# THE INITIAL CONDITIONS OF RIEMANN-LIOUVILLE AND CAPUTO DERIVATIVES


**Manuel Ortigueira**

*UNINOVA and Dep. of Electrical Engineering*

*Faculdade de Ciências e Tecnologia da Universidade Nova de Lisboa*

*Portugal*

*mdo@fct.unl.pt*

**Fernando Coito**

*UNINOVA and Dep. of Electrical Engineering*

*Faculdade de Ciências e Tecnologia da Universidade Nova de Lisboa*

*Portugal*

*fjvc@fct.unl.pt*



**Abstract**:

The initial condition problem for fractional linear system initialisation is restudied in this paper. A general formulation similar to the integer order is presented. The Riemann-Liouville and Caputo initial conditions are interpreted in terms of the general scheme.



**Keywords**:

Initial value problem; fractional derivative; fractional linear system.


## 1 Introduction

The initial value problem is a theme that remains quite up-to-date, even in the classic integer order case (Lunberg et al, 2007). In fact, the computation of the output of a linear system under a given set of initial conditions is an important task in daily applications. Traditionally this task has been accomplished by means of the unilateral Laplace transform (ULT) and the jump formula that is a result of the distribution (generalized function) theory (Ferreira,1997;Hoskins, 1999). The problems found in concrete applications have been addressed and are motivated by the ULT treatment of the origin as presented in the main text books and in the fractional case by the use of derivative definitions that impose specific initial conditions that may not be the most suitable for the problem.

In current literature we find two situations:

- People who consider the Riemann-Liouville (RL) derivative and the associated initial conditions (e.g. Samko et al, 1987; Miller & Ross, 1993; Podlubny, 1999);
- People that use the Caputo (C) derivative that uses integer order derivatives (e.g. Davidson & Essex, 1998; Mainardi and Pagnini, 2003; Jafari and Daftardar-Gejji, 2006).

If x(t) is a causal signal and denoting the Laplace transform by LT we have for the RL case:

$$LT[D^{\alpha}x(t)] = LT\left[ D^{m}[ D^{-(m-\alpha)} x(t)]\right] = s^{\alpha} X(s) - \sum_{i=0}^{m-1} s^{m-i-1} D^{i-m+\alpha}x(0) \qquad (1)$$

where m is the least integer greater than or equal to $\alpha$. In the Caputo case, we have

$$LT[D^{\alpha}x(t)] = LT\left[D^{-(m-\alpha)} [D^{m} x(t)]\right] = s^{\alpha} X(s) - \sum_{i=0}^{m-1} s^{i-m+\alpha} D^{m-i-1}x(0) \qquad (2)$$

In the last years the second approach has been favoured relatively to the first, because it is believed that the RL case leads to initial conditions without physical meaning. This was contradicted by Heymans and Podlubny (2005) that studied several cases and gave physical meaning to the RL initial conditions, by introducing the concept of "inseparable twin". On the other hand, Agrawal (2006) shows that both types of initial conditions can appear. Similar position is assumed by Gorenflo and Mainardi (1997) and Bonilla et al (2007).

In Ortigueira (2003) and Ortigueira and Coito (2007) the problem was faced with all the generality. It is this approach we describe in this paper. It is based on the following assumptions:

- All the involved signals are defined over the whole set of real numbers.
- If the systems are observed for $t>t_0$, $t_0\in R$, our observation window is the Heaviside unit step function, $u(t-t_0)$.
- The initial conditions depend on the past input and output of the system, not on the actual or future.

We will put the Riemann-Liouville and Caputo derivatives in terms of this general frame work and discover which are the equations suitable for RL and C derivatives.

The paper outline proceeds as follows. The initial value problem is treated in three steps by:

stating the initial value problem (section 2 );

presenting the proposed solution (section 2 );

- inserting the RL and C initial conditions in the above general formulation (section 3).

At last, we present some conclusions.

## 2 The Initialization Problem

Let us assume that we have a fractional linear system described by a fractional differential equation like:

$$\sum_{n=0}^{N} a_n D^{\gamma_n} y(t) = \sum_{m=0}^{M} b_m D^{\gamma_m} x(t) \quad \gamma_n < \gamma_{n+1} \qquad (3)$$

where D means derivative and $\gamma_n$ n=0, 1, 2, … are derivative orders that we will assume to be positive real numbers. Usually $a_N$ is chosen to be 1. This equation is valid for every $t\in R$.

As it is well known, the solution of the above equation has two terms: the forced (or evoked) and free (or spontaneous). This second term depends only on the state of the system at the reference. This state constitutes or is related to the initial conditions. These are the values at t=0 of variables in the system and associated with stored energy. It is the structure of the system that imposes the initial conditions, not the eventual way of computing the derivatives.

The instant where the initial conditions are taken is very important, but it has not received much attention. In most papers, people don't care and use t=0. This happens in most mathematical books and papers (see the references in Lunberg et al). Others use $t=0^+$, motivated by the requirement of continuity of the functions for $t\geq 0$ and the initial value theorem. However and as pointed out by Lunberg et al (2007), we must retain the initial conditions at $t=0^-$, because the initial conditions represent the past of the system and do not have any relation with the future inputs. But this obliges us to change the ULT definition by starting the integration at $t=0^-$, instead of t=0, as it is customary. However the ULT has several disadvantages:

- It forces us to use only causal signals.
- Some of its properties lose symmetry, e.g. the translation and the derivation/integration properties.
- It does not treat easily the case of impulses located at t=0 (Hoskins, 1999).
- In the fractional case, it imposes on us the same set of initial conditions as the Riemann-Liouville case that can be a constraint.

To avoid these problems we apply the Bilateral Laplace Transform (BLT).

## 3 The General Approach

To fully understand our reasoning framework we must return back to the equation (3) and assuming we want to compute the system output under a given set of initial conditions. These initial conditions resulted from previous input signal has stopped at some instant, $t=t_p$, in the past. Thus the initial condition problem amounts to find the set of initial conditions for which the system described by model (3) with input v(t) presents the output w(t) (figure 1). This leads us to conclude that our initial conditions must verify:

$$y^{(\gamma_n)}(t_0) = y^{(\gamma_n)}(t_p) = y^{(\gamma_n)}(t_p-), \; n=0, \ldots, N$$

and

$$x^{(\gamma_n)}(t_0) = x^{(\gamma_n)}(t_p) = x^{(\gamma_n)}(t_p-), \; n=0, \ldots, M$$

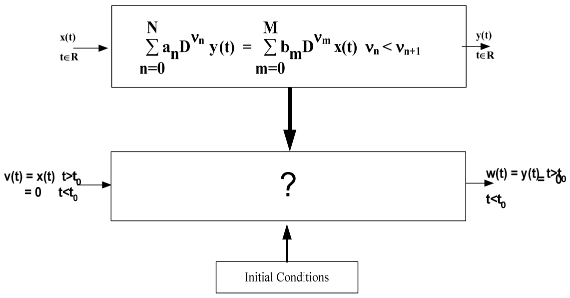


*Figure 1 – The initial condition problem*

These values were retained by the system and are going to influence the output when we excite the system with a new input at a given instant $t_0 > t_p$. As we said before, this forces us to conclude that the initial conditions are not influenced by this new input. Simultaneously it shows the impropriety of using the ULT, because the value of the integral does not depend on what happens at an isolated point, if the function assumes a finite value.

To deal with this case, we will consider t=0 as initial instant and functions with the general format

$$y(t) = \sum_{n=0}^{N} f_n(t)\, t^{\gamma_n} u(t) \qquad (4)$$

where $0 < \gamma_n < \gamma_{n+1}$ and the functions $f_n(t)$ (n=0, …, N) and their derivatives of orders less than or equal to $\gamma_N$ are assumed to be regular at t=0.

We will use a step by step differentiation to make the initial values appear and understand their meaning. Let y(t) be a signal given by (4) and consider the sequence $\beta_n$ by:

$$\beta_n = \gamma_n - \sum_{k=0}^{n-1} \beta_k\,,\ \ \beta_0=\gamma_0 \qquad (5)$$

A step by step procedure is used to derive the main result.

a) According to our assumptions $\beta_0$ is the least real for which $\lim_{t\to 0} \frac{y(t)}{t^{\beta_0}}$ is finite and nonzero. It exists because the $f_n(t)$ are regular at t=0. Its value is $\frac{y^{(\beta_0)}(0)}{\Gamma(\beta_0+1)}$. All the derivatives $D^{\alpha}y(t)$ $(\alpha<\beta_0)$ are continuous at t=0 and assume a zero value. The $\beta_0$ order derivative assumes the value $y^{(\beta_0)}(0)$ and we can construct the function

$$\varphi^{(\beta_0)}(t) = [y(t).u(t)]^{(\beta_0)} - y^{(\beta_0)}(0)u(t) \qquad (6)$$

that is continuous and assumes a zero value at t=0.

b) Now, $\beta_1$ is the least real for which $\lim_{t\to 0} \frac{\varphi^{(\beta_0)}(t)}{t^{\beta_1}}$ is finite and nonzero. Let it be $\frac{y^{(\beta_0+\beta_1)}(0)}{\Gamma(\beta_1+1)}$. Thus the $\beta_1$ order derivative of $\varphi^{(\beta_0)}(t)$ is given by:

$$\varphi^{(\beta_0+\beta_1)}(t) = [y(t).u(t)]^{(\beta_0+\beta_1)} - y^{(\beta_0)}(0)\,\delta^{(\beta_1-1)}(t) - y^{(\beta_0+\beta_1)}(0)\,u(t) \qquad (7)$$

which is again continuous at t=0.

c) Considering $\beta_2$ to be the least real for which $\lim_{t\to 0} \frac{\varphi^{(\beta_0+\beta_1)}(t)}{t^{\beta_2}}$ is finite and nonzero. Let it be $\frac{y^{(\beta_0+\beta_1+\beta_2)}(0)}{\Gamma(\beta_2+1)}$. Thus

$$\varphi^{(\beta_0+\beta_1+\beta_2)}(t) = [y(t).u(t)]^{(\beta_0+\beta_1+\beta_2)} - y^{(\beta_0)}(0)\,\delta^{(\beta_1+\beta_2-1)}(t) - y^{(\beta_0+\beta_1)}(0)\,\delta^{(\beta_2-1)}(t) - y^{(\beta_0+\beta_1+\beta_2)}(0)u(t) \qquad (8)$$

is again continuous at t=0.

d) Repeating this procedure yields a function:

$$\varphi^{(\gamma_N)}(t) = [y(t).u(t)]^{(\gamma_N)} - \sum_{0}^{N-1} y^{(\gamma_m)}(0)\,\delta^{(\gamma_N-\gamma_m-1)}(t) \qquad (9)$$

that is not continuous at t=0, but which can be made continuous if we subtract it $y^{(\gamma_N)}(0)u(t)$. Using this procedure in both members of equation (3) leads to the initial condition complete equation

$$\sum_{i=0}^{N} a_i\,.[y(t).u(t)]^{(\gamma_i)} = \sum_{i=0}^{M} b_i\,.[x(t).u(t)]^{(\gamma_i)} + \sum_{i=1}^{N} a_i\,.\sum_{0}^{i-1} y^{(\gamma_m)}(0)\,\delta^{(\gamma_i-\gamma_m-1)}(t) - \sum_{i=1}^{M} b_i \sum_{0}^{i-1} x^{(\gamma_m)}(0)\,\delta^{(\gamma_i-\gamma_m-1)}(t) \qquad (10)$$

Equation (10) states the general formulation of the initial value problem solution.

## 4 Special Cases

### 4.1 Riemann-Liouville

The left Riemann-Liouville fractional derivative as it is commonly presented can be represented by the following double convolution (Ortigueira et al, 2005)

$$f_{RL}^{(\alpha)}(t) = \delta_+^{(n)}(t) * \left\{ f(t) * \delta_+^{(-\nu)}(t) \right\}$$

where $\alpha = n - \text{V}$, $\delta_+^{(n)}(t)$ is the $n^{th}$ derivative of the Dirac impulse, and

$$\delta_+^{(-\nu)}(t) = \frac{t^{\nu-1}}{\Gamma(\nu)} u(t),\ 0 < \nu < 1$$

In terms of the operator D, we can write:

$$f_{RL}^{(\alpha)}(t) = D\{D[D \ldots D^{-V}]\}f(t)$$

So, we have an integration (negative order derivative) followed by a sequence of N order one derivatives. This leads to $\beta_0 = \gamma = -\text{V}$ and $\beta_i = 1$, and $\gamma_i=\gamma+i$, for I = 1, …, N. Then,

$$\varphi^{(N+\gamma)}(t) = [y(t).u(t)]^{(N+\gamma)} - \sum_0^{N-1} y^{(m+\gamma)}(0)\delta^{(N-1-m)}(t) \quad (11)$$

and

$$LT[\varphi^{(N+\gamma)}(t)] = s^{N+\gamma}Y(s) - \sum_0^{N-1} y^{(m+\gamma)}(+)\, s^{N-m-1} \quad (12)$$

With $\alpha=N+\gamma$, this relation can be rewritten as:

$$LT[\varphi^{(\alpha)}(t)] = s^{\alpha}Y(s) - \sum_0^{N-1} y^{(\alpha-1-i)}(0)\, s^{i} \qquad (13)$$

that is the current Riemann-Liouville solution. With the above set orders, we obtain for the initial condition complete equation

$$\sum_{n=0}^{N} a_n D^{\gamma+n} y(t) = \sum_{m=0}^{M} b_m D^{\gamma+m} x(t) + $$
$$+ \sum_{i=1}^{N} a_i \,.\sum_0^{i-1} y^{(\gamma+m)}(0)\delta^{(i-m-1)}(t) -$$
$$- \sum_{i=1}^{M} b_i \sum_0^{i-1} x^{(\gamma+m)}(0)\delta^{(i-m-1)}(t) \qquad (14)$$

From this result, we immediately conclude that the RL initial conditions are suitable for solving equations of the following format:

$$\sum_{n=0}^{N} a_n D^{\gamma+n} y(t) = \sum_{m=0}^{M} b_m D^{\gamma+m} x(t) \qquad (15)$$

that is a very restrict class.

### 4.2 Caputo

Similarly to the RL case, the left Caputo fractional derivative as it is commonly presented can be represented by the following double convolution (Ortigueira et al, 2005

$$f_C^{(\alpha)}(t) = \left\{ f(t) * \delta_+^{(N)}(t) \right\} * \delta_+^{(-\nu)}(t)$$

In terms of the operator D, we can write:

$$f_C^{(\alpha)}(t) = D^{-V}\{D[D \ldots D]\}f(t)$$

corresponding to a sequence of N order one derivatives and an integration. The Caputo case is not in the framework considered in section 3. In fact, we considered there that the $\gamma_n$ (n=0, …,, N) is an increasing sequence. In Caputo differentiation, we have $\gamma_n = n$ for (n=0, …, N-1) and $\gamma_N = N-\varepsilon$ with $0<\varepsilon<1$. However, the integration does not introduce non zero initial conditions, we have:

$$\varphi^{(\gamma_N)}(t) = [y(t).u(t)]^{(\gamma_N)} - \sum_0^{N} y^{(i)}(0)\delta^{(N-i-1-\varepsilon)}(t) \quad (16)$$

or, putting $\alpha=N-\varepsilon$;

$$\varphi^{(\alpha)}(t) = [y(t).u(t)]^{(\alpha)} - \sum_0^{N} y^{(i)}(0)\delta^{(\alpha-i-1)}(t) \qquad (17)$$

that is the usual way of presenting the C derivative. With this result and following a procedure similar to the one used in the Caputo case, we can write:

$$D^{N-\varepsilon}y(t) + \sum_{n=0}^{N-1} a_n D^n y(t) = b_0 D^{M-\varepsilon}x(t) + \sum_{m=0}^{M-1} b_m D^n x(t) +$$
$$+ \sum_{i=1}^{N} a_i \,.\sum_0^{i-1} y^{(j)}(0)\delta^{(N-j-1-\varepsilon)}(t) -$$
$$- \sum_{i=1}^{M} b_i \sum_0^{i-1} x^{(j)}(0)\delta^{(N-j-1-\varepsilon)}(t) \qquad (18)$$

So and as in the RL case, the C derivative is suitable for dealing with equations with the general format :

$$D^{N-\varepsilon}y(t) + \sum_{n=0}^{N-1} a_n D^n y(t) = b_0 D^{M-\varepsilon}x(t) + \sum_{m=0}^{M-1} b_m D^n x(t) \qquad (19)$$

that represents again a very restrict class of systems.

### 4.3 The rational order case

If all the orders in (3) are rational we can always put them as multiple of a given rational $\gamma$:

$$\gamma_i=i\gamma, \text{ for } i=0,1, \ldots, N.$$

We have: $\beta_0=0$, $\beta_i=\gamma$, for i=1, …, N-1. Then, (9) will be transformed into

$$\varphi^{(n\gamma)}(t) = [y(t).u(t)]^{(n\gamma)} - \sum_0^{n-1} y^{(m\gamma)}(0)\delta^{(n-i)\gamma-1}(t) \quad (20)$$

that inserted in (3), gives

$$\sum_{i=0}^{N} a_i .[y(t).u(t)]^{(i\gamma)} = \sum_{i=0}^{M} b_i .[x(t).u(t)]^{(i\gamma)} +$$

$$+ \sum_{i=1}^{N} a_i . \sum_{j=0}^{i-1} y^{(j\gamma)}(0).\delta^{((i-j)\gamma-1)}(t) -$$

$$- \sum_{i=1}^{M} b_i \sum_{j=0}^{i-1} x^{(j\gamma)}(0).\delta^{((i-j)\gamma-1)}(t) \qquad (21)$$

This is also valid even if $\gamma$ is not rational. This means that (20) represents a very large class of systems. This equation can be solved using the BLT.

### 5 Conclusions

We presented a general approach to the solution of the initial condition problem that appears quite naturally and is independent from the way the derivatives are computed. We looked into the Riemann-Liouville and Caputo derivatives from this general point of view and obtained the classes of the equations suitable to be solved be means of Riemann-Liouville and Caputo derivatives. We conclude that they constitute very restrict sets.

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