

# SYNCHRONIZATION OF CHAOTIC SYSTEMS IN UNIDIRECTIONAL RING NETWORKS WITH DELAY

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## Abstract

In this paper, we consider the synchronization problem in unidirectional ring networks of chaotic systems with time-delay coupling. Firstly, we introduce the notion of strict semi-dissipativity and show by using the small-gain theorem that the trajectories of coupled systems satisfying the strict semi-dissipativity are bounded. Then we derive a sufficient condition for synchronization of the systems coupled in a unidirectional way by using a stability criterion for delay systems.

## Key words

Synchronization, Chaotic systems, Time-delays

## 1 Introduction

Synchronization phenomena are of interest to researchers in applied physics, biology, social sciences, engineering (Pecora and Carroll, 1990.; Strogatz and Stewart, 1993.; Nijmeijer and Mareels, 1997; Pikovsky *et al.*, 2001). More recently, applications of these phenomena to engineering have also been considered and analyzed via control theory (Huijberts *et al.*, 2007; Nijmeijer and Rodriguez-Angeles, 2003; Oguchi and Nijmeijer, 2005; Pogromsky *et al.*, 2002). On the other hand, in practical situations, time-delays caused by signal transmission affect the behavior of coupled systems. It is therefore important to study the effect of time-delay in existing synchronization schemes. Although the effect of time-delay in the synchronization of coupled systems has been investigated both numerically and theoretically by a number of researchers, these works concentrate on synchronization of systems with a coupling term typically described by  $K(x_i(t - \tau) - x_j(t - \tau))$  or  $K(Cx_i(t - \tau) - Cx_j(t - \tau))$  (Amano *et al.*, 2006) and there are only few results for the case in which the coupling term is described by  $K(x_i(t) - x_j(t - \tau))$ . The former requires that a feedback in each system has the same length of time-delay as the transmittal delay, while the latter does not need such a delayed feedback. For the latter case, however, as the cou-

pling term does not vanish in synchronous condition, even if uncoupled each system is bounded, the coupled systems are not necessarily bounded. Therefore the synchronization problem has seldom been studied. We have already considered the problem for two chaotic systems bidirectionally coupled with the coupling term  $K(x_i(t) - x_j(t - \tau))$  and derived sufficient conditions for synchronization (Yamamoto *et al.*, 2007; Oguchi *et al.*, 2007).

In this paper we consider synchronization of identical chaotic systems which are unidirectionally coupled in a ring with state/output feedback described by  $K(y_i(t) - y_j(t - \tau))$ . First, we introduce the notion of strict semi-dissipativity and show by using the small-gain theorem that coupled systems satisfying the strict semi-dissipativity are bounded. Then we derive a sufficient condition for synchronization of the systems coupled with unidirectional communication by using the stability criterion for delay systems.

## 2 Preliminaries

In this section, we review some results derived in our previous work (Yamamoto *et al.*, 2007). Throughout this paper,  $\|\cdot\|$  denotes the Euclidean norm. For a vector function  $v(t) : [0, \infty) \rightarrow \mathbb{R}^m$ , if  $\|v\|_\infty \triangleq \sup_{t \geq 0} \|v(t)\| < \infty$ , then we denote  $v \in \mathcal{L}_\infty^m$ . Define a continuous norm by  $\|\phi\|_c \triangleq \max_{-\tau \leq \theta \leq 0} \|\phi(\theta)\|$  for a vector function  $\phi : [-\tau, 0] \rightarrow \mathbb{R}^n$ .

### 2.1 Semi-passivity and semi-dissipativity

Consider the nonlinear system

$$\dot{x}_i(t) = f_i(x_i, u_i), \quad y_i(t) = h_i(x_i) \quad (t \geq 0) \quad (1)$$

with state  $x_i \in \mathbb{R}^n$ , input  $u_i \in \mathbb{R}^m$ , output  $y_i \in \mathbb{R}^m$ ,  $f_i : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  and  $h_i : \mathbb{R}^n \rightarrow \mathbb{R}^m$ .

According to semi-passivity as defined in (Pogromsky *et al.*, 2002), we introduce strict semi-passivity and

strict semi-dissipativity as follows.

**Definition 1 (strict semi-passivity).** System (1) is said to be strictly semi-passive, if there exist a  $C^1$ -class function  $V_i : \mathbb{R}^n \rightarrow \mathbb{R}$ , class  $\mathcal{K}_\infty$  functions  $\underline{\alpha}_i(\cdot)$ ,  $\bar{\alpha}_i(\cdot)$  and  $\alpha_i(\cdot)$  satisfying

$$\begin{aligned} \underline{\alpha}_i(\|x_i\|) &\leq V_i(x_i) \leq \bar{\alpha}_i(\|x_i\|) \\ \dot{V}_i(x_i) &\leq -\alpha_i(\|x_i\|) - H(x_i) + y_i^T u_i \end{aligned}$$

for all  $x_i \in \mathbb{R}^n$ ,  $u_i \in \mathbb{R}^m$ ,  $y_i \in \mathbb{R}^m$ , where the function  $H(x_i)$  satisfies the following condition:

$$\|x_i\| \geq \eta_i \quad \Rightarrow \quad H(x_i) \geq 0$$

for a positive real number  $\eta_i$ .

**Definition 2 (strict semi-dissipativity).** System (1) is said to be strictly semi-dissipative with respect to the supply rate  $w_i(u_i, y_i)$ , if there exist a  $C^1$ -class function  $V_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}$  and class  $\mathcal{K}_\infty$  functions  $\underline{\alpha}_i(\cdot)$ ,  $\bar{\alpha}_i(\cdot)$  and  $\alpha_i(\cdot)$  satisfying

$$\underline{\alpha}_i(\|x_i\|) \leq V(x_i) \leq \bar{\alpha}_i(\|x_i\|) \quad (2)$$

$$\dot{V}_i(x_i) \leq -\alpha_i(\|x_i\|) + w_i(u_i, y_i) - H(x_i) \quad (3)$$

for all  $x_i \in \mathbb{R}^n$ ,  $u_i \in \mathbb{R}^p$ ,  $y_i \in \mathbb{R}^m$ , where the function  $H(x_i)$  satisfies

$$\|x_i\| \geq \eta_i \quad \Rightarrow \quad H(x_i) \geq 0 \quad (4)$$

for some positive real number  $\eta_i$ .

**Remark 1.** The system is strictly semi-passive if the supply rate  $w_i(u_i, y_i) = y_i^T u_i$  for all  $u_i \in \mathbb{R}^m$ .

For a strictly semi-dissipative system, the following lemma can be proved in a similar way as the argument of the input-to-state stability (ISS) in (Isidori, 1999).

**Lemma 1.** Suppose that system (1) is strictly semi-dissipative with respect to  $w(u_i, y_i) \leq \beta_i(\|u_i\|)$  where  $\beta_i \in \mathcal{K}$ . This means that there exists a  $C^1$ -class function  $V_i : \mathbb{R}^n \rightarrow \mathbb{R}$  such that

$$\underline{\alpha}_i(\|x_i\|) \leq V_i(x_i) \leq \bar{\alpha}_i(\|x_i\|) \quad (5)$$

$$\dot{V}_i(x_i) \leq -\alpha_i(\|x_i\|) - H(x_i) + w(u_i, y_i) \quad (6)$$

where  $\alpha_i$  is a class- $\mathcal{K}_\infty$  function and  $H(x_i)$  has the property (4). Then the trajectories  $x_i(t)$  of the system (1) satisfy the following inequality for any  $u_i \in \mathcal{L}^m$  and the initial state  $x_{i0}$ .

$$\limsup_{t \rightarrow \infty} \|x_i(t)\| \leq \max\{\gamma_i(\limsup_{t \rightarrow \infty} \|u_i(t)\|), \rho_i(\eta_i)\} \quad (7)$$

where  $\rho_i$  and  $\gamma_i$  are defined by

$$\rho_i(\cdot) = \underline{\alpha}_i^{-1} \circ \bar{\alpha}_i(\cdot), \quad (8)$$

$$\gamma_i(\cdot) = \underline{\alpha}_i^{-1} \circ \bar{\alpha}_i \circ \alpha_i^{-1} \circ \kappa_i \beta_i(\cdot) \quad (9)$$

with  $\kappa > 1$ .

## 2.2 Convergency of coupled systems

Consider the following two systems

$$\dot{x}_i(t) = f_i(x_i, u_i), \quad y_i(t) = C_i x_i \quad (t \geq 0) \quad (10)$$

where  $x_i \in \mathbb{R}^n$ ,  $u_i = \text{col}(u_{i1}, \dots, u_{il_i}) \in \mathbb{R}^m$ ,  $y_i \in \mathbb{R}^m$ ,  $C_i \in \mathbb{R}^{m \times n}$  and  $f_i : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  with initial conditions  $x_i(0) = x_{0i}$  for  $i = 1, 2$ , respectively.

Suppose that each system (10) is strictly semi-dissipative, respectively. Then from Lemma 1, each system has the property (7). Now, we consider the case in which these two systems are coupled by the following inputs containing time-delay,

$$\begin{cases} u_1(t) = y_2(t - \tau) = C_2 x_2(t - \tau) \\ u_2(t) = y_1(t - \tau) = C_1 x_1(t - \tau). \end{cases} \quad (11)$$

where  $\tau(t)$  is a constant delay and the initial conditions of  $x_i$  for  $i = 1, 2$  are respectively given by

$$\begin{aligned} x_i(\theta) &= \phi_i(\theta) \quad (-h_1 \leq \theta \leq 0) \\ x_i(0) &= \phi_i(0) = x_{0i} \end{aligned}$$

where  $\phi_i : [-\tau, 0] \rightarrow \mathbb{R}^n$ .

Define class- $\mathcal{K}$  functions as

$$\begin{aligned} \pi_{12}(r) &= \gamma_1(\sigma_{\max}(C_2) \cdot r) \\ \pi_{21}(r) &= \gamma_2(\sigma_{\max}(C_1) \cdot r) \end{aligned} \quad (r \geq 0) \quad (12)$$

where  $\gamma_i(\cdot)$  are defined by (9) and  $\sigma_{\max}(\cdot)$  denotes the maximum singular value of a matrix.

Then we obtain the following lemma.

**Lemma 2.** For coupled system (10) with the coupling term (11), if the functions  $\pi_{12}(\cdot)$  and  $\pi_{21}(\cdot)$  in (12) satisfy

$$\pi_{12} \circ \pi_{21}(r) < r \quad \text{for all } r > 0, \quad (13)$$

then the trajectories  $x_1(t)$  and  $x_2(t)$  satisfy

$$\limsup_{t \rightarrow \infty} \|x_1(t)\| \leq \max\{\pi_{12} \circ \rho_2(\eta_2), \rho_1(\eta_1)\} \quad (14)$$

$$\limsup_{t \rightarrow \infty} \|x_2(t)\| \leq \max\{\pi_{21} \circ \rho_1(\eta_1), \rho_2(\eta_2)\} \quad (15)$$

where  $\rho_i$  is defined by (8) and  $\eta_i$  satisfies (4).

### 3 Synchronization in Unidirectional Ring Networks

We consider  $N$  identical systems:

$$\Sigma_i : \begin{cases} \dot{x}_i(t) = Ax_i + f(x_i) + Bu_i \\ y_i(t) = Cx_i & (t \geq 0) \\ x_i(\theta) = \phi_i(\theta) & (-\tau \leq \theta \leq 0) \end{cases} \quad (16)$$

for  $i = 1, \dots, N$ , where  $x_i \in \mathbb{R}^n$ ,  $u_i \in \mathbb{R}^m$ ,  $y_i \in \mathbb{R}^m$ ,  $A \in \mathbb{R}^{n \times n}$ ,  $B \in \mathbb{R}^{n \times m}$  and  $C \in \mathbb{R}^{m \times n}$ . In addition,  $f : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is Lipschitz continuous and  $\phi_i : [-\tau, 0] \rightarrow \mathbb{R}^n$  with  $\tau > 0$ .

Throughout this paper, we assume that each system (16) is strictly semi-passive and these systems are coupled by the following controller

$$u_i(t) = K_{ij}(y_i(t) - y_j(t - \tau)) \quad (17)$$

for  $(i, j) = \{(1, N), (2, 1), \dots, (N, N - 1)\}$ , where  $\tau > 0$  is constant delay and  $K_{ij} = K_{ij}^T \leq 0$ . This network has an unidirectional ring structure. In addition, if  $m = n$  and  $C$  is nonsingular, the coupling means a full state coupling. While, if  $m < n$ , it is an output coupling. Here we formulate synchronization of coupled systems as follows.

**Definition 3.** *If there exist a positive real number  $r$  such that the trajectories  $x_i(t)$  of the systems (16) with initial conditions  $\phi_i$  such that  $\|\phi_i - \phi_j\|_c \leq r$  satisfy  $\|x_i(t) - x_j(t)\| \rightarrow 0$  as  $t \rightarrow \infty$  for all  $i, j$ , then the coupled systems (16) and (17) are asymptotically synchronized.*

In this paper, we derive a synchronization condition for systems coupled in an unidirectional ring network.

#### 3.1 Boundedness of Coupled Systems

We show under suitable assumptions the boundedness of the coupled systems (16) and (17). Firstly, we consider the case  $m = n$  and assume the non-singularity of the matrix  $C$ . Note that strictly semi-passive systems with the coupling terms (17) are strictly semi-dissipative. The fact can be shown as follows.

For the systems (16), there exist some  $C^1$ -class function  $V : \mathbb{R}^n \rightarrow \mathbb{R}$ , class- $\mathcal{K}_\infty$  functions  $\underline{\alpha}(\cdot)$ ,  $\overline{\alpha}(\cdot)$  and  $\epsilon(\cdot)$  satisfying

$$\underline{\alpha}(\|x_i\|) \leq V(x_i) \leq \overline{\alpha}(\|x_i\|) \quad (18)$$

$$\dot{V}(x_i) \leq -\epsilon(\|x_i\|) - H(x_i) + y_i^T u_i \quad (19)$$

for all  $x_i \in \mathbb{R}^n$ ,  $u_i \in \mathbb{R}^m$ ,  $y_i \in \mathbb{R}^m$ . Substituting (17)

into (19), we obtain

$$\begin{aligned} \dot{V}(x_i) &\leq -\epsilon(\|x_i\|) - H(x_i) + y_i^T K_{ij}(y_i - y_j(t - \tau)) \\ &\leq -\epsilon(\|x_i\|) - H(x_i) + \frac{1}{2} y_i^T K_{ij} y_i \\ &\quad - \frac{1}{2} y_j(t - \tau)^T K y_j(t - \tau) \\ &\leq -\alpha(\|x_i\|) - H(x_i) + \beta(\|y_j(t - \tau)\|) \end{aligned} \quad (20)$$

where

$$\begin{aligned} \alpha(r) &= \epsilon(r) - \frac{1}{2} \lambda_{\max}(K_{ij}) \sigma_{\min}^2(C) r^2 \\ \beta(r) &= -\frac{1}{2} \lambda_{\min}(K_{ij}) r^2 \end{aligned} \quad (21)$$

for  $r > 0$ . Since the matrix  $K_{ij}$  is negative semi-definite, the functions  $\alpha(\cdot)$  and  $\beta(\cdot)$  in (21) belong to class  $\mathcal{K}_\infty$ . Therefore the systems (16) with (17) can be seen as a strictly semi-dissipative systems with respect to  $\beta(\|v_i\|)$  with inputs  $v_i = y_j(t - \tau)$ .

From the property, we obtain the following result by using Lemma 1.

**Theorem 1.** *For  $N$  systems (16) coupled by (17), define class- $\mathcal{K}$  functions as*

$$\pi_{ij}(r) := \gamma_i(\sigma_{\max}(C)r)$$

for  $r \geq 0$ , where  $\gamma_i$  are defined in Lemma 1. If the functions  $\pi_{ij}(\cdot)$  satisfy

$$\pi_{NN-1} \circ \dots \circ \pi_{1N}(r) < r \text{ for all } r > 0,$$

then the trajectories of the coupled systems converge to the bounded set

$$\Omega = \{x_i \in \mathbb{R}^n \mid \|x_i\| \leq \rho_i(\eta_i)\}. \quad (22)$$

This theorem can be proved in a similar way as the proof of Lemma 2 (Yamamoto *et al.*, 2007).

If all coupling gains are identical, that is  $K_{ij} = K \leq 0$ , then the functions  $\gamma_i$  are also identical, and the above result is simplified as follows.

**Corollary 1.** *If  $K_{ij} = K \leq 0$  and  $\pi_{ij}(r) = \pi := \gamma(\sigma_{\max}(C)r) < r$  for all  $r \geq 0$ , then the trajectories of the coupled systems converge to the bounded set*

$$\Omega = \{x_i \in \mathbb{R}^n \mid \|x_i\| \leq \rho(\eta_i)\}. \quad (23)$$

Next, we consider the case of output coupling, that is  $m < n$ . Then we assume that  $CB$  is non-singular.

From the non-singularity of  $CB$ , the system (16) can be transformed to the following normal form (Lozano *et al.*, 2000).

$$\dot{y}_i(t) = a(y_i, z_i) + CBu_i \quad (24)$$

$$\dot{z}_i(t) = q(y_i, z_i) \quad (25)$$

for  $i = 1, \dots, N$ , where  $z_i \in \mathbb{R}^{n-m}$  and  $[y_i^T \ z_i^T]^T = \Phi x_i$  for a nonsingular matrix  $\Phi \triangleq [C^T \ D^T]^T$  with  $D \in \mathbb{R}^{(n-m) \times n}$  such that  $DB = 0$  and functions  $a : \mathbb{R}^m \times \mathbb{R}^{n-m} \rightarrow \mathbb{R}^m$  and  $q : \mathbb{R}^m \times \mathbb{R}^{n-m} \rightarrow \mathbb{R}^{n-m}$  are Lipschitz continuous.

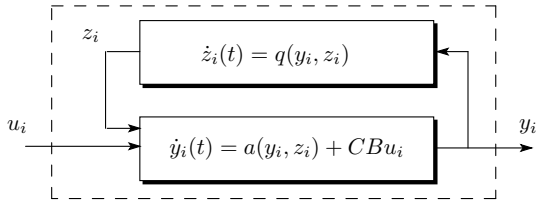


Figure 1. Decomposition of a cell system

Here the first equation can be recognized as a system with input  $v_i^T = (u_i^T, z_i^T)$  and output  $y_i$ . At this point, we assume the followings: (i) The system (24) is strictly semi-dissipative with respect to the supply rate  $w(v_i, y_i) \leq \beta_y(\|z_i\|) + y_i^T u_i$ , where  $\beta_y \in \mathcal{K}$ , i.e. there exist a positive definite  $C^1$ -class functions  $V_y$ , class  $\mathcal{K}_\infty$  functions  $\underline{\alpha}_y, \bar{\alpha}_y$  and  $\epsilon_y$  satisfying

$$\begin{aligned} \underline{\alpha}_y(\|y_i\|) &\leq V_y(y_i) \leq \bar{\alpha}_y(\|y_i\|) \\ \dot{V}_y(y_i) &\leq -\epsilon_y(\|y_i\|) - H_y(y_i) + \beta_y(\|z_i\|) \\ &\quad + y_i^T u_i \end{aligned} \quad (26)$$

for all  $y_i \in \mathbb{R}^m, u_i \in \mathbb{R}^m, z_i \in \mathbb{R}^{n-m}$ , where the function  $H_y(y_i)$  satisfies that  $H_y(y_i) \geq 0$  if  $\|y_i\| \geq \eta_y$  for some positive real number  $\eta_y$ . (ii) The system (25) is strictly semi-dissipative with respect to the supply rate  $w(y_i, z_i) \leq \beta_z(\|y_i\|)$ , where  $\beta_z \in \mathcal{K}$ , i.e. there exist a positive definite  $C^1$ -class functions  $V_z$ , class  $\mathcal{K}_\infty$  functions  $\underline{\alpha}_z, \bar{\alpha}_z$  and  $\alpha_z$  satisfying

$$\begin{aligned} \underline{\alpha}_z(\|z_i\|) &\leq V_z(z_i) \leq \bar{\alpha}_z(\|z_i\|) \\ \dot{V}_z(z_i) &\leq -\alpha_z(\|z_i\|) - H_z(z_i) + \beta_z(\|y_i\|) \end{aligned}$$

for all  $z_i \in \mathbb{R}^{n-m}, y_i \in \mathbb{R}^m$ , where the function  $H_z(z_i)$  satisfies that  $H_z(z_i) \geq 0$  if  $\|z_i\| \geq \eta_z$  for some positive real number  $\eta_z$ .

Substituting (17) for (26), we obtain

$$\begin{aligned} \dot{V}_y(y_i) &\leq -\alpha_i(\|y_i\|) - H_y(y_i) + \beta_y(\|z_i\|) \\ &\quad + \beta_{ij}(\|y_j(t - \tau)\|) \end{aligned}$$

where  $\alpha_i(r) = \epsilon_y(r) - \frac{1}{2}\lambda_{max}(K_{ij})r^2$  and  $\beta_{ij}(r) = -\frac{1}{2}\lambda_{min}(K_{ij})r^2$ . Then, from lemma 1, we obtain the following theorem.

**Theorem 2.** For all  $r > 0$  and  $i, j (1 \leq i < j \leq N)$ , if

$$\gamma_y \circ \gamma_z(r) < r \quad \text{and} \quad \gamma_{ij}(r) < r \quad (27)$$

hold, then the trajectories of the system (16) converge to the set

$$\Omega := \{x_i \in \mathbb{R}^n \mid \|y_i\| \leq s_y \text{ and } \|z_i\| \leq s_z\} \quad (28)$$

where  $s_y := \max\{\rho_y(\eta_y), \gamma_y \circ \rho_z(\eta_z)\}$

$$s_z := \max\{\rho_z(\eta_z), \gamma_z \circ \rho_y(\eta_y)\}$$

$$\rho_y(\cdot) = \underline{\alpha}_y^{-1} \circ \bar{\alpha}_y(\cdot), \quad \gamma_y(\cdot) = \rho_y \circ \alpha_y^{-1} \circ \kappa \beta_y(\cdot),$$

$$\rho_z(\cdot) = \underline{\alpha}_z^{-1} \circ \bar{\alpha}_z(\cdot), \quad \gamma_z(\cdot) = \rho_z \circ \alpha_z^{-1} \circ \kappa \beta_z(\cdot).$$

### 3.2 Synchronization Conditions

Next we consider a condition for synchronization of the coupled systems (16) with (17).

Define the synchronization error  $e(t)$  as

$$e(t) := [e_{12}(t)^T, e_{13}(t)^T, \dots, e_{1N}(t)^T]^T \in \mathbb{R}^{(N-1)n} \quad (29)$$

with  $e_{1k}(t) = x_1(t) - x_k(t)$  for  $k = 2, \dots, N$ . Then  $\lim_{t \rightarrow \infty} \|e(t)\| \rightarrow 0$  means all systems asymptotically synchronize. Therefore we consider a condition in which the dynamics of  $e$  has  $e = 0$  as an asymptotically stable equilibrium. In general, however, the dynamics of  $e$  does not have  $e = 0$  as a solution for arbitrary coupling gains and delays in (17). In order to give the network a symmetric structure, we set the coupling gains and the delays:  $K_{ij} = K, \tau_{ij} = \tau$ . Then, the dynamics of the error  $e(t)$  has  $e = 0$  as an equilibrium solution. In addition, note that  $x_1$  converges to  $\Omega$  of the discussion in the preceding section.

By using the boundedness of the trajectory  $x_1$ , we derive a synchronization condition. For the unidirectional ring network, the synchronization error dynamics linearized around  $e = 0$  is given by

$$\dot{e}(t) = A_0(x_1)e(t) + A_1e(t - \tau) \quad (30)$$

where  $A_0(x_1) = I \otimes \hat{A}(x_1)$  with

$$\hat{A}(x_1) = A + BKC + D(x_1)$$

for  $D(x_1) \triangleq \left( \frac{\partial f(x)}{\partial x} \right) \Big|_{x=x_1}$  and  $A_1$  is defined as

$$A_1 = \Lambda \otimes BKC$$

with  $\Lambda = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ -1 & \ddots & \vdots & \vdots \\ & & \ddots & 0 & 1 \\ 0 & & & -1 & 1 \end{bmatrix}$  and the notation  $\otimes$  denotes

the Kronecker product. If  $e = 0$  is asymptotically stable,  $e(t)$  asymptotically converges to zero and synchronization is accomplished. Therefore the synchronization problem can be reduced to the stability problem for the above retarded system. By using the Lyapunov-Krasovskii theorem, we obtain the following synchronization condition.

**Theorem 3.** For all  $x_1 \in \Omega$  given by (23) or (28), if there exist positive definite matrices  $P, Q, Z \in \mathbb{R}^{(N-1)n \times (N-1)n}$  and matrices  $Y, W \in \mathbb{R}^{(N-1)n \times (N-1)n}$  satisfying the following LMI, then  $e = 0$  of the error dynamics is asymptotically stable.

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & -\tau Y & \tau A_0^T Z \\ \Gamma_{12}^T & \Gamma_{22} & -\tau W & \tau A_1^T Z \\ -\tau Y^T & -\tau W^T & -\tau Z & 0 \\ \tau Z A_0 & \tau Z A_1 & 0 & -\tau Z \end{bmatrix} < 0 \quad (31)$$

where  $\Gamma_{11} = PA_0(x_1) + A_0^T(x_1)P + Y + Y^T + Q$ ,  $\Gamma_{12} = PA_1 - Y + W^T$  and  $\Gamma_{22} = -Q - W - W^T$ .

As the LMI (31) is affine with respect to the system matrices  $A_0(x_1)$  and  $A_1$ , this result can be extended to a stability criterion for the polytopic systems.

Since  $x_1$  is bounded, each element of  $D(x_1)$  is also bounded. As a result, the approximated error dynamics (30) can be rewritten by the following polytopic system:

$$\dot{e}(t) = \sum_{i=1}^m p_i A_0^i e(t) + A_1 e(t - \tau)$$

where  $A_0^i = A_{0i} + D_i$  are constant matrices and  $p_i(x_1) \in [0, 1]$  are polytopic coordinates satisfying the convex sum property  $\sum_{i=1}^m p_i(x_1) = 1$ . Using the ‘‘vertex systems’’, we can obtain the following polytopic linear differential inclusion (PLDI)

$$\dot{e}(t) \in \text{Co} \left\{ A_0^1 e(t) + A_1 e(t - \tau), \dots, A_0^m e(t) + A_1 e(t - \tau) \right\} \quad (32)$$

where Co denotes a convex hull. Therefore we can obtain the following stability criterion.

**Theorem 4.** Consider the PLDI (32). If there exist positive definite matrices  $P, Q^i, Z \in \mathbb{R}^{(N-1)n \times (N-1)n}$  and matrices  $Y^i, W^i \in \mathbb{R}^{(N-1)n \times (N-1)n}$  for  $i = 1, \dots, m$  satisfying the following LMI, then  $e = 0$  of the error dynamics is asymptotically stable.

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12}^i & -\tau Y^i & \tau A_0^{iT} Z \\ \Gamma_{12}^{iT} & \Gamma_{22}^i & -\tau W^i & \tau A_1^T Z \\ -\tau Y^{iT} & -\tau W^{iT} & -\tau Z & 0 \\ \tau Z A_0 & \tau Z A_1 & 0 & -\tau Z \end{bmatrix} < 0 \quad (33)$$

where  $\Gamma_{11}^i = PA_0^i + A_0^{iT}P + Y^i + Y^{iT} + Q^i$ ,  $\Gamma_{12}^i = PA_1 - Y^i + W^{iT}$  and  $\Gamma_{22}^i = -Q^i - W^i - W^{iT}$ .

Using Theorem 4, we can check the stability of  $e = 0$  by solving a finite number of LMIs.

**Corollary 2.** Suppose that the linearized synchronization error dynamics (30) is described by the PLDI (32). If there exist positive definite matrices  $P, Q^i, Z \in \mathbb{R}^{(N-1)n \times (N-1)n}$  and matrices  $Y^i, W^i \in \mathbb{R}^{(N-1)n \times (N-1)n}$  for  $i = 1, \dots, m$  satisfying the LMI (33), then the synchronization of coupled systems is asymptotically accomplished.

## 4 Example

Consider a unidirectional ring network of Lorenz systems ( $N = 6$ ).

$$\dot{x}_i(t) = \begin{bmatrix} \sigma(x_{i2} - x_{i1}) \\ rx_{i1} - x_{i2} - x_{i1}x_{i3} \\ -bx_{i3} + x_{i1}x_{i2} \end{bmatrix} + Bu_i, \quad y_i = Cx_i \quad (34)$$

where  $\sigma = 10$ ,  $r = 28$ ,  $b = 8/3$  and  $B^T = C$ . At this stage, we assume that  $m = n$  and  $C = I_{3 \times 3}$  and the coupled system forms an unidirectional ring network with  $K_{21} = K_{32} = K_{43} = K_{54} = K_{65} = K_{16} := K = -kI_{3 \times 3} < 0$  and  $\tau_{ij} = \tau$  in (17).

We consider a storage function  $V(\tilde{x}_i) = \frac{1}{2} \tilde{x}_i^T \tilde{x}_i$ , where  $\tilde{x}_i = [x_{i1} \ x_{i2} \ x_{i3} - \sigma - r]^T$ . Then the derivative along the trajectory of each system is given by  $\dot{V}(\tilde{x}_i) = -\alpha(\|\tilde{x}_i\|) - H(\tilde{x}_i) + \beta(\|\tilde{x}_j\|)$  for  $i = 1, 2, 3, 4, 5, 6$  and  $j = 6, 1, 2, 3, 4, 5$ , where the functions  $\alpha(\cdot)$ ,  $\beta(\cdot)$  and  $H(x)$  are defined as  $\alpha(\|\tilde{x}_i\|) = (\frac{k}{2} + \epsilon)\|\tilde{x}_i\|^2$ ,  $\beta(\|\tilde{x}_i\|) = \frac{k}{2}\|\tilde{x}_i\|^2$  and

$$\begin{aligned} H(\tilde{x}_i) = & (\sigma - \epsilon)\tilde{x}_{i1}^2 + (1 - \epsilon)\tilde{x}_{i2}^2 - \frac{b^2(\sigma + r)^2}{4(b - \epsilon)} \\ & + (b - \epsilon)(\tilde{x}_{i3} - \frac{b - 2\epsilon}{2(b - \epsilon)}(\sigma + r))^2, \end{aligned}$$

and  $0 < \epsilon < 1$ . Then  $\gamma_{ij}(\cdot)$  is given by  $\gamma_{ij}(r) = \sqrt{\frac{\kappa \frac{k}{2}}{(k/2 + \epsilon)}} r$ , where  $\kappa > 1$ . Since  $C = I$ ,  $\pi(\cdot) = \gamma(\cdot)$ . Therefore, for any finite number  $k > 0$ , setting  $\kappa$  sufficiently close to 1,  $\pi(r) < r$  holds for all  $r > 0$ . Furthermore, for  $\epsilon = 0.01$ , the minimum  $\eta$  satisfying  $H(\tilde{x}_i) \geq 0$  is given by  $\eta = 39.4$  and  $\rho(\cdot)$  is the identity map. So the bounded set is given by

$$\Omega = \{\tilde{x}_i \in \mathbb{R}^3 \mid \|\tilde{x}_i\| \leq 39.4\} \quad (35)$$

and each trajectory  $x_i(t)$  converges to the set  $\Omega$ . A graphical representation of the bounds is given in Figure 2. The ball represents the computed bounds from eq. (35). Using the obtained bounds, we check the synchronization condition. To reduce computation, the estimated bounds are approximated by a cube and the

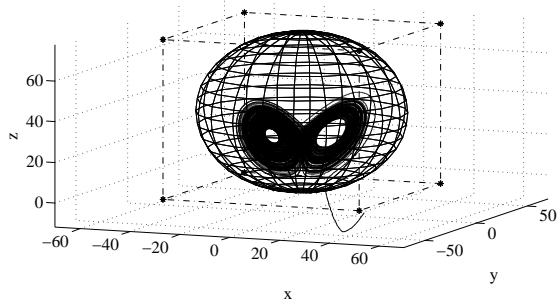


Figure 2. Estimated bounds on the trajectories of the coupled Lorenz system

synchronization error dynamics is rewritten by the following polytopic linear differential inclusion, which consists of 8 vertex systems.

$$\dot{e} \in \text{Co} \left\{ A_0^1 e(t) + A_1 e(t-\tau), \dots, A_0^8 e(t) + A_1 e(t-\tau) \right\}$$

Setting  $k = 30$  and  $\tau = 0.01$ , the LMI condition holds for all  $x_1 \in \Omega$ . Figure 3 shows that the norm of  $e(t)$  converges to zero and synchronization is achieved, and Figure 4 shows that the coupled systems behave chaotically under synchronization.

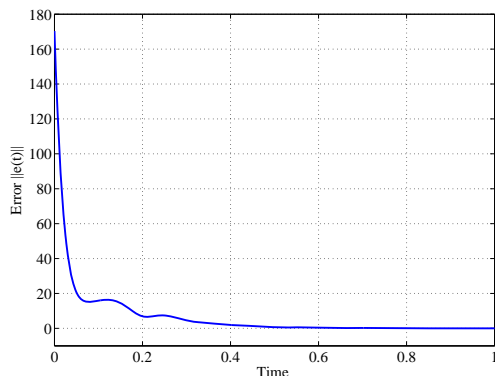


Figure 3. The norm of the error  $\|e(t)\|$

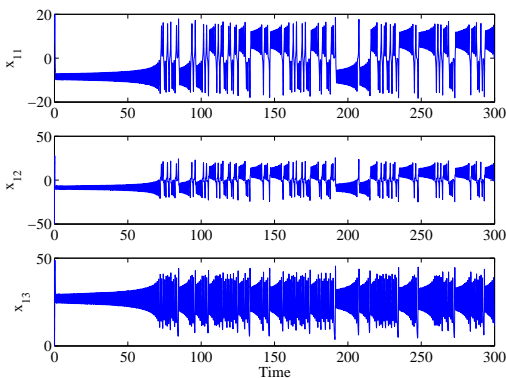


Figure 4. Behavior of system 1

## 5 Conclusion

In this paper, we have considered a sufficient condition for synchronization in an unidirectional ring network of  $N$  chaotic systems with time-delay coupling. For the system, we derived a condition for the boundedness of the strictly semi-dissipative systems by the small-gain theorem. Then, for the coupled systems in an unidirectional ring network, we derived sufficient conditions for synchronization of the systems by the Lyapunov-Krasovskii theorem.

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