bar{u}_{k+1,2} h(t_k, x_{k+1,2}, u_{k+1,2}) \Delta N_{k+1,2}, \\
\bar{p}_{k+1,3} &= \bar{p}_{k,3} + (\bar{x}_{k+1,3} - \omega \bar{p}_{k+1,3} - \kappa \bar{p}_{k+1,3} \bar{x}_{k+1,2} \\
&\quad + C \bar{p}_{k+1,3}(t) \bar{u}_{k+1,3} + \gamma \bar{p}_{k+1,1} \bar{x}_{k+1,1} \\
&\quad + \epsilon \bar{p}_{k+1,2}(t) \bar{x}_{k+1,2})\Delta \\
&\quad + \sum_{i=1}^{3} q_{3i} \Delta W_{k+1,3} \\
&\quad + \bar{u}_{k+1,3} h(t_k, x_{k+1,3}, u_{k+1,3}) \Delta N_{k+1,3}
\end{aligned} \tag{13}$$

where

$$\begin{aligned}
\bar{u}_{k+1,1} &= -A \bar{p}_{k+1,1}(t) h(t_k, x_{k+1,1}, u_{k+1,1}) \Delta N_{k+1,1} \\
\bar{u}_{k+1,2} &= -B \bar{p}_{k+1,2}(t) h(t_k, x_{k+1,2}, u_{k+1,2}) \Delta N_{k+1,2} \\
\bar{u}_{k+1,3} &= -C \bar{p}_{k+1,3}(t) h(t_k, x_{k+1,3}, u_{k+1,3}) \Delta N_{k+1,3}
\end{aligned} \tag{14}$$

3 Strong convergence and boundedness

In this section we study properties of boundedness and convergence of the numerical solutions of system (2). Firstly, we will consider the strong convergence of numerical solutions of system (1) to the exact solution, for which we introduce the following Lipschitz assumption on the drift and diffusion coefficients (see [Platen, 1999]) of equation (1), linear growth assumption on the drift and boundedness of controls assumption:

(H1) The functions $f(t,x,u)$, $g(t,x,u)$ and $h(t,x,u)$ satisfy the following global Lipschitz condition: there exist positive constants C_1, C_2 and C_3, such that

$$||f(t,x,u) - f(t,y,u)|| \leq C_1||x-y||,$$
$$||g(t,x,u) - g(t,y,u)|| \leq C_2||x-y||,$$
$$||h(t,x,u) - h(t,y,u)|| \leq C_3||x-y||,$$

(H2) The functions $f(t,x,u), g(t,x,u)$ satisfy the following linear growth condition: there exist positive constant L, such that

$$\begin{aligned}||f(t,x,u) - f(t,y,u)|| + ||g(t,x,u) - g(t,y,u)|| \\ +||h(t,x,u) - h(t,y,u)|| \\ \leq L(1+||x||^r + ||y||^r)||x-y||\end{aligned}$$

(H3) There exists a constant C_4, such that, for $i = 1, 2, 3$ and $0 \leq t \leq T$,

$$||u_i(t)|| \leq C_4$$

Moreover, we establish the following inequality, which gives a limit on the probability that a stochastic process exceeds any given value during a given time interval $[0,T]$, see [Le Gall, 2013]:

Lemma 1 (Doob martingale inequality). *Let X_t be an real martingale and suppose that the stochastic process is cadlag. Then, for $\forall\, \lambda > 0$ constant and $\forall\, p \geq 1$,*

$$P\left[\sup_{0\leq t\leq T} X_t \geq \lambda\right] \leq \frac{\mathrm{E}[max(X_T^p, 0)]}{\lambda^p}. \qquad (15)$$

The next result says that the Euler-Maruyama numerical solutions of (2) converge strongly to the exact solution if f and g satisfy the local Lipschitz condition and the linear growth condition.

Theorem 1. *Let $x_\Delta(t)$, $\bar{x}_k(t)$ be the solution of equation (11) and the numerical solution of the Euler-Maruyama scheme (2) respectively, then, under (H1) and (H2) assumptions, there exist C positive constant, such that, for any $\Delta \in [0,1]$ and $0 \leq t \leq T$:*

$$E(||x_\Delta(t) - \bar{x}_k(t)||^2|\mathcal{F}_{k(t)}) \leq C(2\Delta)(1 + ||\bar{x}_k(t)||^2)$$

Proof. Assumptions (H1) and (H2) imply that

$$\begin{aligned}
&E(||x_\Delta(t) - \bar{x}_k(t)||^2|\mathcal{F}_{k(t)}) = \\
&E(||\int_{k/t)}^t f(s,\bar{x},u)ds + \int_{k/t)}^t g(s,\bar{x},u)dW(t) \\
&+ \int_{k/t)}^t h(s,\bar{x},u)dN(s)||^2|\mathcal{F}_{k(t)}) \\
&\leq E(||\int_{k/t)}^t ||f(s,\bar{x},u)ds||^2|\mathcal{F}_{k(t)}) \\
&+E(||\int_{k/t)}^t ||g(s,\bar{x},u)dW(s)||^2|\mathcal{F}_{k(t)}) \\
&+E(||\int_{k/t)}^t ||h(s,\bar{x},u)dN(s)|^2|\mathcal{F}_{k(t)}) \\
&\leq 2(\Delta + E(||\int_{k/t)}^t ||h(s,\bar{x},u)dN(s)|^2|\mathcal{F}_{k(t)})) \quad (16)
\end{aligned}$$

On the other hand, we claim that ther exist $C > 0$ such that

$$E||\Delta N_k||^2 \leq C\Delta \qquad (17)$$

Indeed, in virtue of Doob martingale inequality (15), there exist λ such that we have:

$$P\left[\sup_{0\leq t\leq T} |N(t+s) - N(s)| \geq \lambda\right] \leq \frac{\mathrm{E}[(|N(t+s) - N(s)|^2)]}{\lambda^2}. \qquad (18)$$

Furthermore, it is clear that

$$N(t+s) - N(s) \sim \text{Poisson}(\lambda t) \sim \text{Poisson}(kt)$$

whence

$$\mathrm{E}[|N(t+s) - N(s)|^2] = k^2 + k$$

Therefore,

$$P\left[\sup_{0\leq t\leq T} |N(t+s) - N(s)| \geq \lambda\right] \leq \frac{k^2+k}{\lambda^2}.$$

Now, denote by A_k the process

$$A_k = P\left[\sup_{0\leq t\leq T} |N(t+s) - N(s)| \geq \lambda\right]$$

then, A_k satisfies

$$\sum_{k=1}^{\infty} P(A_k) < \infty$$

and, from Borell-Cantelli lemma [Kushner and Yin, 2003], there exist a process N with zero probability such that $\forall w \notin N$: $\exists\, k_1(w)$ such that for $k \geq k_1(w)$:

$$\sup_{0\leq t\leq T} |N(t_{k+1}+s)(w) - N(t_k s)(w)| \leq M.$$

which proves (17). Finally, coming back to the inequality (16) we have:

$$\begin{aligned}
&E(||\int_{k/t)}^t ||h(s,\bar{x},u)dN(s)||^2|\mathcal{F}_{k(t)}||) \\
\leq\ &E(||h(s,\bar{x},u)dN(s)||^2|\mathcal{F}_{k(t)}) \\
=\ &||h(s,\bar{x},u)||^2 E(||\Delta N(s)||^2) \\
\leq\ &C\Delta(1 + ||\bar{x}_k(t)||^2),
\end{aligned}$$

Hence it follows that

$$E(||x_\Delta(t) - \bar{x}_k(t)||^2|\mathcal{F}_{k(t)}) \leq C(2\Delta)(1 + ||\bar{x}_k(t)||^2)$$

and, by application of the discrete Gronwall inequality, we obtain:

$$E(||x_\Delta(t) - \bar{x}_k(t)||^2|\mathcal{F}_{k(t)}) \leq C2\Delta e^{C2\Delta}$$

and, for $\Delta \to 0$, we obtain the strong convergence to the exact solution of equation (11).

Now, we will proof the boundedness of the numerical solution of equation (11).

Theorem 2. *Let $\bar{x}_k(t)$ be the numerical solution of the Euler-Maruyama scheme (12), then, under (H1), (H2) and (H3) assumptions, there exist C positive constant, such that, for $0 \leq t \leq T$:*

$$E(||\bar{x}_k(t)||^2 \leq C$$

Proof. By using the following inequality, for any real numbers d, e, f:

$$||d + e + f||^2 \leq 3(||d||^2 + ||e||^2 + ||f||^2)$$

and taking the expected value:

$$\begin{aligned}
&E||\bar{x}_{k+1,1}||^2 \\
&\leq 3(E||\bar{x}_{k,1} + (\eta\bar{x}_{k+1,1} - \beta\bar{x}_{k+1,2} - \delta\bar{x}_{k+1,1}\bar{x}_{k+1,3} \\
&-A\bar{x}_{k+1,1}u_{k+1,1})\Delta||^2 + E||\alpha_1\Delta W_{k+1,1}||^2 \\
&+E||u_{k+1,1}h(X_{k+1,1}, t_k)\Delta N_{k+1,1}||^2) \\
&\leq 3(2(E||\bar{x}_{k,1}||^2 + E||\eta\bar{x}_{k+1,1} - \beta\bar{x}_{k+1,2} \\
&-\delta\bar{x}_{k+1,1}\bar{x}_{k+1,3} - A\bar{x}_{k+1,1}u_{k+1,1})\Delta||^2) \\
&+E||\alpha_1\Delta W_{k+1,1}||^2 \\
&+E||u_{k+1,1}h(X_{k+1,1}, t_k)\Delta N_{k+1,1}||^2)
\end{aligned}$$

According to hypothesis H1, H2 and H3 we have:

$$E||\bar{x}_{k+1,1}||^2 \leq 12(L + \eta\beta C_1 - \delta LA) + \alpha C_2 + C_3C_4$$

as required.

4 Final considerations

The stochastic fluctuations to which our system is subjected are varied and of diverse nature, so when considering the stochastic integration of equations (1),

$$x = x_0 + \int_0^t f(t, x(t), u(t))dt + \int_0^t g(t, x(t), u(t))dW + x(t)u(t)\int_0^t\int_{\mathbb{R}^n}\gamma(t, x(t-), z)\tilde{N}(dt, dz), \quad (19)$$

we must take into account that three types of integrals are involved: the Riemann integral for the deterministic part, the Itô integral for the Wiener process and the Lebesgue integral for the Lévy process. It is possible to build a stationary Poisson point process to approximately simulate our systems of stochastic differential equations driven by Lévy jumps. According to [Zou and Wuan, 2014], we can write, for a stationary Poisson process $\xi(t)$:

$$\int_{\mathbb{R}^n}\gamma(t, x_i(t-), z)N(dt, dz) = \sum_{0\leq\tau_1<\cdots<\tau_n\leq T}\gamma(\tau_j, \xi(\tau_j)) \quad (20)$$

for $D_\xi = \{\tau_1, \tau_2, \cdots, \tau_n\}$ a domain of definition made using the exponential distribution:

$$\tau_k = F(k) := 1 - e^{-0.25k},\ k = 0, 1, \cdots n, \quad (21)$$

for a Poisson process $\xi(\cdot)$ defined on a new increasingly ordered partition

$$D = D_\xi \cup D_W = \{T_1, T_2, \cdots, T_n = T\}, \quad (22)$$

where $D_W = \{t_1, t_2, \cdots, t_n = T\}$ is the partition of the stochastic process without jumps, so $T_i = \tau_i \in D_\xi$ or $T_i = t_i \in D_W$, as following:

$$\xi(T_k) := \begin{cases} \dfrac{(0.5)^{T_k}e^{-0.5}}{T_k!} & \text{, if } T_k = \tau_k \\ 0 & \text{, if } T_k = t_k. \end{cases}$$

On the other hand, regarding the existence and uniqueness of solutions, our model satisfies the conditions of at most linear growth and Lipschitz continuity, [Oksendal and Sulem, 2005]:

1 . (At most linear growth). There exist a constant $C_1 < \infty$ such that

$$\int_{\mathbb{R}}\sum_{i=1}^3||\gamma_i(t, x(t), z))||^2\nu_k(dz_k) + ||g(t, x, u)||^2 + ||f(t, x, u)||^2 \leq C_1(1 + ||x||^2)\ \forall x \in \mathbb{R}^n$$

2. (Lipschitz continuity). There exist a constant $C_2 < \infty$ such that

$$\int_{\mathbb{R}}\sum_{i=1}^3||\gamma_i(t, x(t), z_i)) - \gamma_i(t, y(t), z_i))||^2\nu_k(dz_k) + ||g(t, x, u) - g(t, y, u)||^2 + ||f(t, x, u) - f(t, y, u)||^2 \leq C_2||x - y||^2\ \forall x, y \in \mathbb{R}^n$$

Indeed, the following calculations and [Romero-Meléndez, Castillo-Fernández, and González-Santos, 2021] guarantee that the above conditions are satisfied. Considering

$$\begin{aligned}
h(t, x, u) &= x(t)u(t)\int_{\mathbb{R}^n}\gamma(t, x_3(t-), z)N(dt, dz) \\
&= x(t)u(t)\sum_{0\leq\tau_1<\cdots<\tau_n\leq T}\gamma(\tau_j, \xi(\tau_j))
\end{aligned}$$

we have:

$$
\begin{aligned}
&||h(t,x,u) - h(t,y,u)|| = \\
&||x(t)u(t)\sum_{\dots}\gamma(\tau_j,\xi(\tau_j)) - y(t)u(t)\sum_{\dots}\gamma(\tau_j,\xi(\tau_j))|| \\
&= \Big(x(t)u(t) - y(t)u(t)\Big)\sum_{\dots}\gamma(\tau_j,\xi(\tau_j)) \\
&\leq ||x(t)u(t) - y(t)u(t)|| \cdot ||\sum_{\dots}\gamma(\tau_j,\xi(\tau_j))|| \\
&\leq ||u(t)|| \cdot ||x(t) - y(t)|| \cdot ||\sum_{\dots}\gamma(\tau_j,\xi(\tau_j))|| \\
&\leq C_4||x(t) - y(t)|| \cdot \sum_{\dots}||\gamma(\tau_j,\xi(\tau_j))|| \\
&\leq C_4 q^k ||x(t) - y(t)||
\end{aligned}
$$

for some constants C_4, q and k, where we have used that the control is bounded and γ is an exponential random variable. So, according to [Oksendal and Sulem, 2005], for our model the existence and uniqueness of solutions of system (2) are guaranteed.

5 Conclusion

In this paper, we have considered the stochastic and controlled Lotka-Volterra one-predator-two-prey model with jumps. Considering the numerical solutions via Euler-Maruyama scheme, we have proved the boundedness and strong convergence for this kind of solutions, assuming standard linear growth and Lipschitz conditions on the drift and diffusion terms.

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