VARIATIONAL OPTIMALITY CONDITION IN CONTROL OF HYPERBOLIC SYSTEMS WITH BOUNDARY DELAY PARAMETERS

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Abstract

An optimal control problem of a first-order hyperbolic system is studied, in which a boundary condition at one of the ends is determined from a controlled system of ordinary differential equations with constant state lag. Control functions are bounded and measurable functions. The system of ordinary differential equations at the boundary is linear in state. However the matrix of coefficients depends on control functions. Therefore, the optimality condition of Pontryagin's maximum principle type in this problem is a necessary, but not a sufficient optimality condition. In this paper, the problem is reduced to an optimal control problem of a special system of ordinary differential equations. The proposed approach is based on the use of an exact formula of the cost functional increment. The reduced problem can be solved using a wide range of effective methods used for optimization problems in systems of ordinary differential equations. Problems of this kind arise when modeling thermal separation processes, suppression of mechanical vibrations in drilling, wave processes and population dynamics.

Key words

Hyperbolic systems, boundary delay, variational optimality condition, reduction of optimal control problems.

1 Introduction

Delay differential equations are a special kind of differential equations in which an unknown function and its derivatives enter at different values of the argument. The delay can be due to a variety of reasons, for example, the presence of inertia of dynamic systems, the limited velocity of processes, etc.

The study of optimal control problems for delay systems began almost immediately after the first results in the optimal control theory of classical dynamical systems were obtained. In 1961, G. L. Kharatishvili [Kharatishvili, 1961] generalized the Pontryagin's maximum principle to the case of processes with delay. A little bit later, R. Gabasov and S. V. Churakova [Gabasov, 1967] proved the existence of optimal controls in the control problem for a system of ordinary differential equations (ODEs) with delay argument. In [Frankena, 1975], a general optimal control problem was considered, which includes differential equations with delay argument, under restrictions on both phase and control variables. The necessary optimality conditions were obtained, in which the Lagrange multipliers appear. In 2011, G. V. Bokov [Bokov, 2011] formulated an optimal control problem that contains a constant time delay both in the phase variable and in the control variable, and also proved the necessary optimality condition using the needle variation, substantiated the maximum principle in a problem with an infinite time interval. An interesting modern area of research is a design of identifiers for systems with a known and unknown time-delays [Furtat, 2020].

The study of optimal control problems for partial differential equations with delay mainly follows the path of applying the approaches developed earlier for ODEs. Here we can note the results in the field of existence and uniqueness [Teo, 1979], optimality conditions like the maximum principle [Sadek, 1990], numerical methods [Mai, 2017].

This article considers an optimal control of a first-order hyperbolic system, in which the boundary condition at one of the ends is determined from a controlled system of ODEs with a constant state delay. Problems of this kind arise when modeling thermal separation processes [Demidenko, 2006], suppression of mechanical vibrations in drilling [Bresch-Pietri, 2016], wave processes [Souhaile, 2021] , population dynamics [Piazzera, 2004], etc. Admissible controls are bounded and measurable functions. The system of ODEs on the boundary is linear in state, but the matrix of coefficients at phase variables depends on control functions. Therefore, the optimality condition of Pontryagin's maximum principle type in this problem is a necessary but not sufficient optimality condition. In this regard, the same methods are usually applied to solve such problems as for general nonlinear optimal control problems.

The main result of the work is in reduction of the original problem to the optimal control problem of a system of ODEs. The proposed approach is based on the use of an exact (without remainder terms) increment formula of the cost functional. The corresponding statement is formulated as a variational optimality condition. An example illustrating the reduction process is given. Note that the reduced problem has the following structure. The control system is linear in state, but the matrix of coefficients at phase variables depends on the control functions. The reduced problem can be solved using a wide range of efficient methods used for this class of optimal control problems in systems of ODEs. This approach was proposed in [Arguchintsev, 2021] for classic optimal control problems with fixed boundary conditions and without delay. Two symmetric variational conditions were proved. Delay parameters lead to only one variational optimality condition.

2 Problem statement

We consider a system

$$x_t + A(s,t)x_s = \Phi(s,t)x + \bar{f}(s,t), \tag{1}$$

$$(s,t) \in \Pi, \ \ \Pi = S \times T, \ \ S = [s_0, s_1], \ \ T = [t_0, t_1].$$

Here $x = x(s,t)$ is $n-$ dimensional vector-function, $\mathbf{A} = \mathbf{A}(s,t)$ is a matrix of order $(n \times n)$, We suppose that system (1) is written in an invariant form, that is $\mathbf{A}$ is a diagonal matrix. Diagonal elements $a_i(s,t)$ of A are of constant sign in Π:

$$a_i(s,t) > 0, \ \ i = 1, 2, \ldots, m_1;$$

$$a_i(s,t) = 0, \ \ i = m_1 + 1, m_1 + 2, \ldots, m_2;$$

$$a_i(s,t) < 0, \ \ i = m_2 + 1, m_2 + 2, \ldots, n.$$

We consider two subvectors

$$x^+ = (x_1, \ldots, x_{m_1}), \ x^- = (x_{m_2+1}, \ldots, x_n),$$

which correspond to the positive and negative diagonal elements of matrix A.

Let us introduce the initial-boundary conditions for the hyperbolic system (1). Note that the boundary condition at $s = s_0$ is determined from a controlled system of linear ordinary differential equations with a constant state delay.

$$\frac{dx^+(s_0,t)}{dt} = N(u(t),t)x^+(s_0, t-\alpha) + b(u(t),t), \tag{2}$$

$$x(s,t_0) = x^0(s), \ \ s \in S, \quad x^-(s_1,t) = \nu(t), \ \ t \in T,$$
$$x^+(s_0,t) = q(t), \ \ t \in [-\alpha; t_0]; \ \ \alpha > 0,$$

where α is a constant delay.

We consider bounded and measurable r - dimensional control vector functions $u(t)$ on T satisfying almost everywhere the inclusion-type restrictions

$$u(t) \in U \subset E^r, \quad t \in T, \tag{3}$$

U is a compact set.

It is required to find an admissible control that delivers a minimum to the objective functional

$$J(u) = \int_s \langle c(s), x(s,t_1)\rangle ds, \quad u \in U. \tag{4}$$

Here $\langle , \rangle$ means a classic scalar product in a finite-dimensional Euclidean space of a corresponding dimension.

The problem (1) – (4) is considered under the following assumptions.

1) Diagonal elements $a_i(s,t)$ of matrix $\mathbf{A}$ are continuous and continuously differentiable in Π.

2) Functions $x^0(s), \ \nu(t), \ q(t)$ are continuous on S and T, respectively, and satisfy the matching conditions

$$\nu(t_0) = (x^0(s_1))^-, \quad q(t) = (x^0(s_0))^+.$$

3) Functions $\Phi(s,t), \ \bar{f}(s,t), \ N(u,t), \ b(u,t)$ and $c(s)$ are continuous in aggregate of its arguments on $S \times T, S \times T, U \times T, U \times T$ and S, respectively.

For any admissible control, there is a unique generalized solution of the initial-boundary value problem (1) – (2), which is continuous in Π function [Rozhdestvensky, 1968] (pp. 63–69, 90–94). Each component of the solution $x_i, i = 1, 2, \ldots, n,$ is continuously differentiable along the characteristics of (1). The continuity of the solution is guaranteed by above assumptions on the parameters of the problem and the matching conditions. These conditions do not guarantee existence of a classical solution in the rectangle Π. This requires the fulfillment of higher-order matching conditions closely related to the hyperbolic system itself [Godunov, 1979]. Thus, instead of the left side of (1) we will consider a differential operator

$$\left(\frac{dx}{dt}\right)_A = \left(\left(\frac{dx_1}{dt}\right)_A, \left(\frac{dx_2}{dt}\right)_A, \ldots, \left(\frac{dx_n}{dt}\right)_A\right),$$

where $(dx_i/dt)_A$ are derivatives of the corresponding state vector component along the corresponding family of characteristics.

Note that the above assumptions are not sufficient for existence and uniqueness of optimal control problem (1) – (4) solution. However our aim is not in solving this problem but it is in reduction to an optimal control problem for a system of ODE equations.

3 Variational optimality condition

The problem under consideration is a linear one with respect to x. However the classical Pontryagin's maximum principle is not a sufficient optimality condition in this problem. This is explained by the dependence of the matrix of coefficients $N(u,t)$ in (2) on controls. Therefore, to solve such problems, methods developed for the general nonlinear case are usually used. In this paper, we use the technique previously applied in [Arguchintsev, 2021] for problems without lag parameters.

Consider two arbitrary different admissible processes: $\{u, x\}$ and $\{\tilde{u} = u + \triangle u, \tilde{x} = x + \triangle x\}$. Let's write a system in increments

$$\left(\frac{d\triangle x}{dt}\right)_A = \Phi(s,t)\triangle x,$$

$$\triangle x^+(s_0,t) = 0,\ t \in [-\alpha; t_0];\ \ \triangle x(s,t_0) = 0,\ \ s \in S,$$
$$\triangle x^-(s_1,t) = 0,\ t \in T;$$

$$\begin{aligned}&\triangle x^+_t(s_0,t) = N(\tilde{u},t)\tilde{x}^+(s_0,t-\alpha)\\&-N(u,t)x^+(s_0,t-\alpha) + \triangle b(u,t),\end{aligned} \tag{5}$$

$\triangle b(u,t) = b(\tilde{u},t) - b(u,t)$. We apply the following representation for the right side of (5):

$$\begin{aligned}&\triangle x^+_t(s_0,t) = \triangle_{\tilde{u}} N(u,t)\tilde{x}^+(s_0,t-\alpha)\\&+N(u,t)\triangle x^+(s_0,t-\alpha) + \triangle b(u,t),\end{aligned}$$

where $\triangle_{\tilde{u}} N(u,t) = N(\tilde{u},t) - N(u,t)$. We write the increment of functional in the form

$$\triangle J(u) = J(\tilde{u}) - J(u) = \int_S \langle c(s), \triangle x(s,t_1)\rangle ds. \tag{6}$$

In formula (6), we add zero terms

$$\int_\Pi\int \langle \psi(s,t), \left(\frac{d\triangle x}{dt}\right)_A - \Phi(s,t)\triangle x\rangle ds dt,$$

$$\int_T \langle p(t), \triangle x^+_t(s_0,t) - \triangle_{\tilde{u}} N(u,t)\tilde{x}^+(s_0,t-\alpha)$$

$$-N(u,t)\triangle x^+(s_0,t-\alpha) - \triangle b(u,t)\rangle dt,$$

where $\psi(s,t),\ \ p(t)$ are while arbitrary vector functions of dimension n and m_1, respectively, having the same analytic properties as the corresponding functions $x(s,t),\ x^+(t)$.

Integrating by parts we obtain

$$\triangle J(u) = \int_S \langle c(s), \triangle x(s,t_1)\rangle ds$$

$$+\int_S [\langle \psi(s,t_1), \triangle x(s,t_1)\rangle - \langle \psi(s,t_0), \triangle x(s,t_0)\rangle] ds$$

$$-\int_\Pi\int \langle \left(\frac{d\psi}{dt}\right)_A + A_s\psi, \triangle x(s,t)\rangle ds dt$$

$$+\int_T [\langle \psi(s_1,t), A(s_1,t)\triangle x(s_1,t)\rangle$$

$$-\langle \psi(s_0,t), A(s_0,t)\triangle x(s_0,t)\rangle] dt$$

$$+\langle p(t_1), \triangle x^+(s_0,t_1)\rangle - \langle p(t_0), \triangle x^+(s_0,t_0)\rangle$$

$$-\int_T \langle p_t, \triangle x^+(s_0,t)\rangle dt$$

$$-\int_\Pi\int \langle \psi(s,t), \Phi(s,t)\triangle x\rangle ds dt$$

$$-\int_T \langle p(t), \triangle_{\tilde{u}} N(u,t)\tilde{x}^+(s_0,t-\alpha)$$

$$+N(u,t)\triangle x^+(s_0,t-\alpha) + \triangle b(u,t)\rangle dt.$$

Here we use a generalized integration by parts formula for the term

$$\int_\Pi\int \langle \psi(s,t), \left(\frac{d\triangle x}{dt}\right)_A - \Phi(s,t)\triangle x\rangle ds dt.$$

This formula is used for the generalized solution concept in [Potapov, 1983].

Next, consider the following expression:

$$\int_T \langle p(t), N(u,t)\triangle x^+(s_0,t-\alpha)\rangle dt$$

$$= \int_{t_0-\alpha}^{t_0} \langle p(\tau+\alpha), N(u(\tau+\alpha),\tau+\alpha)\triangle x^+(s_0,\tau)\rangle d\tau$$

$$+\int_{t_0}^{t_1-\alpha} \langle p(\tau+\alpha), N(uu(\tau+\alpha), \tau+\alpha)\triangle x^+(s_0,\tau)\rangle d\tau.$$

Here $\tau = t-\alpha, \tau \in [t_0-\alpha, t_1-\alpha]$. Let's return to variable t. Taking into account the fact that the first term of this expression is equal to zero, we get

$$\int_T \langle p(t), N(u,t)\triangle x^+(s_0, t-\alpha)\rangle dt$$

$$= \int_{t_0}^{t_1-\alpha} \langle p(t+\alpha), N(u(t+\alpha), t+\alpha)\triangle x^+(s_0,t)\rangle dt.$$

We require that the functions $\psi(s,t),\ p(t)$ be solutions of the following adjoint problems

$$\left(\frac{d\psi}{dt}\right)_A + \mathbf{A}_s\psi = -\Phi^T(s,t)\psi,\ \psi(s,t_1) = -c(s),$$
$$\psi^+(s_1,t) = 0;\ \ \psi^-(s_0,t) = 0,\ \ t \in T; \tag{7}$$

$$p_t = \begin{cases} -N^T(u(t+\alpha), t+\alpha)p(t+\alpha) - \psi(s_0,t), \\ \qquad\qquad\qquad\qquad t \in [0; t_1-\alpha], \\ -\psi(s_0,t), \qquad\qquad t \in [t_1-\alpha; t_1]; \end{cases} \tag{8}$$

$$p(t_1) = 0;\ \ p(t) \equiv 0,\ \ t > t_1.$$

Here $A^+(s,t)$ is a diagonal submatrix of A, composed of its positive elements, ψ^+ is a subvector of ψ, composed of its first m_1 components. Then the formula for the increment of the functional takes the form

$$\triangle J(u) = -\int_T \langle p(t), \triangle_{\tilde{u}} N(u,t)\tilde{x}^+(s_0, t-\alpha)$$
$$+\triangle b(u,t)\rangle dt. \tag{9}$$

(9) is the exact increment formula (without remainder terms) for any pair of admissible processes, while the original ODE system (2) is integrated on the perturbed control.

The resulting increment formula allows us to reduce the original problem of optimal control of a hyperbolic system to the optimal control problem for the system of ODEs

$$I(v) = -\int_T \langle p(t,u), N(v(t),t)$$
$$-N(u((t),t))z(t-\alpha, v)$$
$$+b(v(t),t) - b(u(t),t)\rangle dt \to \ \min, \tag{10}$$

$$\dot{z} = N(v(t),t), z(t-\alpha) + b(v(t),t),\ \ t \in T;$$
$$z(t) = 0,\ t \in [t_0-\alpha, t_0];\ \ v(t) \in U. \tag{11}$$

This result enables us to formulate a new variational optimality condition (as opposed to traditional finite-dimensional optimality conditions).

Theorem. A control $u^*(t)$ is optimal in the problem (1)–(4) if and only if the function $v^* = u^*(t)$ is optimal in problem (10) – (11) for any fixed admissible $u(t)$.

It follows from (9) that the optimal value of the functional in the original problem is given by

$$J(v^*) = J(u) + I(v^*).$$

Note that the proved theorem is true for any fixed admissible function $u(t)$. It is due to the fact that the increment formula has been proved for an arbitrary pair of admissible controls. We did not use any local control variations.

4 Reduction scheme

The following solution scheme based on the theorem can be proposed.

1. An arbitrary initial admissible control $u = u(t)$ is specified. Then we calculate a solution $p = p(t,u)$ of adjoint problems (7) – (8) corresponding to this control.
2. We solve an auxiliary optimal control problem (10) – (11) for the system of ordinary differential equations. Its solution will be a solution of the problem (1) – (4).

The following example illustrates this reduction scheme.

In $\Pi = S \times T = [0;4] \times [0;4]$ we consider an optimal control problem

$$x_{1t} + x_{1s} = (s+2)\cos t,$$

$$x_{2t} - 2x_{2s} = x_2 - 2e^s,$$

$$x_{1t}(0,t) = u \cdot x_1(0,t-1),\ \ u(t) \in U = [0,2];$$

$$x_1(0,t) = 0,2\cdot t,\ \ t \in [-1;0];$$

$$x_2(4,t) = t^2,\ \ x_1(s,0) = 0,\ \ x_2(s,0) = s-4;$$

$$J(u) = \int\limits_S [2x_1(s,4) + 3x_2(s,4)]\,ds \to min,\ \ u \in U.$$

Here

$$\Phi(s,t) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ \ N(u,t) = u,\ \ b(u,t) = 0.$$

Adjoint problems (7)–(8) have the following form:

$$\psi_{1t} + \psi_{1s} = 0, \quad \psi_{2t} - 2\psi_{2s} = -\psi_2,$$

$$\psi_1(s,4) = -2, \;\; \psi_2(s,4) = -3,$$

$$\psi_1(4,t) = 0; \;\; \psi_2(0,t) = 0;$$

$$p_t = \begin{cases} -p(t+1)\cdot u(t+1) - \psi_1(0,t), & t \in [0;3), \\ -\psi_1(0,t), & t \in [3;4], \end{cases}$$

$$p(4) = 0.$$

We choose an admissible control $u(t) \equiv 0$ for all $t \in [0,4]$.

Then we solve (7) – (8):

$$\psi_1(s,t) = \begin{cases} 0, \; t < s, \\ -2, \; t \geq s. \end{cases}$$

$$p(t) = 2t - 4, \;\; t \in [0;4].$$

The problem is reduced to the following optimal control problem:

$$I(v) = \int_0^4 (4-2t)\cdot v(t)\cdot z(t-1,v)\,dt \to min,$$

$$z_t = v\cdot z(t-1), \;\; z(t) = 0,2\cdot t, \;\; t \in [-1;0], \;\; z(0) = 0;$$

$$v(t) \in [0;2].$$

For solving the reduced optimal control problem, one can use a wide set of modern optimal control methods (for example, see reviews [Golfetto, 2012; Biral (2016); Srochko (2021)]).

5 Conclusions

We considered an optimal control problem by a special type of hyperbolic equations containing delay parameters in boundary conditions. The proposed approach allows to reduce this problem to an optimal control problem by a system of ordinary differential equations. Our further goal is to extend this approach to the case of quadratic cost functionals.

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