LIMIT SHAPES OF REACHABLE SETS OF SINGULARLY PERTURBED LINEAR CONTROL SYSTEMS

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Abstract
We study shapes of reachable sets of singularly perturbed linear control systems. The fast component of a phase vector is assumed to be governed by a hyperbolic linear system. We show that the shapes of reachable sets have a limit as the parameter of singular perturbation tends to zero. We also find a sharp estimate for the rate of convergence. A precise asymptotics for the support function of the normalized reachable sets is presented.

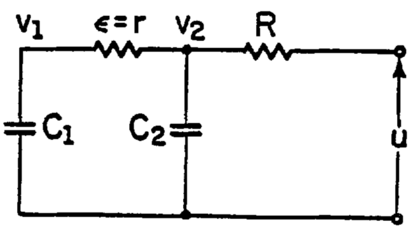
Key words
Singular linear dynamic system, reachable set.

1 Singular Linear Dynamic Systems

Consider the following dynamic system

$$\begin{aligned} \dot{x} &= Ax + By + Fu, \\ \varepsilon\dot{y} &= Cx + Dy + Gu, \ u \in U, \end{aligned} \tag{1}$$

where $\varepsilon > 0$ is a small parameter. Traditionally, the components $x \in \mathbb{X} = \mathbf{R}^m$ and $y \in \mathbb{Y} = \mathbf{R}^k$ of a phase vector $z = (x, y)$ are said to be slow and fast, respectively. The feasible motions start from zero at zero time: $z(0) = 0$. The dynamic systems of this form arise in abundance from physics and engineering: For instance, assume that in the RC-network drawn below the resistance $r = \varepsilon$ is much smaller than R. Then the network is described by dynamic equations of the form (1), where $x = \frac{C_1 v_1 + C_2 v_2}{C_1 + C_2}$ is the slow variable, and $y = v_2$ is the fast one.



All the system data, i.e., the matrices $A, \ldots, G$, and the set $U \subset \mathbb{U} = \mathbf{R}^r$ are functions of time and the parameter ε. In order to avoid unnecessary complications the sets U are assumed to be central symmetric convex bodies: $U = -U$, and U has nonempty interior. We hold the regularity and controllability hypotheses as stated in Sections 3, 4, and make the hyperbolicity assumption: For any t the matrix $D|_{\varepsilon=0}$ has no purely imaginary eigenvalues.

2 Problem Statement

Given an interval of time $[0, T]$, we study the reachable set $\mathcal{D}_\varepsilon(T)$ of system (1) as $\varepsilon \to 0$. Recall that the reachable set of a control dynamic system is the set of the ends at time T of all admissible trajectories. The reachable sets are central symmetric convex bodies in the phase space $\mathbb{V} = \mathbb{X} \times \mathbb{Y}$.

The issue on the limit behavior of reachable sets as $\varepsilon \to 0$ was addressed in [Dontchev and Slavov, 1988] under the assumption that $D|_{\varepsilon=0}$ is a stable matrix for any t. The main result is that the sets $\mathcal{D}_\varepsilon(T)$ have a limit with respect to the Hausdorff metric as $\varepsilon \to 0$, and the rate of convergence is $O(\varepsilon^\alpha)$, where $0 < \alpha < 1$ is arbitrary.

In our recent paper [Goncharova and Ovseevich, 2009], under the same assumptions we have shown that the rate of convergence is $\varepsilon \log 1/\varepsilon$. Moreover, we have isolated the main term of the form $\mathbf{c}\, \varepsilon \log 1/\varepsilon$ in the asymptotics for the support function of $\mathcal{D}_\varepsilon(T)$ so that the remainder is $o(\varepsilon \log 1/\varepsilon)$. Now, we are to extend these results to the unstable, hyperbolic case. The direct generalization of the above mentioned results is false: the sets $\mathcal{D}_\varepsilon(T)$ have no, in general, a natural limit. However, the notion of a shape of a convex body [Ovseevich, 1991] is a sure remedy, and allows us to state and prove a similar asymptotics for shapes of the reachable sets.

Our methods are based on the exact decomposition of slow, stable fast, and unstable fast variables. We also rely heavily on the averaging principle (ergodic theorem) for a progressive motion on a torus.

3 Regularity Assumptions

The singular control systems we consider belong to the following class:

$$\varepsilon \dot{z} = \mathfrak{A}z + \mathfrak{B}u,\ u \in U, \tag{2}$$

where $z \in \mathbb{V} = \mathbf{R}^n$, the system matrix $\mathfrak{A}$ and the matrix $\mathfrak{B}$ have the forms $\mathfrak{A} = \mathfrak{A}_0 + \varepsilon\mathfrak{A}_1$ and $\mathfrak{B} = \mathfrak{B}_0 + \varepsilon\mathfrak{B}_1$, respectively, the matrices $\mathfrak{A}_0$, $\mathfrak{B}_0$ do not depend on ε, and are Lipschitz continuous with respect to (wrt) t, while $\mathfrak{A}_1$ and $\mathfrak{B}_1$ are uniformly bounded in $[0, T] \times [0, \varepsilon_0]$. The set U of control vectors is a convex compact in $\mathbb{U}$. Its support function $h = H_U$ admits the decomposition $h_0 + \varepsilon h_1$, where $h_0 = h|_{\varepsilon=0}$ depends on t in the Lipschitz continuous way, and h_1 is uniformly bounded on the unit sphere. Remind that the support function of a convex compact set $U \subset \mathbb{U}$ is given by the formula: $H_U(\zeta) = \sup_{u\in U}\langle u, \zeta\rangle$, where $\zeta \in \mathbb{U}^*$, and uniquely defines the set U.

One can show that the Lipschitz continuity assumption is essential. There are examples, where all the system parameters are Hölder continuous with respect to time (with any positive exponent less than 1), but the rate of convergence is greater than $\varepsilon \log 1/\varepsilon$.

By default, any function is Borel measurable.

4 Controllability Condition

We assume that the following sufficient condition for controllability of system (1) is met:

Consider the *reduced* slow-fast system

$$\dot{x} = (A - BD^{-1}C)x + (F - BD^{-1}G)u, \tag{3}$$
$$\dot{y} = D(\tau)y + G(\tau)u, \tag{4}$$

where the parameter $\varepsilon = 0$, and the coefficients of the fast system are frozen at any $\tau \in [0, T]$. The condition due to Sannuti [Sannuti, 1977] requires that the systems (3) and (4) without control constraints are controllable.

If system (1) is controllable, then the reachable sets $\mathcal{D}_\varepsilon(T)$ are convex bodies, i.e., have non-empty interior.

The Sannuti controllability condition is stronger than proper controllability of system (1). When studying convergence of shapes of the reachable sets this condition is employed to ensure that the limit (properly normalized) reachable set is a body.

5 Shapes of Reachable Sets

The reachable sets $\mathcal{D}_\varepsilon(T)$ can be regarded [Ovseevich, 1991] as elements of the metric space $\mathbb{B}$ of central symmetric convex bodies in the phase space $\mathbb{V}$ with the Banach–Mazur distance ρ:

$$\rho(\Omega_1, \Omega_2) = \log(t(\Omega_1, \Omega_2)t(\Omega_2, \Omega_1)),$$
$$t(\Omega_1, \Omega_2) = \inf\{t \geq 1 : t\Omega_1 \supset \Omega_2\}.$$

The general linear group $GL(\mathbb{V})$ naturally acts on the space $\mathbb{B}$ by isometries. The factorspace $\mathbb{S}$ is called the space of shapes of central symmetric convex bodies, where the shape $\operatorname{Sh}\Omega \in \mathbb{S}$ of a convex body $\Omega \in \mathbb{B}$ is the orbit $\operatorname{Sh}\Omega = \{g\Omega : \det g \neq 0\}$ of the point Ω with respect to the action of $GL(\mathbb{V})$. The Banach–Mazur factormetric

$$\rho(\operatorname{Sh}\Omega_1, \operatorname{Sh}\Omega_2) = \inf_{g\in GL(\mathbb{V})} \rho(g\Omega_1, \Omega_2)$$

makes $\mathbb{S}$ into a compact metric space. The convergence of the reachable sets $\mathcal{D}_\varepsilon(T)$ and their shapes is understood in the sense of the Banach–Mazur metric. For two asymptotically equal functions with values in the space of convex bodies or the space of their shapes, the following notations are used: $\Omega_1(T) \sim \Omega_2(T)$, if $\rho(\Omega_1(T), \Omega_2(T)) \to 0$ as $T \to \infty$, and similarly $\operatorname{Sh}\Omega_1(T) \sim \operatorname{Sh}\Omega_2(T)$, if $\rho(\operatorname{Sh}\Omega_1(T), \operatorname{Sh}\Omega_2(T)) \to 0$ as $T \to \infty$. The convergence of convex bodies may be also understood in the sense of convergence of their support functions. The equivalence of the two definitions of convergence of convex bodies is established by the following lemma [Figurina and Ovseevich, 1999]:

Lemma 1. *A sequence $\Omega_i \in \mathbb{B}$ converges to $\Omega \in \mathbb{B}$ in the sense of the Banach–Mazur metric if and only if the corresponding sequence of the support functions $H_{\Omega_i}(\xi)$ converges to the support function $H_\Omega(\xi)$ pointwise and is uniformly bounded on the unit sphere σ in the dual space $\mathbb{V}^*$. The rates of convergence are the same: $\rho(\Omega_i, \Omega) \sim \sup_{\xi\in\sigma} |H_{\Omega_i}(\xi) - H_\Omega(\xi)|$ as $i \to \infty$.*

6 Splitting Dynamic System

Following [Kokotovich, 1984] we can simplify a regular system of the form (2) by using gauge transformations. In other words, we can apply a substitution $z = Xw$, where X is an invertible matrix, and get a new control system $\varepsilon\dot{w} = \mathfrak{A}'w + \mathfrak{B}'u$ such that

$$\mathfrak{A}' = X^{-1}\mathfrak{A}X - \varepsilon X^{-1}\dot{X}, \text{ and } \mathfrak{B}' = X^{-1}\mathfrak{B}. \tag{5}$$

An important observation is that if X is Lipschitz continuous in t, and has the form $X = X_0 + \varepsilon X_1$, where the invertible matrix X_0 does not depend on ε, while X_1 is uniformly bounded, then the transformation (5) preserves the regularity assumptions as stated in Section 3. Indeed, one can define the decomposition $\mathfrak{A}' = \mathfrak{A}'_0 + \varepsilon\mathfrak{A}'_1$ into the regular and small parts as follows: $\mathfrak{A}'_0 = X_0^{-1}\mathfrak{A}_0 X_0$, and

$$\mathfrak{A}'_1 = \frac{1}{\varepsilon}(X^{-1}\mathfrak{A}_0 X - X_0^{-1}\mathfrak{A}_0 X_0) + \\ + X^{-1}\mathfrak{A}_1 X - X^{-1}\dot{X}.$$

We will call this kind of gauge transformations the Lipschitz continuous ones. In other words, the condition means that X_0 is Lipschitz continuous wrt t, and X_1 is Lipschitz continuous in t with the Lipschitz constant of order $O(1/\varepsilon)$. These transformations do not change the shapes of reachable sets.

We aim at reducing the system matrix to a block-diagonal form to separate slow and fast variables. Our investigation is based upon the resulting split dynamic system.

Theorem 1. *Suppose that system* (1) *is regular. Then the system can be reduced by a Lipschitz continuous gauge transformation to the regular split form*

$$\begin{aligned} \dot{x} &= \widetilde{F}u, \\ \varepsilon\dot{y}_\pm &= \widetilde{D}_\pm y_\pm + \widetilde{G}_\pm u, \ u \in U, \end{aligned} \tag{6}$$

where for each t *the matrix* $\widetilde{D}_\pm|_{\varepsilon=0}$ *is strictly unstable/stable according to the index* $+/-$. *If the Sannuti condition of controllability holds for* (1)*, then the same is true for the split system* (6).

Here, a matrix is said to be (strictly) stable/unstable if the real parts of its eigenvalues are negative/positive.

In our previous paper [Goncharova and Ovseevich, 2009] the stable case was addressed, and then we used an approximate splitting which is insufficient in the unstable case. Here, we present a sketch of the proof.

System (1) can be presented in the form (2), where $z = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathfrak{A} = \begin{pmatrix} \varepsilon A & \varepsilon B \\ C & D \end{pmatrix}$, and $\mathfrak{B} = \begin{pmatrix} \varepsilon F \\ G \end{pmatrix}$.

The transformation is made in a sequence of steps: We do approximate splitting with a remainder of order $O(\varepsilon)$. For this we use the lower-triangular transformation $X = \begin{pmatrix} 1 & 0 \\ S & 1 \end{pmatrix}$, where $S = -D^{-1}C$, and 1 stands for a unit matrix. Then, the system matrix $\mathfrak{A}$ takes the form $\mathfrak{A} = \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} + O(\varepsilon)$. Furthermore, we apply a Lipschitz continuous gauge transformation Y in the space $\mathbb{Y}$ of fast variables such that the matrix $Y^{-1}DY$ has the block-diagonal form $Y^{-1}DY = \begin{pmatrix} D_+ & 0 \\ 0 & D_- \end{pmatrix}$, where the matrices $D_\pm$ are strictly unstable/stable according to the index $+/-$ provided that ε is small enough. Such a transformation Y does exist due to the hyperbolicity assumption. Thus, we can bring the system matrix $\mathfrak{A}$ to the regular form

$$\mathfrak{A} = \bar{\mathfrak{A}} + \varepsilon H, \tag{7}$$

where the block-diagonal matrix

$$\bar{\mathfrak{A}} = \left.\begin{pmatrix} D_+ & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D_- \end{pmatrix}\right|_{\varepsilon=0}$$

is Lipschitz continuous in t, and the matrix H is uniformly bounded. Then we kill the remainder εH in (7) by the lower and upper block-triangular transformations $R_\pm = 1 + \varepsilon\varphi^\pm$, where

$$\varphi^+ = \begin{pmatrix} 0 & \varphi_{+0} & \varphi_{+-} \\ 0 & 0 & \varphi_{0-} \\ 0 & 0 & 0 \end{pmatrix}, \quad \varphi^- = \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{0+} & 0 & 0 \\ \varphi_{-+} & \varphi_{-0} & 0 \end{pmatrix}.$$

In fact, we will not succeed completely, just make the remainder into a block-diagonal matrix of the same structure as the above $\bar{\mathfrak{A}}$ is. To this end the functions φ^- and φ^+ should be defined as solutions to special singular differential equations with strictly stable, respectively, unstable (for small ε) operators. By nonlinear version of Levinson–Tikhonov theorem the Cauchy problems with suitable initial conditions have uniformly bounded solutions such that $\varepsilon\dot{\varphi}^+$ and $\varepsilon\dot{\varphi}^-$ are bounded. This implies that the operators R_+, R_- are uniformly Lipschitz continuous wrt t.

By this point we reduced the original system by a Lipschitz continuous gauge transformation to the (regular) form

$$\begin{aligned} \dot{x} &= A'x + F'u, \\ \varepsilon\dot{y}_\pm &= D'_\pm y_\pm + G'_\pm u, \ u \in U, \end{aligned}$$

where the matrix $D'_\pm$ is strictly unstable/stable according to the index $+/-$. To bring the system to the form (6) it remains to kill A'. For this one can use the transformation $x \mapsto \Phi'(0,t)x$, where $\Phi'(s,t)$ is the fundamental matrix of the system $\dot{x} = A'x$.

A straightforward examination reveals that the Sannuti controllability condition is preserved under the gauge transformations used in the proof.

As far as we are interested in the shapes $\operatorname{Sh}\mathcal{D}_\varepsilon(T)$ of reachable sets, gauge transformations do not matter. We can assume in advance that the subject under consideration is the reachable set to the split system (6). That's why we drop tildes in what follows.

7 Normalization of Reachable Sets

To study the shapes $\operatorname{Sh}\mathcal{D}_\varepsilon(T)$ we find a matrix multiplier $N_\varepsilon = N_\varepsilon(T)$ such that the normalized reachable sets $\Omega_\varepsilon(T) = N_\varepsilon(T)\mathcal{D}_\varepsilon(T)$ possess a good limit behavior as $\varepsilon \to 0$.

For the split system (6), we can easily write down such a multiplier:

$$N_\varepsilon(T)(x, y_+, y_-) = (x, \Psi_{\varepsilon+}(0,T)y_+, y_-), \tag{8}$$

where $\Psi_{\varepsilon\pm}$ is the fundamental matrix of the unstable/stable system $\varepsilon\dot{y}_\pm = D_\pm y_\pm$.

8 Limit Objects

Denote by $H_\varepsilon(\xi, \eta_\pm)$ the support function of the normalized reachable set $\Omega_\varepsilon(T)$ to system (6), where $\varepsilon > 0$, and ξ, $\eta_\pm = (\eta_+, \eta_-)$ are dual to the variables x, $y_\pm$. Let $H_{U_t}(\zeta)$ be the support function of the set $U = U_t$, and $h = h_t = H_{U_t}|_{\varepsilon=0}$. Define the functions $h_i = h_{T_i}$, where $i \in \{+,-\}$, and $T_+ = 0$, $T_- = T$. Similarly, define the matrices $f_i = F(T_i)^*$, $g_i = G_i(T_i)^*$, $i \in \{+,-\}$, and $d_\pm = \mp D_\pm(T_\pm)^*$. In these definitions we assume $\varepsilon = 0$.

Finally, define the function

$$H_0(\xi,\eta_\pm) = \int_0^T h_t(F(t)^*\xi)\,dt + \\ + \sum_{i\in\{+,-\}} \int_0^\infty h_i(g_i e^{d_i t}\eta_i)\,dt,$$

where $F(t)$ is an abbreviation for $F(t)|_{\varepsilon=0}$. This is the support function of a convex compact $\Omega_0(T)$.

The set $\Omega_0(T) \subset \mathbb{X}\times\mathbb{Y}$ consists of pairs (x,y) such that

$$y = -D^{-1}Cx + w, \tag{9}$$

x runs over the reachable set at the instant T of the system

$$\begin{aligned} &\dot{x} = (A - BD^{-1}C)x + (F - BD^{-1}G)u, \\ &x(0) = 0,\ u(t) \in U_t, \end{aligned} \tag{10}$$

and $w = (w_+, w_-)$, where w_+ and w_- belong to the reachable sets in an infinite time of the systems

$$\begin{aligned} &\dot{w}_+ = -D_+(0)w_+ + G_+(0)u_+, \\ &w_+(0) = 0,\ u_+(t) \in U_0, \end{aligned} \tag{11}$$

$$\begin{aligned} &\dot{w}_- = D_-(T)w_- + G_-(T)u_-, \\ &w_-(0) = 0,\ u_-(t) \in U_T. \end{aligned} \tag{12}$$

Note that due to the Sannuti condition for controllability of the split system (6) the function $H_0(\xi,\eta_\pm)$ is strictly positive as soon as $(\xi,\eta_\pm) \neq 0$. This means exactly that the set $\Omega_0(T)$ is a body.

9 Main Results

Now we can state our main results, which imply, in particular, that the set $\Omega_0(T)$ is the limit normalized reachable set:

$$\Omega_0(T) = \lim_{\varepsilon\to 0} \Omega_\varepsilon(T).$$

Theorem 2. *Let $H_\varepsilon(\xi,\eta_\pm)$ be the support function of the normalized reachable set $\Omega_\varepsilon(T)$ to system (6), then $H_\varepsilon(\xi,\eta_\pm) \to H_0(\xi,\eta_\pm)$ uniformly on compacts as $\varepsilon \to 0$. Moreover, we have the (uniform) asymptotic equivalence:*

$$H_\varepsilon(\xi,\eta_\pm) = H_0(\xi,\eta_\pm) + O(\varepsilon\log 1/\varepsilon) \text{ as } \varepsilon \to 0.$$

Since the limit set is a convex body, we can restate Theorem 2 in the language of shapes:

Theorem 3. *The shapes $\operatorname{Sh}\mathcal{D}_\varepsilon(T)$ have the limit $\operatorname{Sh}\Omega_0(T)$ as $\varepsilon \to 0$. Moreover, the Banach–Mazur distance $\rho(\operatorname{Sh}\mathcal{D}_\varepsilon(T), \operatorname{Sh}\Omega_0(T))$ is $O(\varepsilon\log 1/\varepsilon)$.*

Further refinement of our main asymptotic result consists in finding the remainder in Theorem 2 in a more precise form $\mathbf{C}(\xi,\eta_\pm)\varepsilon\log 1/\varepsilon + o(\varepsilon\log 1/\varepsilon)$. We can do this under an extra assumption that the support function h is C^1-smooth outside the origin. This analytic assumption is equivalent to the geometric one of strict convexity of control sets U.

For any homogeneous of degree zero and continuous function Φ of $\zeta \neq 0$ consider the (ergodic) average

$$\operatorname{Av}\Phi\,(\eta_i) = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau \Phi(g_i e^{d_i t}\eta_i)\,dt, \tag{13}$$

where i stands for $+$ or $-$. One can show, with some effort, that this limit always exists. In particular, by putting $\Phi_i = \left\langle f_i\xi, \frac{\partial h_i}{\partial\zeta}\right\rangle$ we can define the averages $\operatorname{Av}\left\langle f_i\xi, \frac{\partial h_i}{\partial\zeta}\right\rangle(\eta_i)$. Now, we can define the functions

$$\mathbf{c}_i(\xi,\eta_i) = \frac{1}{\Lambda_i}\left(\operatorname{Av}\left\langle f_i\xi, \frac{\partial h_i}{\partial\zeta}\right\rangle(\eta_i) - h_i(f_i\xi)\right), \tag{14}$$

where $\Lambda_i = \Lambda(\eta_i)$ is the absolute value of the first Lyapunov exponent of the function $t \mapsto |e^{d_i t}\eta_i|$. Remind that the first Lyapunov exponent of a function $f(t)$ is the upper limit $\overline{\lim}_{t\to+\infty} \frac{\log|f(t)|}{t}$. In our case the limit coincides with the real part of an eigenvalue of d_i. Finally, we put

$$\mathbf{C}(\xi,\eta_\pm) = \mathbf{c}_+(\xi,\eta_+) + \mathbf{c}_-(\xi,\eta_-).$$

Theorem 4. *Assume that the support functions $h_\pm(\zeta)$ are C^1-smooth outside the origin. Then, if the argument $(\xi,\eta_\pm)$ is fixed, the following asymptotics holds for the support function of the normalized reachable set $\Omega_\varepsilon(T)$ of system (6):*

$$H_\varepsilon(\xi,\eta_\pm) = H_0(\xi,\eta_\pm) + \mathbf{C}(\xi,\eta_\pm)\,\varepsilon\log 1/\varepsilon + \\ + o\,(\varepsilon\log 1/\varepsilon). \tag{15}$$

10 Ideas for Proofs

Our arguments are based upon the explicit representation

$$H_\varepsilon(\xi,\eta_\pm) = \int_0^T H_{U_t}\Big(F(t)^*\xi + \frac{1}{\varepsilon}G_+(t)^*\Psi_{\varepsilon+}(0,t)^*\eta_+ \\ + \frac{1}{\varepsilon}G_-(t)^*\Psi_{\varepsilon-}(T,t)^*\eta_-\Big)\,dt \tag{16}$$

of the support function of the normalized reachable set. Here, the argument ε of the functions H_{U_t}, $F(t)$, $G_\pm(t)$ is omitted.

To understand the limit behavior of (16) we use the idea going back at least to [Dontchev and Veliov, 1983], which, basically, says that the reachable set of a linear control system can be asymptotically (as $\varepsilon \to 0$) decomposed in such a way that the stable, unstable and neutral parts of the reachable set are formed by using controls supported on nonoverlapping intervals of time. In particular, these parts are asymptotically independent so that the entire limit reachable set becomes the Cartesian product of its stable, unstable and neutral components.

We divide the time interval $I = [0, T]$ into the three subintervals $I_+ = [0, \delta_+]$, $I_0 = [\delta_+, T - \delta_-]$, and $I_- = [T - \delta_-, T]$, where $\delta_\pm$ are positive parameters such that $\delta_\pm \to 0$, while $\delta_\pm/\varepsilon \to \infty$ as $\varepsilon \to 0$. The controls supported on the "long" interval I_0 are responsible for the "slow" part of the reachable set, while the controls supported on the "short" intervals I_+ and I_- form the "fast" part of it. The proper choice of $\delta_\pm$ is crucial for the accuracy of approximation. We choose $\delta_\pm$ comparable with $\varepsilon \log 1/\varepsilon$.

11 Example

The above results can be illustrated by a simple example of a singularly perturbed unstable linear system:

$$\begin{aligned} \dot{x} &= u \\ \varepsilon\dot{y} &= y + u, \end{aligned}$$

where x, y, u are scalars, and $|u| \leq 1$. The support function $H_\varepsilon(\xi, \eta)$, where $\eta = \eta_+$, of the normalized reachable set $\Omega_\varepsilon(T)$ takes the form:

$$H_\varepsilon(\xi, \eta) = \int_0^T \left| \xi + \frac{1}{\varepsilon} e^{-t/\varepsilon} \eta \right| dt.$$

An easy calculation reveals that in this case the difference of the support functions of the prelimit and limit normalized reachable sets equals

$$\begin{aligned} \Delta H &= H_\varepsilon(\xi, \eta) - H_0(\xi, \eta) = \\ &= -2t_\varepsilon|\xi| - |\eta|(2e^{-t_\varepsilon/\varepsilon} - e^{-T/\varepsilon}) \end{aligned}$$

provided that $\xi\eta < 0$. Here, $t_\varepsilon = \varepsilon \log \frac{1}{\varepsilon} \frac{|\eta|}{|\xi|}$. Thus, for fixed ξ, η in this range, the difference ΔH has the form $-2|\xi|\varepsilon \log 1/\varepsilon + C\varepsilon + r$, where C is a constant, and the remainder r is exponentially small as $\varepsilon \to 0$. This proves that the estimates given in Theorems 2, 3 are sharp, and conform with Theorem 4.

12 Optimal Control Problem

Consider an optimal control problem

$$\phi(z(T)) \to \inf,$$

where $z(T) = (x(T), y(T)) \in \mathcal{D}_\varepsilon(T)$ is the terminal point of a feasible trajectory of system (1), and $\mathcal{D}_\varepsilon(T)$ is the reachable set. The scalar function ϕ is assumed to be Lipschitz continuous. We associate a "normalized" problem

$$\phi(N_\varepsilon(T)z(T)) \to \inf,\ z \in \mathcal{D}_\varepsilon(T) \qquad (17)$$

with the above one. In (17) $N_\varepsilon(T)$ is a matrix factor such that the normalized reachable sets $\Omega_\varepsilon(T) = N_\varepsilon(T)\mathcal{D}_\varepsilon(T)$ converge as $\varepsilon \to 0$. Note that in the stable case the normalization is superfluous. For systems in the split form (6), an approriate normalizing matrix multiplier can be given by (8). Define the "limit" minimization problem

$$\phi(z) \to \inf,\ z \in \Omega_0, \qquad (18)$$

where Ω_0 is the limit normalized reachable set. Denote by v_ε, v_0 the minimal values in problems (17), (18). Now we can restate Theorems 2, 3 in the form of an asymptotic relation

$$v_\varepsilon = v_0 + O(\varepsilon \log 1/\varepsilon).$$

It is instructive to compare the latter formula with the basic result of [Dontchev and Veliov, 1983].

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