From an Equilibrium to Quasiperiodicity in Non-Smooth Systems

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Abstract— **Considering a two-dimensional system of nonautonomous differential equations with discontinuous right-hand sides describing the behavior of a DC/DC converter with pulse-width modulated control, the paper demonstrates how a two-dimensional invariant torus can arise from a stable equilibrium point. We determine the chart of dynamical modes and show that there is a region of parameter space in which the system has a single stable node equilibrium point. Under variation of the parameters, this equilibrium may collide with a discontinuity boundary between two smooth regions in the phase space. When this happens, one can observe a variety of different bifurcation scenarios. One scenario is the continuous transformation of the stable equilibrium into a stable period-1 focus. A second is the transformation of the stable node equilibrium into an unstable period-1 focus, and the associated formation of a two-dimensional (ergodic or resonant) torus.**

I. Introduction

Many practical problems lead us to consider dynamical systems that are piecewise-smooth. Examples of such systems include switching circuits, mechanical systems with friction or impacts, and models of certain managerial and economic systems.

The phase trajectories of these systems are "sewed" together from separate smooth parts. As a parameter is varied, one of the arcs of a periodic trajectory may become tangent to a sewing surface, i.e., to a surface that divides the phase space into domains of different dynamics or crosses through the border of this surface. When this happens, the Floquet multipliers of the orbit can change abruptly, leading to a special class of nonlinear dynamic phenomena known as border-collision bifurcations [1], [2], [3], [4], [5].

A simple type of border-collision bifurcation is represented by the continuous transformation of a solution from one type into another with preservation of the cycle period. Here, the solution type is determined by the number of sections from which the cycle is sewed up [5]. However, more complicated phenomena are also possible, including new types of direct transitions to chaos, and the merging or disappearance of solutions of different types [6], [7], [8]. Border-collision related bifurcations also include sliding and grazing bifurcations [8], [9]

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Continuous-time piecewise-smooth systems also display bifurcations similar to the bifurcations of the equilibrium that one observes in smooth systems. Examples of such bifurcations include saddle-node, transcritical, pitchfork and Hopf-like bifurcations. In a recent paper [10], a detailed mathematical study of different types of non-smooth bifurcations scenarios for equilibrium points was performed. For each of the classical smooth bifurcations, a discontinuous bifurcation transition was found to exist, and this results was illustrated with appropriate low-dimensional examples.

The main feature of discontinuous bifurcations of equilibria is a jump of one eigenvalue (or a pair of eigenvalues) over the imaginary axis when the parameter passes through the bifurcation point. For nonsmooth continuous-time systems, Leine *et al.* [10] presented interesting examples of Hopf-like and Hopf-pitchfork bifurcation transitions. In both cases, a stable periodic orbit arises from the stable equilibrium point similarly to what one observes in smooth systems. However, for the discontinuous Hopf bifurcation, a pair of eigenvalues jumps over the imaginary axis.

Piecewise-smooth systems such as switching circuits [6], [11] and impact oscillators [12] can also display quasiperiodic behavior. In a couple of recent papers [13], [14] we have shown that border-collision bifurcations can lead to the birth of a stable closed invariant curve associated with quasiperiodic or phase-locked periodic dynamics. This transition resembles the well-known Neimark-Sacker bifurcation. However, rather than through a continuous crossing of a pair of complex-conjugate multipliers of the periodic orbit through the unit circle, the border-collision bifurcation involves a jump of the multipliers from the inside to the outside of the this circle.

Together with the results obtained by Leine *et al.* [10], this leads to the question if non-smooth systems can exhibit a combined bifurcation in which an invariant torus arises directly from the stable equilibrium point. The purpose of the present paper is to examine the mechanism of invariant torus birth under such conditions.

II. Description of the system

Let us consider the two-dimensional continuous-time piecewise-smooth nonautonomous system:

$$\dot{x} = \lambda_1 \left(x - \mathrm{K_F} \right); \quad \dot{y} = \lambda_2 \left(y - \mathrm{K_F} \right), \tag{1}$$

$$\mathrm{K_F} = \frac{1}{2} \left(1 + \mathrm{sign}(\xi) \right),$$

$$\xi = x(\tau) - \vartheta y(\tau) + \frac{q}{2\Omega} - \eta(t), \; \eta(t) = \frac{q}{\alpha\Omega} \left(t - \tau \right),$$

$$\vartheta = \lambda_1 / \lambda_2 .$$

These equations constitute the model of a DC/DC converter with pulse-width modulated control. A detailed description of the converter circuit, the functioning of the converter, and its areas of application can be found in our previous publications [6]. Here, one can also find an explanation of the model equations.

The (dimensionless) dynamic variables x and y are linear combinations of the currents and voltages in the converter filter, and ξ is a linear combination of the variables x and y. The function ξ represents the error signal, i. e., the deviation of the converter output voltage from its desired value at the beginning of each ramp cycle. $\lambda_{1,2}$, are the eigenvalues of the matrix for system (1). Based on the electronic parameters of a typical converter, we have chosen $\lambda_1 \approx -0.977$; $\lambda_2 \approx -0.232$; $q \approx 35.606$.

Operation of the converter is characterized by a cyclic switching of the circuit topology. The nonlinearity of the system is directly related to the switching processes as controlled by the applied pulse-width modulation.

The switching function $\mathrm{K_F}$ reacts to changes in the sign of the differences between the error signal ξ and the ramp function $\eta(t)$. The sawtooth function η is periodically repeated ramp function with the ramp period 1 , i. e., $\eta(t+1) = \eta(t)$.

The parameter q controls the amplitude of the sawtooth function and the value of the reference signal. $\tau = [t] = k - 1, k = 1, 2, \dots$ is the discrete time variable, $[t]$ being defined as a function that is equal to the integer value of its argument. α is an amplification constant and Ω is the normalized input voltage to the converter. In the following bifurcation analysis we shall use α and Ω as control parameters.

III. Bifurcation analysis

By integrating the equations of motion for the continuous-time system (1) ramp period by ramp period, investigation of this system can be reduced to the analysis of the two-dimensional piecewise-smooth stroboscopic map [6], [13].

$$\begin{gathered} x_k = e^{\lambda_1} x_{k-1} + e^{\lambda_1(1-z_k)} - e^{\lambda_1}; \\ y_k = e^{\lambda_2} y_{k-1} + e^{\lambda_2(1-z_k)} - e^{\lambda_2}, \\ k = 1, 2, \dots \end{gathered} \tag{2}$$

Here the variable z_k can be determined according to the expressions:

$$z_k = \begin{cases} 0, & \varphi_{k-1} \le 0; \\ \dfrac{\alpha\Omega}{q}\varphi_{k-1}, & 0 < \varphi_{k-1} < \dfrac{q}{\alpha\Omega}; \\ 1, & \varphi_{k-1} \ge \dfrac{q}{\alpha\Omega}\,, \end{cases}$$

$$\varphi_{k-1} = x_{k-1} - \vartheta y_{k-1} + \frac{q}{2\Omega}, \quad 0 \le z_k \le 1.$$

In the following investigations the parameters α and Ω are varied within the limits: $12.0 \le \alpha \le 40.0$ and $4.6 \le \Omega \le 7.6$.

When $0 < \Omega < \frac{q}{2(\vartheta-1)}(1 - 2/\alpha)$, oscillations that may arise at the beginning of a transient are damped. The value of $\mathrm{K_F}$ is constant and equal to unity, and the system behavior

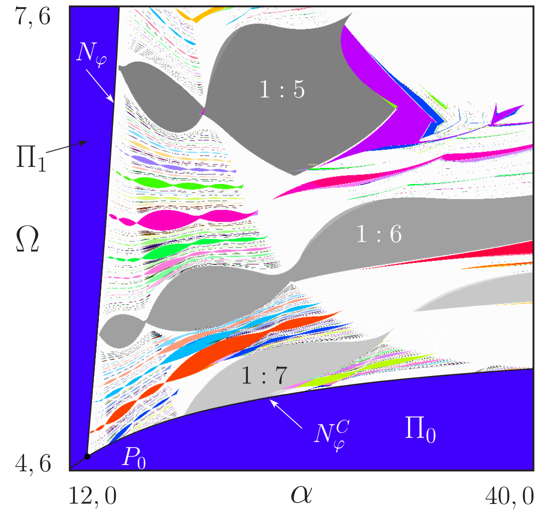


Fig. 1. Chart of dynamical modes in the parameter plane (α,Ω) for a DC/DC converter. Π_0 is the domain where only the stable equilibrium $(x_+, y_+) = (1, 1)$ exists. N_φ is line of Neimark–Sacker bifurcation. N_φ^C is the curve for the birth of oscillations from an equilibrium point. Π_1 is the domains of stability for period-1 orbit. White domains represent quasiperiodic or high-periodic cyclic solutions.

is described by the set of linear autonomous differential equations

$$\begin{gathered} \dot{x} = \lambda_1(x-1); \quad \dot{y} = \lambda_2(y-1); \\ x(\tau) - \vartheta y(\tau) + \frac{q}{2\Omega} - \eta(t) > 0 \end{gathered}$$

with a single equilibrium point $(x_-, y_-) = (1, 1)$. The eigenvalues $\lambda_{1,2}$ of the Jacobian matrix for this system are real and negative. Hence, the equilibrium is a stable node. Note that this equilibrium point for the original system (1) will be represented as a fixed point in the map (2). Other fixed points in the map represent periodic cycles in the original system.

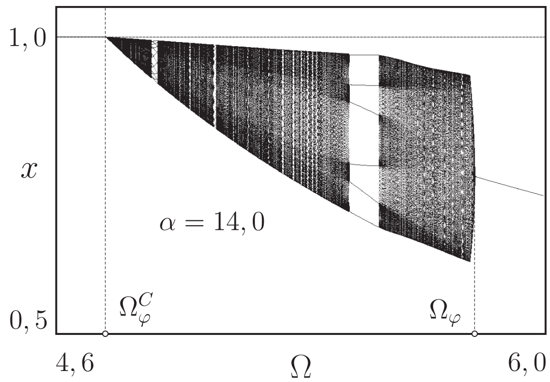
Oscillations arise when $1-\vartheta+\frac{q}{2\alpha\Omega}(\alpha-2) > 0$. Depending on the parameters α and Ω these oscillations can be periodic with a period multiple to the period of the external action, or they can be quasiperiodic or chaotic. The curve in the parameter plane (α, Ω) where oscillations arise from the equilibrium point is determined by the equation

$$1 - \vartheta + \frac{q}{2\alpha\Omega}(\alpha - 2) = 0. \tag{3}$$

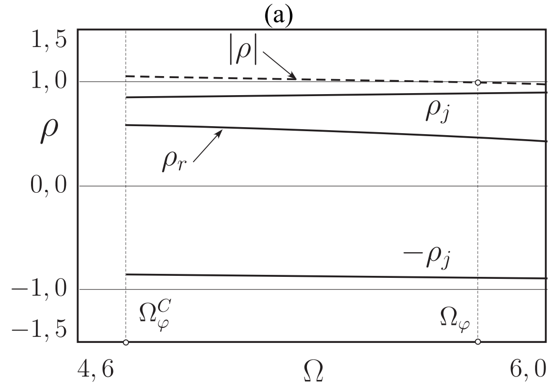
We will denote this curve as N_φ^C. For parameter values immediately above the bifurcation point, the oscillation amplitude is small, and the amplitude grows linearly as the system moves away from the bifurcation curve N_φ^C.

Figure 1 represents the chart of the dynamical modes within the plane of control parameters (α, Ω) where the bifurcation boundary (3) is marked as N_φ^C and N_φ is a line of Neimark–Sacker bifurcation. Below the curve N_φ^C (within domain Π_0) the system (1) has the single stable equilibrium $(x_-, y_-) = (1, 1)$.

The Neimark–Sacker bifurcation curve N_φ is supported by the curve N_φ^C in the point P_φ of codimension two. The



(a)



(b)

Fig. 2. Birth of quasiperiodic orbit from a stable equilibrium through a border-collision bifurcation on the curve N_φ^C.The section $\{(\alpha, \Omega) : \alpha = 14; 4.6 < \Omega < 6.0\}$. (a) Bifurcation diagram.(b) Multiplier diagram for the fixed point. $\Omega_\varphi^C \approx 4.746$ is the bifurcation value of the parameter Ω calculated according to expression (3).Ω_φ is the bifurcation point for the Neimark–Sacker bifurcation. $\Omega_\varphi \approx 5.801$.

coordinates of this point in the parameter plane (α, Ω) are determined by:

$$\alpha_* = \frac{(\lambda_1 - \lambda_2)(1 - e^{\lambda_1+\lambda_2})}{\lambda_1\lambda_2(e^{\lambda_1} - e^{\lambda_2})} + 2; \tag{4}$$

$$\Omega_* = \frac{q\lambda_2(1 - e^{\lambda_1+\lambda_2})}{2(\lambda_1 - \lambda_2)(1 - e^{\lambda_1+\lambda_2}) + 2\lambda_1\lambda_2(e^{\lambda_1} - e^{\lambda_2})}.$$

Between the curves N_φ^C and N_φ one can see a large number of periodic windows that correspond to resonance tongues. On the chart of dynamical modes, only the largest resonance tongues are visible. Some of these tongues are supported by the bifurcation curve N_φ^C.

Above the bifurcation curve (3), the map (2) has a fixed point corresponding to a period-1 switching cycle of the continuous-time system (1).

This fixed point can be found from the equation for one time iterated map (2). The stability of the fixed point is determined by the conditions:

$$\begin{cases} e^{\lambda_1+\lambda_2}\left(1 + \dfrac{\lambda_1\alpha\Omega(e^{\lambda_1 z} - e^{\lambda_2 z})}{qe^{(\lambda_1+\lambda_2)z}}\right) < 1; \\ \dfrac{e^{\lambda_1(1-z)} - e^{\lambda_1}}{1 - e^{\lambda_1}} - \vartheta\,\dfrac{e^{\lambda_2(1-z)} - e^{\lambda_2}}{1 - e^{\lambda_2}} \\ +\dfrac{q}{2\Omega\alpha}(\alpha - 2z) = 0. \end{cases} \tag{5}$$

When passing through the boundary (3) into the domain Π_0, the fixed point is replaced by the equilibrium point $(x_-, y_-) = (1, 1)$, and along the curve (3), the equation for the fixed point has the unique solution $x_* = 1$, $y_* = 1$. The eigenvalues $\rho_{1,2}$ of the Jacobian matrix for this solution (multipliers of the period-1 cycle) are complex-conjugate:

$$\rho_{1,2} = \rho_r \pm j\rho_j,$$

$$\rho_r = \frac{e^{\lambda_1} + e^{\lambda_2}}{2}; \quad \rho_j = \sqrt{\gamma(\alpha - 2) + \mu},$$

$$\mu = e^{\lambda_1+\lambda_2} - \frac{1}{4}(e^{\lambda_1} + e^{\lambda_2})^2;$$

$$\gamma = \frac{\lambda_1\lambda_2(e^{\lambda_1} - e^{\lambda_2})}{2(\lambda_1 - \lambda_2)}, \quad \alpha \gtrsim 2.7.$$

Along the bifurcation curve (3), the map (2) thus has a fixed point $(x_*, y_*) = (1, 1)$ with complex-conjugate multipliers.

As one can see, the real part of the multipliers $\rho_{1,2}$ does not depend on the parameters α, Ω but is constant $\rho_r \approx 0.5847$. Therefore, the stability of the fixed point $(x_*, y_*) = (1, 1)$ is determined only by the imaginary part of the multipliers. The condition for stability is determined from (5) as:

$$e^{\lambda_1+\lambda_2} + \lambda_1\frac{e^{\lambda_1} - e^{\lambda_2}}{1 - \vartheta + 0.5q/\Omega} < 1 \tag{6}$$

or

$$e^{\lambda_1+\lambda_2} + \lambda_1\frac{e^{\lambda_1} - e^{\lambda_2}}{2\vartheta - 2}(\alpha - 2) < 1.$$

From the condition (6) and the equation (3) one can see, that while changing the parameters α and Ω along the curve (3), the fixed point loses stability as we pass the point P_φ (see (4)), when a complex conjugated pair of multipliers crosses the unit circle.

Depending on the parameter values we may therefore observe two different bifurcation scenarios. The first option is the smooth transformation of the stable equilibrium (x_-, y_-) into a stable period-1 focus. This type of bifurcation takes place on that part of the curve N_φ^C that falls to the left of the point P_φ.

The second option is that the stable node equilibrium is transformed smoothly into an unstable focus. This bifurcation, which occurs along that part of the bifurcation curve N_φ^C that falls to the right of the point P_φ, results in the birth of a closed invariant curve for the discrete map (2). In the time-continuous system (1), the stable equilibrium ceases to exist and is replaced by an unstable period-1 focus cycle with Floquet multipliers that fall outside of the unit circle.

If the rotation number is irrational, the invariant curve is densely filled with points of the trajectory (the Poincaré section is a closed smooth curve), and the dynamics is quasiperiodic. Figure 2 shows the bifurcation and multiplier diagrams to illustrate the birth of a quasiperiodic orbit from the stable equilibrium.

When the rotation number is rational, the closed invariant curve contains a pair of cycles, one of which is stable, while the other is a saddle. The invariant curve itself is the union

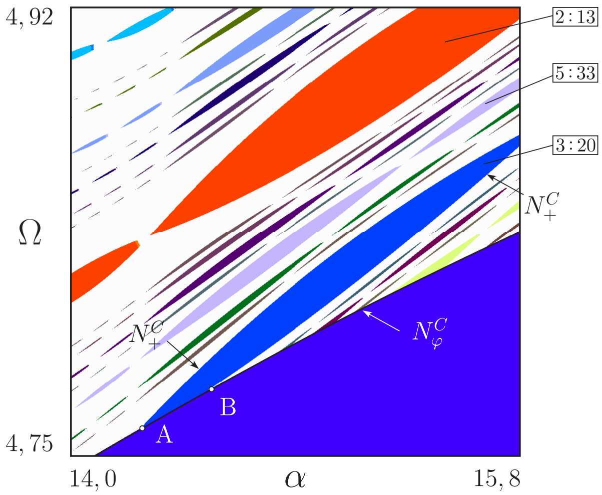


Fig. 3. Part of the chart of the dynamical modes near the 3 : 20. N_{+}^{C} are the border-collision fold bifurcation curves. A and B are codimension two points in which the curves N_{+}^{C} are supported by the line N_{φ}^{C}. At the transition across the segment AB saddle and stable cycles situated on the closed invariant curve are born from an equilibrium point.

of the unstable manifolds of the saddle orbit with the stable periodic orbit. Let us finally consider a few additional aspects of the transition from the equilibrium to a phase-locked invariant curve. Figure 3 shows part of the chart of dynamical modes containing the 3 : 20 resonance tongue. It is well-known that the resonance tongues in piecewise-smooth systems are bounded by border-collision fold bifurcation curves. In Fig. 3 such curves are denoted as N_{+}^{C}. As one can see from from the figure, the bifurcation boundaries N_{+}^{C} are supported by the curve N_{φ}^{C} in the codimension two points A and B. In the points of the segment AB a pair of cycles (stable and saddle) softly arise from the equilibrium point. These cycles are situated on the closed invariant curve. The diagrams in the Fig. 2 show that the typical size of the invariant curve (its ”diameter”) grows linearly from zero as the system moves away from the bifurcation point.

Figure 4 illustrates the birth of the invariant curve in the transition across the segment AB of the bifurcation curve N_{φ}^{C}. Here solid lines correspond to the stable cycle and dashed lines to the saddle. Figure 5 shows the phase portrait of the dynamical system (2) within the 3 : 20 resonance tongue for the parameter values $\alpha = 14.5$ and $\Omega = 4.78$. Black circles in this figure mark the points of the stable cycle and the white circles show the points of the saddle cycle. Here $W_{\pm}^{U}$ and $W_{\pm}^{S}$ are unstable and stable manifolds of the saddle periodic orbit, respectively.

IV. Conclusion

We considered the model of a DC/DC converter with pulse-width modulated control. The behavior of such a converter is described by a two-dimensional set of piecewise-linear nonautonomous differential equations. The chart of dynamical modes in the parameter plane (α, Ω) for this system was obtained through a detailed numerical and analytical study. When the value of parameter Ω is small enough,

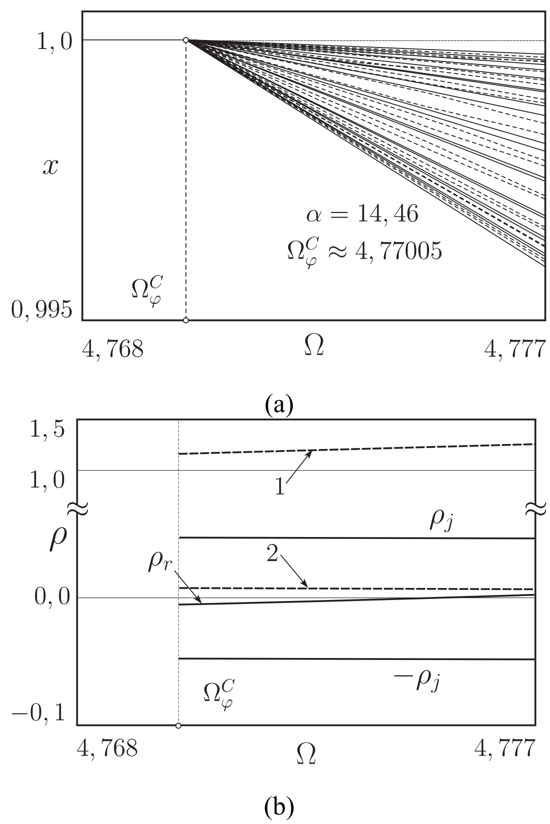


Fig. 4. Birth of stable and saddle period-20 cycles situated on the closed invariant curve from a stable equilibrium through a border-collision bifurcation on the curve N_{φ}^{C}. (a) Bifurcation diagram. Here solid lines correspond to the stable cycle and dashed lines to the saddle. (b) Multiplier diagram for 20-cycle. Solid lines mark the multipliers of the stable cycle and dashed lines 1 and 2 show the multipliers of the saddle cycle.

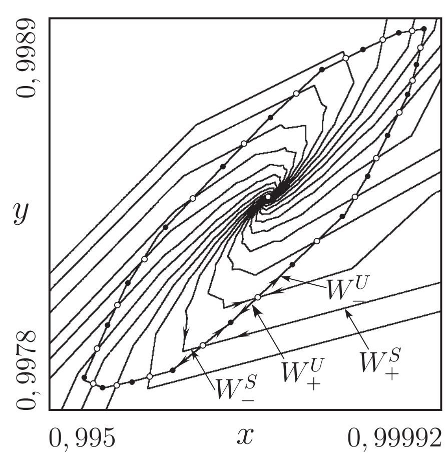


Fig. 5. Phase portrait of the map (2) within the 3 : 20 resonance tongue for the parameter values $\alpha = 14.5$ and $\Omega = 4.78$. Here, the black circles mark the stable cycle and the white circles mark the saddle ones. $W_{\pm}^{U}$ and $W_{\pm}^{S}$ are unstable and stable manifolds, respectively.

the continuous-time representation has a single stable node equilibrium. We showed that, under parameter variation, this equilibrium can undergo a novel type of border-collision bifurcation, in which a quasiperiodic or a phase-locked invariant torus softly arises from the stable equilibrium state.

It is important to note that the phenomena observed for the system (1) are not restricted to a single point in parameter space, but occur along that part of the bifurcation curve N_{φ}^{C} that falls to the right of the point P_{φ} (see Fig. 1). To the left of P_{φ} we observe a direct transition from the stable node equilibrium point to a stable period-1 focus. We also note that the above bifurcation scenarios do not exhaust the spectrum of strange bifurcation phenomena that our converter system can display.

The bifurcation phenomena observed for the system (1) are distinguished from a classic Hopf bifurcation by the following characteristics:

First, the transition is connected with the disappearance of the stable equilibrium point, when it collides with a discontinuity boundary between two smooth regions in the phase space. It is not connected with the loss stability of equilibrium point as it occurs in the classic Hopf bifurcation.

Second, the disappearance of the equilibrium point gives rise to two types of bifurcation behavior:

(i) the stable equilibrium disappears and is replaced by a stable period-1 orbit, the amplitude of which is growing linearly from zero as the system moves away from the bifurcation point. Similar to the discontinuous Hopf-like bifurcations, studied by Leine *et al.* [10], the stable equilibrium turns into an unstable focus, and this equilibrium is surrounded by the stable periodic orbit. In the considered case, a unique stable periodic-1 orbit exists after the bifurcation;

(ii) rather than as a stable node, the periodic cycle arising in the bifurcation is born as an unstable focus surrounded by a resonant or ergodic torus.

However, more complicated bifurcation phenomena are also possible, including multiple choice bifurcations [15], [16] in which several attractors “surrounding an unstable focus period-1 cycle” are created simultaneously as the stable equilibrium point disappears, direct transition from a stable node equilibrium to chaotic attractor.

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