SYNCHRONIZATION EFFECTS RELATED TO NEIGHBORHOOD SIMILARITY IN A RANDOM NETWORK OF NON-IDENTICAL OSCILLATORS

Celso Freitas
Associate Laboratory for Computing and Applied Mathematics - LAC
Brazilian National Institute for Space Research - INPE
Brazil
cbnfreitas@gmail.com

Elbert Macau
Associate Laboratory for Computing and Applied Mathematics - LAC
Brazilian National Institute for Space Research - INPE
Brazil
elbert.macau@inpe.br

Ricardo Viana
Department of Physics
Federal University of Paraná - UFPR
Brazil
viana@fisica.ufpr.br

Abstract

We explore in this article complex networks of non-identical oscillators. More specifically, we focus on the impact of Similar or Dissimilar neighborhoods over synchronization measures. Maybe contrary to the intuitive idea, our numerical simulations show that the more homogeneous is a network, the higher tend to be the coupling strength required to phase-lock. In addition, if the coupled oscillatory system is composed of heterogeneous variants, then less coupling strength is required for phase-lock and larger values of order parameter are observed, which means that the fixed phase synchronization is closer to full synchronization.

Key words

Synchronization, complex networks, Kuramoto model, non-identical oscillators.

1 Introduction

Many social and biological studies about multi-agents systems reveal that their members tend to select similar peers to interact in a myriad of processes. However, this is not a general rule. Ref. [Wedekind et al. 1995] shows that some genetically determined odor components like MHC (major histocompatibility complex) can be important in mate choice. Women tend to score male odor as more pleasant when they differed from their own MHC, maybe as a mechanism to enhance DNA diversity. Ref. [Hamm 2000] presented strong evidences that adolescents did not choose friends with identical orientations: some aspect are indeed important while others are almost uncorrelated. Thus, Nature seems to favor *Similar, Neutral or Dissimilar* (neighborhood) patterns to achieve distinct objectives.

We focus on an analogy of these concepts within the non-identical phase-oscillator Kuramoto model with local mean field coupling, which is one of the main paradigms to describe collective behavior and synchronization [Pikovsky et al. 2003]. This model is interesting because it approximates the dynamics of any nonlinear oscillators near its limit cycle, under weak mutual interaction. Besides, there are a number of applications from different areas based on it [Strogatz 2001; Arenas et al. 2008], which highlight the role that synchronization plays. Among the works of this research field, we cite [Sun et al. 2009], based on Master Stability Function for nearly identical oscillators. [Brede 2008] proposes an algorithm to obtain optimized networks related to local and global synchronization. Our main contribution is to analyze how the heterogeneity among the connected oscillators in the networks modifies some key synchronization features. If there are only two connected oscillators, it is known that the relation between dissonance and coupling strength determines synchronization.

In this paper, we introduce the novel total dissonance measure for vertex weighted graphs. It comprises the dissonance among every two neighbor nodes in the graph. Thus, we construct Similar and Dissimilar networks with optimization tools and show that these measure yields deeply influences the critical coupling strength for phase-locking.

2 Model

We consider a system of N phase oscillator coupled through a simple and connected graph, whose dynamic of oscillator $i = 1, \dots, N$ is given by the following ordinary equation

$$\dot{\theta}_i = \omega_i + \frac{\varepsilon}{d_i} \sum_{j=1}^{N} A_{ij} \sin(\theta_j - \theta_i) \,, \qquad (1)$$

where $\omega = (\omega_1, \ldots, \omega_N) \in \mathbb{R}^N$ are the *natural frequencies* of each oscillator. The *coupling strength* $\varepsilon \geq 0$ is a system parameter that adjusts the intensity of attraction between neighbor oscillators. The symmetrical *coupling graph* is expressed by its adjacency $N \times N$ matrix A, so that $A_{ii} = 0$; $A_{ij} = 1$, if oscillators i, j are connected; and $A_{ij} = 0$, otherwise. The vertex degree is denoted by $d_i = \sum_j^N A_{ij}$.

Our new measure, the *total dissonance*, is defined as

$$\nu_{\text{Total}} = \frac{1}{N}\sqrt{\sum_{i,j=1}^{N} A_{ij}(\omega_i - \omega_j)^2}. \qquad (2)$$

Of course, $\nu_{\text{Total}} = 0$ if and only all oscillators are identical oscillators. Otherwise, if we obtain permutations of ω such the associated ν_{Total} is closer to its minimum or maximum, then we have the Similar or the Dissimilar configuration, respectively. The overall maximum and minimum values of ν_{Total}

We consider Erdös-Rényi (ER) networks and mean degree 2. Different sizes of small to medium networks are considered. For each size of network, a single choice of natural frequencies with zero mean is randomly draw from an uniform distribution over $[-\pi, \pi]$. Negative values of natural frequencies mainly arise due to change of variables to obtain zero mean ω, which is convenient from the mathematical point of view [Pikovsky et al. 2003]. Of course, ν_{Total} is smooth related to ω. However, we are interest *only* in permutations of ω, which correspond to a discontinuous problem requiring combinatorial optimization techniques. So, an simulated annealing algorithm is applied to obtain Similar (minimization of ν_{Total}) or Dissimilar (maximization of ν_{Total}) networks. Neutral patterns correspond to non-optimized patterns, with ν_{Total} near the middle value between extremes. Fig. 1 illustrates the result of these optimizations.

The norm of the global mean field, the *order parameter*, will be denoted by

$$R(\theta) = \left| 1/N \sum_{i=1}^{N} \mathrm{e}^{\mathrm{i}\,\theta_i} \right|.$$

This measure R may range from 0 to 1, indicating that the ensemble gradually changes from null global mean field, where the sum of all phasors $\mathrm{e}^{\mathrm{i}\,\theta_i}$ cancel out, until full synchronization, where $\theta_1 = \ldots = \theta_N$, respectively.

Under this context, if ε is large enough, analytical results like in [Jadbabaie et al. 2004] guarantee convergence to phase-lock, that is all oscillators evolve together as a rigid body and $R(\theta(t))$ converges. However, we are interested here in much smaller values for the coupling strength ε than the lower bound given by that work. For a given choice of parameters and initial conditions, we call ε_{PL} the smallest value of coupling strength ε such that R converges. In addition, we define R_{PL} as the correspondent value of R. Both ε_{PL} and R_{PL} are numerically evaluated in this work. A discussion about analytical bounds for ε_{PL} may be found at [Chopra et al. 2009]. Note also that R and ε_{PL} does not come into play in the optimization process. We are interested in the correlation between Similar, Neutral and Dissimilar configurations, obtained through an optimization algorithm, and the synchronization quantifiers R and ε_{PL} which makes use of the coupling graphs but requires numerical integration.

3 Numerical Simulation and Conclusion

We present here results for 100 different ER graphs and $N = 10, 50, 100$. Three version of each combination of graph A and natural frequencies ω are studied: without optimization (Neutral), with small ν_{Total} (Similar) and with high ν_{Total} (Dissimilar). Tab. 1 shows a summary of the measures ν_{Total} obtained.

Network	Mean and Standard Deviation of ν_{Total}		
	Similar	Neutral	Dissimilar
ER 10	1.190 0.085	1.960 0.178	2.500 0.048
ER 50	0.327 0.016	0.768 0.042	1.120 0.016
ER 100	0.245 0.008	0.545 0.019	0.777 0.008

Table 1. Total Dissonance ν_{Total} for each size of network considered.

So far, observe that both mean value and standard deviation of ν_{Total} become smaller as N increases.

The numerical integration is performed with random initial condition $\theta^0 \in \mathbb{R}^N$, with uniform distribution over the unit circle. We emphasize that there is no correlation between θ^0 and ω. A different initial condition is considered for each network. Successive integrations were performed with increasing values of ε until phase-lock, that is $\varepsilon = \varepsilon_{\text{PL}}$. An Adams-Bashforth-Moulton Method for numerical integration with fixed step size $h = 0.01$ was applied. A large enough transient time was suppressed from the data of at least 2.10^3 units of time.

Tab. 2 and 3 present the main result of this work. Note that if ν_{Total} increases, then both the mean and standard deviation of ε_{PL} decreases, which is counter intuitive, since $\varepsilon_{\text{PL}} = 0$ implies identical oscillators, so any positive value of ε yields phase-lock. Furthermore, the mean values of ε_{PL} increase with the increment of N for all cases. However, this increment becomes smaller for larger values of N. For all configuration patterns, the critical coupling strength ε_{PL} did not grow with the same order as the population size N. If ν_{Total} increases, then the mean value of R_{PL} also increases, but the standard deviation of R_{PL} decreases.

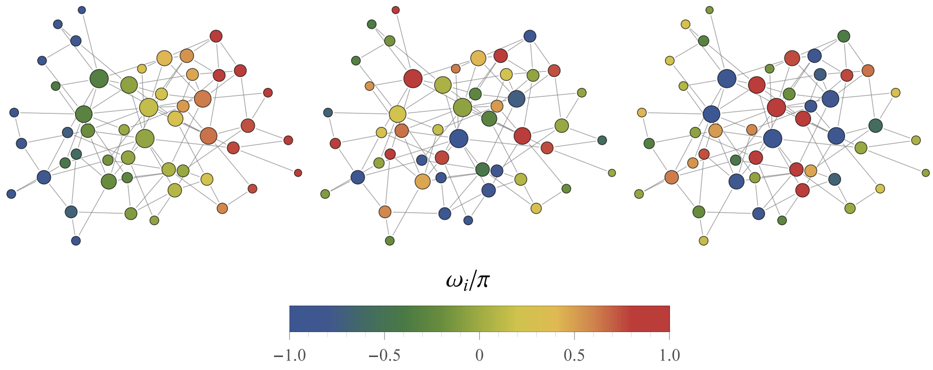

Figure 1. Different configurations of ER networks with $N = 50$: (Left) Similar, $\nu_{\text{Total}} = 0.32$; (Middle) Random, $\nu_{\text{Total}} = 0.78$; and (Right) Dissimilar $\nu_{\text{Total}} = 1.11$.

Network	Mean Standard Deviation of ε_{PL}		
	Similar	Neutral	Dissimilar
ER 10	5.75 0.95	4.76 0.66	3.88 0.26
ER 50	10.30 1.24	6.42 0.87	4.19 0.26
ER 100	11.70 1.24	6.76 0.97	4.50 0.30

Table 2. Critical coupling for phase-lock ε_{PL} and its order parameter R_{PL}.

Network	Mean Standard Deviation of R_{PL}		
	Similar	Neutral	Dissimilar
ER 10	0.59 0.07	0.79 0.04	0.86 0.02
ER 50	0.51 0.09	0.79 0.05	0.88 0.02
ER 100	0.54 0.08	0.79 0.05	0.89 0.02

Table 3. Critical coupling for phase-lock ε_{PL} and its order parameter R_{PL}.

So, our simulations suggest that Dissimilar networks require lesser coupling strengths to achieve fixed phase synchronization. Moreover, this synchronization is closer to full synchronization in comparison with networks with smaller ν_{Total}. This result may be important for instance to technological applications with fixed coupling topology and inner dynamics, since we shows how it is possible to enhance synchronization just by exchanging the oscillators positions in the system. Experiments with other graph topologies exploring these and other synchronization phenomena related to total dissonance are already been prepared for further publications.

Acknowledgements

We would like to thank the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - CAPES (Process: BEX 10571/13-2), CNPq and FAPESP (grant 2011/50151-0) for financial support.

References

Arenas, A., Díaz-Guilera, A., Kurths, J., Moreno, Y., and Zhou, C. (2008). Synchronization in complex networks. *Physics Reports*, 469(3):93–153.

Brede, M. (2008). Locals vs. global synchronization in networks of non-identical kuramoto oscillators. *The European Physical Journal B-Condensed Matter and Complex Systems*, 62(1):87–94.

Hamm, J. V. (2000). Do birds of a feather flock together? the variable bases for african american, asian american, and european american adolescents' selection of similar friends. *Developmental psychology*, 36(2):209.

Jadbabaie, A., Motee, N., and Barahona, M. (2004). On the stability of the kuramoto model of coupled nonlinear oscillators. In *American Control Conference, 2004. Proceedings of the 2004*, volume 5, pages 4296–4301. IEEE.

Pikovsky, A., Rosenblum, M., and Kurths, J. (2003). *Synchronization: A Universal Concept in Nonlinear Sciences*. Cambridge Nonlinear Science Series. Cambridge University Press.

Strogatz, S. H. (2001). Exploring complex networks. *Nature*, 410(6825):268–276.

Sun, J., Bollt, E. M., and Nishikawa, T. (2009). Master stability functions for coupled nearly identical dynamical systems. *EPL (Europhysics Letters)*, 85(6):60011.

Wang, K., Fu, X., and Li, K. (2009). Cluster synchronization in community networks with nonidentical nodes. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 19(2):–.

Wedekind, C., Seebeck, T., Bettens, F., and Paepke, A. J. (1995). Mhc-dependent mate preferences in humans. *Proceedings of the Royal Society of London. Series B: Biological Sciences*, 260(1359):245–249.

Chopra, N. and Spong, M.W. (2009). On Exponential Synchronization of Kuramoto Oscillators. *Automatic Control, IEEE Transactions on*, 54(2):353-357.