STOCHASTIC RESONANCE IN COUPLED BISTABLE SYSTEMS

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Abstract
We consider a system of two coupled bistable systems driven by both periodic and noise sources, focusing mainly on stochastic resonance (SR). In the absence of coupling, we found two critical damping parameters: one for the onset of resonances, and another for which theses resonances are optimum. We demonstrate that the absence of resonances in the weak coupling regime, is solely due to the presence of chaos in the system. Turning on the coupling, we found that the strong coupling regime induces a coherence that manifests itself by the matching of the signal to noise ratios of both subsystems. Finally, we demonstrate that our system does not synchronize for any coupling parameter.

Key words
Stochastic Resonance, Chaos, Coupled stochastic oscillators, synchronization

1 Introduction

Among a large variety of phenomena which has been attracting researchers in coupled nonlinear systems over several decades, synchronization [Pikovsky, Rosenblum and Kurths, 2001], chaos and bifurcations structures [Kozlowski, Parlitz and Lauterborn, 1995] have been the most prominent. The phenomenon of stochastic resonance however has been thouroughly and mostly explored in single oscillator system [Gammaitoni, Hänggi, Jung and Marschensoni, 1998], including chemical reactions, bistable ring lasers, semiconductors devices, and mechanoreceptor cells in the tail of the grayfish. This now well-established effect requires three fundamental ingredients: (i) a weak coherent signal, (ii) a noise source which is inherent to the system or which is added externally to the signal, and (iii) an energetic activation barrier. In the absence of noise, the signal should be weak enough such that the effect of signal-induced switching must not be observed. Likewise, the noise-induced switching should not be appreciable in the absence of the signal. It is the interplay of both the signal and the noise that results in a sharp enhancement of the power spectrum within a narrow range about the forcing frequency. This observation was explained by relating the forcing frequency with the switch rate (Kramer's rate) of the unperturbed system [Benzi, Parisi, Sutera, and Vulpiani, 1989]. To distinguish this to the dynamical resonance, one speaks of stochastic resonance (SR). Due to its simplicity and robustness, SR has been implemented by mother nature on almost every scale, thus attracting interdisciplinary interest from physicists, geologists, engineers, biologists and medical doctors, who nowadays use it as an instrument for their specific purposes[Wellens, Shatokhin and Buchleitner, 2004]. The first experimental observation of SR was performed while investigating the noise dependence of the spectral line of an ac-driven Schmitt-Trigger [Fauve and Heslot, 1983]. Since then SR has grown into a rapidly developing, interdisciplinary field of research, with numerous experimental observations in biological, laser, electronic and even quantum systems.

Although SR has been largely explored in various dynamical systems [Gammaitoni, Hänggi, Jung and Marschensoni, 1998; Wellens, Shatokhin and Buchleitner, 2004], little has been done for coupled stochastic systems [Bulsara and Schmera, 1993; Neiman and Schimansky-Geier, 1995; Gandhimathi, Rajasekar and Kurths, 2006; Anishchenko, Astakhov, Neiman, Vadivasova and Schimansky-Geier, 2007]. The case of coupled underdampled stochastic bistable systems, which is ours, has hitherto not yet been considered.

In this paper, we demonstrate the constructive role of noise assisted by a weak signal in a system in which chaos plays a role. Such a system has been considered by [Neiman and Schimansky-Geier, 1995] where SR has been studied. Here we revisit the same system but weakly damped, i.e., the inertia plays a major role, thereby rendering the system richer in that chaos is likely to show up for some parameters values. The coupling parameter, the noise strength and the signal's frequency and amplitude are the key parameters. The paper is organized as follows: section 2 is devoted to the description of the model, while section 3 and 4 present our results and discussions, and section 5 concludes the paper.

2 The model system

The system that we are interested in, consists of two mutually coupled bistable underdamped oscillators which are forced by a periodic signal and statistically independent noise sources. This system is thus governed by the following dimensionless stochastic differential equations,

$$\ddot{x} = -\gamma\dot{x} + \frac{dV(x)}{dx} + k(y-x) + \sqrt{2D}\xi_x(t) + F(t) \quad (1)$$

$$\ddot{y} = -\gamma\dot{y} + \frac{dV(y)}{dy} - k(y-x) + \sqrt{2D}\xi_y(t) + F(t) \quad (2)$$

where k is the coupling strength, γ the damping parameter, D the noise intensity of two independent Gaussian white noise $\xi_x(t)$ and $\xi_y(t)$

$$\langle \xi_x(t)\xi_y(t')\rangle = 2\, D\, \delta_{xy}(t-t'), \quad (3)$$

which are uncorrelated with zero-mean. The driving signal, $F(t) = A_0\cos(\Omega\, t + \Phi)$, is characterized by the amplitude A_0, the frequency Ω and the phase Φ. The potentials of the two subsystems $V(x) = a_1\, x^2/2 - b_1\, x^4/4$ and $V(y) = a_2\, y^2/2 - b_2\, y^4/4$ are sketched in Fig. 1, with parameters set to $a_1 = b_1 = 1$ and $a_2 = 1, b_2 = 1.5$. This choice, that determines the Kramer's rates, leads to two different activation barrier energies $\Delta V_x = 0.17$ and $\Delta V_y = 0.25$. For the purposes of stochastic resonance, we fix the driving amplitude A_0 smaller than the above activation barrier energies, so to avoid switchings that are due solely to the driving force. The relaxation frequencies of the two subsystems are thus identical and equal to $\omega_x = \omega_y = \sqrt{2a_1}$. To allow for adiabatic driving, we set the modulation frequency smaller than the relaxation one, say $\Omega = \omega_x/20$. Considering the subsystem y, for zero noise $D = 0$ and for a moderate damping $\gamma = 0.25$, Figure 2 gives a vivid picture of no-switching with $A_0 = 0.1 < \Delta V_y$ (a), and switching with $A_0 = 0.15 > \Delta V_y$ (b).

We note in passing that considerable efforts have been put to understand the mechanism of stochastic resonance in a single oscillator, Eq. 1 or Eq. 2 with $k = 0$. It has been demonstrated that such a system may exhibit a new type of SR [Stocks, Stein and McClintock, 1993; Dykman, 1993; Kang, Xu and Xie, 2003], due to the approximate coincidence between the lowest-energy eigenfrequency and the driving frequency. This has been pointed out as a general phenomenon in all underdamped nonlinear oscillators. Conditions for the coexistence of both resonances have also been obtained.

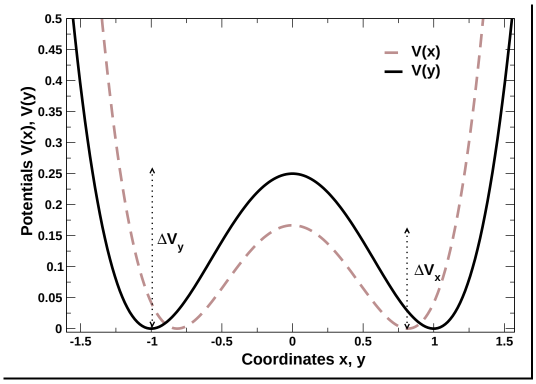

Figure 1. Potentials of the two subsystems V(x) (dashed) and V(y) (solid), with $\Delta V_x = 0.17$ and $\Delta V_y = 0.25$ their activation barrier energies, respectively.

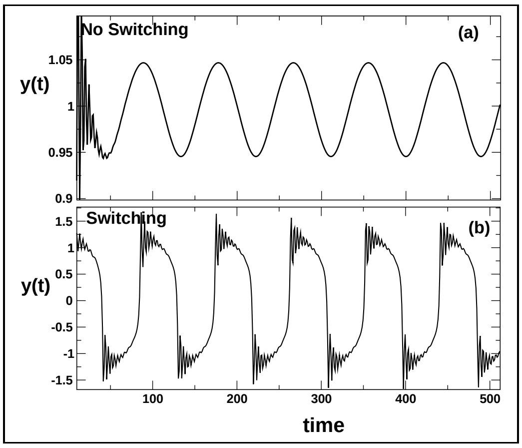

Figure 2. Protypical pictures of no-switching with $A_0 = 0.10$ (a) and switching with $A_0 = 0.50$ (b) of the subsystem y with $\Delta V_y = 0.25$ and for $k = 0.0, D = 0.0, \gamma = 0.25$.

It is also worth recalling that when the dissipation is effectively strong as compared to the inertia, system Eqs. 1, 2 reduce to the first order stochastic differential equations. Our system will be treated numerically using standard techniques. In what follows, we set Φ to zero, and explore the phenomena of stochastic resonance on uncoupled subsystems, for weak as well as strong limits of dissipation.

3 Stochastic resonance for uncoupled subsystems

Here only the noise and the driving force are present, thereby allowing for the study of the stochastic resonance. For this purpose, We consider two situations where (a) $A_0 = 0.10$ and, (b) $A_0 = 0.15$ that do not alllow switching in the absence of noise. Note also that in the absence of the driving, the stochastic switching time scale which is characterized by the Kramer's rates, $\Gamma_{K_{x,y}} \propto \exp(-\Delta V_{x,y}/D)$, is too long due to the weakness of the white Gaussian noise employed here, i.e. the noise only can not also induce switching. The time scale of switching being $1/\Gamma_{K_{x,y}}$, the time series for $D \in (0, 0.5)$ (not shown) do not exhibit any switching. When both the noise and the driving force are applied, the signal to noise ratio (SNR) is indeed the good candidate commonly used for detecting the constructive role of noise. Figure 3 shows SNR of both uncoupled subsystems in the weak damping regime $\gamma = 0.1$ (a) and in the strong one $\gamma = 0.75$ (b). In each panel, the lower curves in black are for $A_0 = 0.1$ while the upper ones in brown are for $A_0 = 0.15$, where SNR_x is in solid and SNR_y in dashed. It turns out that the cooperative effect of noise and driving force does not show up for the weaker dissipation regime (a) where chaos may be present. A systematic study for the dissipation, ranging from very small to very large values of γ has revealed two critical values, namely γ_{res} for which resonances appear, and γ_{opt} for which optimum resonances are reached. Fig. 4 depicts SNR_x for various values of γ as displayed on the panel. We found in this case that $\gamma_{res} = 0.08$, while $\gamma_{opt} = 0.5$. Finally the Lyapunov exponent, a good indicator of chaos in dynamical systems, has been plotted as function of γ in Figure 5. This clearly confirms that chaos is present in the weak damping regime and is thus responsible of the destruction of the resonances in the system. The mechanism preventing the appearance of resonances in that regime is a topic of its own and will be published elsewhere. At this point, the question that naturally arises is what would be the impact of the coupling on the phenomena observed here.

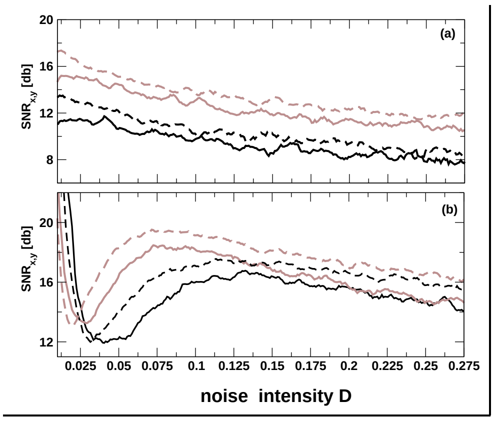

Figure 3. $SNR_{x,y}$ of the two uncoupled subsystems for $\gamma = 0.1$ (a) and $\gamma = 0.75$ (b). In each panel, black-solid (SNR_x) and black-dashed (SNR_y) for $A_0 = 0.1$, and brown-solid (SNR_x) and brown-dashed (SNR_y) for $A_0 = 0.15$. Unlike (a), resonances are clearly seen in (b).

4 Influence of the coupling parameter

Here the coupling is switched on and we want to see how the SNR studied in the above section can be affected. In this case where both subsystems are coupled, the synchronization phenomena is another important issue and will also be addressed.

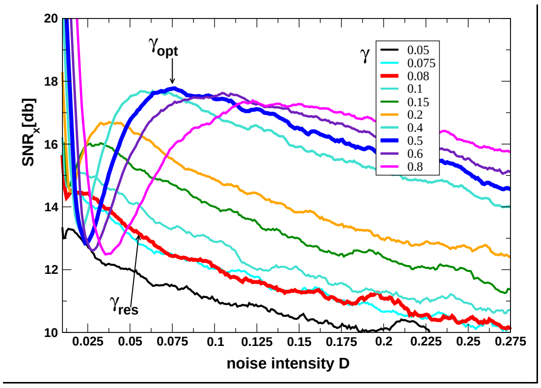

Figure 4. SNR_x as function of γ for $A_0 = 0.15$ and $k = 0.0$, showing the critical values for resonances $\gamma_{res} = 0.08$ and for the optimum of the resonances that can be reached $\gamma_{opt} = 0.5$.

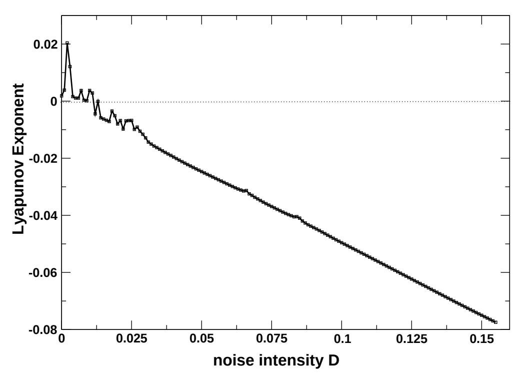

Figure 5. Lyapunov exponent of the subsystem x, for $k = 0$, $D = 0$, and $A_0 = 0.15$, indicating the presence of chaos in the weak dissipation regime.

4.1 Stochastic resonance

The stochastic resonance (SR), as already introduced in the preceding section, is essentially based on the exploration of the power spectra of subsystems $\bar{x}(\omega)$ and $\bar{y}(\omega)$. Because of the coupling, another quantity of interest is the coherence function defined as

$$\Gamma^2 = \frac{|S_{xy}(\omega)|}{S_{xx}(\omega)S_{yy}(\omega)} \tag{4}$$

where $S_{xy}(\omega)$ is the cross spectrum of processes $x(t)$, $y(t)$ and $S_{xx}(\omega)$, $S_{yy}(\omega)$ are the power spectra of $x(t)$, $y(t)$, respectively. Likewise the power spectrum of the collective processes $u(t) = x(t) + y(t)$ which reads

$$S_{uu}(\omega) = S_{xx}(\omega) + 2\,S_{xy}(\omega) + S_{yy}(\omega) \tag{5}$$

is also of interest. Here $S_{xy}(\omega)$ is the real part of the cross-spectrum between $x(t)$ and $y(t)$. The coherence function measures the coherence of the signals. This quantity reaches unity in case both processes become coherent.

Figs. 6 and 7 show the signal to noise ratio of the system as function of the coupling constant for parameters that in the absence of the coupling showed both no-resonances (Fig. 3(a)) and resonances (Fig. 3(b)). It turns out that the coupling has no influence on the resonances phenomena – these are well preserved for the whole range of the coupling constant. Remarkably, as the coupling increases, the SNR of both subsystems comes closer and closer and becomes identical at the stronger limit. Similarly the coherence Γ^2 (not shown) exhibits the same trend. As the coupling constant increases, this quantity goes to unity, indicating an optimum coherence of the system. What would this strong limit of coupling mean for the synchronization of the two subsystems?

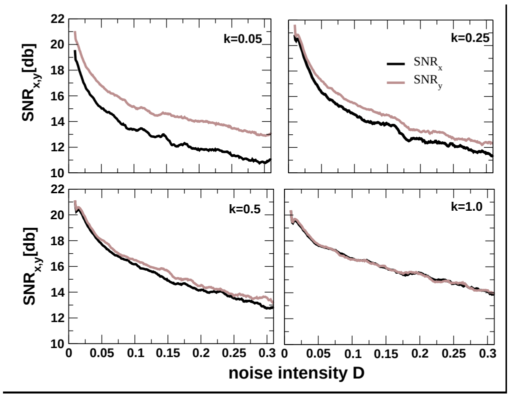

Figure 6. SNR$_{x,y}$ as function of the coupling constant at the weak dissipation limit $\gamma = 0.05$, for $A_0 = 0.15$. (a) k=0.05, (b) k=0.25, (c) k=0.5, (d) k=1.0.

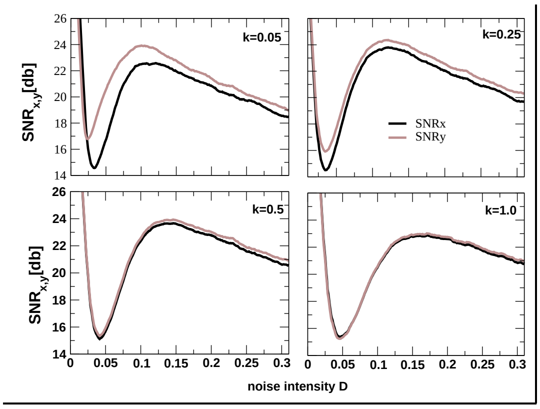

Figure 7. SNR$_{x,y}$ as function of the coupling constant at the strong dissipation limit $\gamma = 0.05$, for $A_0 = 0.15$. (a) k=0.05, (b) k=0.25, (c) k=0.5, (d) k=1.0.

4.2 Synchronization

Here we wish to find out in which parameter ranges the system may synchronize. To proceed we consider the following quantity

$$L(t) = \sqrt{(x(t) - y(t))^2 + (\dot{x}(t) - \dot{y}(t))^2} \qquad (6)$$

which is also proven as a good measure of the synchronization [Pikovsky, Rosenblum and Kurths, 2001]. We have thus played with the coupling parameter and no synchronization state has been achieved, even at the very strong coupling that show strong coherence on the SNR$_{x,y}$, see Figs. 6 and 7 for $k = 1$. Fig. 8 shows an example of the time series of the measure of the synchronization $L(t)$, for $A_0 = 0.15$, $\gamma = 1.0$, and $k = 1.0$. Similar outputs are found throughout the whole range of the coupling parameter, demonstrating that synchronization in our system is not reached as $L(t)$ does not converge to zero. What makes this synchronization difficult to achieve is presumably not only because the two subsystems are topologically not identical, see Fig. 1, but also because the system is nondeterministic due to the presence of noise. The opposite happens in deterministic coupled nonlinear systems where it is commonly known that the strong coupling enforces the synchronization of both subsystems [Vincent, Kenfack, Njah and Okinlade, 2005; Vincent and kenfack, 2008].

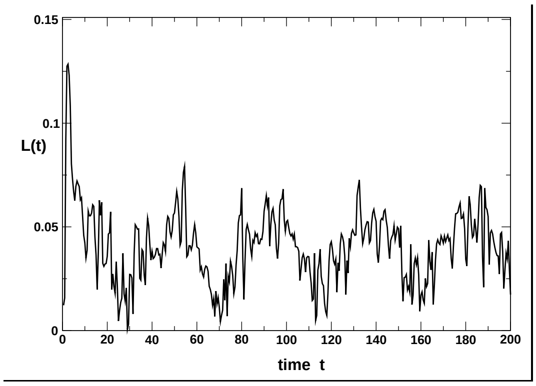

Figure 8. Measure of synchronization $L(t)$ as function of time in the strong coupling regime $k = 1$, for $A_0 = 0.15$, and $\gamma = 0.75$. This is a signature of nonsynchronization that prevails in the entire system.

5 conclusion

We have investigated the dynamics of two coupled periodically driven bistable systems submerged into a certain amount of noise. The phenomenon of stochastic resonance which has been central was already considered in a similar coupled bistable systems but at the limit of very strong damping [Neiman and Schimansky-Geier, 1995], i.e. the overdamped one in which chaos is not present. On one hand, we dealt with the dynamics of the uncoupled systems in which we showed that there are two important damping parameters; one that indicates the threshold for the appearance of resonances and another one for which these resonances are optimum. We further showed that the weak damping regime prohibits resonances, that is the role of noise is not constructive. With the help of the Lyapunov exponent, we found that this non appearance of resonances is due to the presence of chaos in the system. On the other hand when the coupling is turned on, we found that these resonances are in general not affected. The overall behavior is preserved. However, the strong coupling regime induces both signal to noise ratios to match, thereby showing a very high coherence. Along the same lines, exploring the whole range of the coupling parameter, the synchronization of the two subsystems has not been reached contrary to the coupled nonlinear deterministic systems in the strong coupling regime. The influence of the phase Φ of the driving on the rich phenomena observed, which is a future topic, will certainly contribute additional insights into the understanding of our system.

Acknowledgments

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