PHYSCON 2017, Florence, Italy, 17–19 July, 2017

PARAMETER OPTIMIZATION FOR ESTIMATION OF LINEAR NON-STATIONARY SYSTEMS

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Abstract

A problem of the optimal parameters choice for the best state estimation of the linear system subject to uncertain perturbations is considered. The problem is interpreted as a differential game for the Riccati equation that arises in the process of solution of the uncertain minimax estimation. The game is realized by two players: the first player (an observer) can choose some matrices of the system at any instant of time in order to minimize the diameter of the informational set at the end of the observation interval. The second player (an opponent of the observer) tries to maximize the diameter choosing the matrices which are multipliers at perturbations. All the choosing parameters are limited to compact sets in appropriate spaces of matrices. The perturbations in the system are subjected to integral constraints. The payoff of the game is the Euclidean norm of the inverse Riccati matrix at the end of the process. A specific case of the problem with constant matrices is considered. Methods of minimax optimization, the theory of optimum control, and the theory of differential games are used. Examples are also given.

Key words

observations' control, parameters optimization, differential games.

1 Introduction

State estimation problems for uncertain determinate linear systems are well examined at present, [Kurzhanski and Vályi, 1996; Kurzhanski and Varaiya, 2014]. The main mathematical apparatus is connected here with the theory of control and estimation under uncertainty. In the special case of estimation with integral constraints for perturbations, the basic relations are quite similar to the equations of the well-known Kalman filter. But in the determinate theory the main object of investigation is the informational set. The diameter of this set may serve as the quality characteristic of the observation process. The first player (an observer) tries to minimize the diameter, and the second player (an opponent of the observer) aims to prevent it. Both players can choose the parameters that lie in compact sets of matrices at any instant of time. Thus, the problem is reduced to a differential game for the Riccati equation of the process. As the diameter of the informational set is proportional to the Euclidean norm of the inverse Riccati matrix, the mentioned value is taken as the payoff of the game. We consider the differential game in the class 'counterstrategy/strategy' and use the approach connected with the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation, [Krasovskii and Subbotin, 1988; Subbotin, 1999; Fleming, Soner, 2006].

We offer two ways that overcome the lack of the Lipshitz conditions and suggest a numerical scheme for the solution of the problem. Note that problems of observations' control were considered in different aspects in [Grigoryev et al., 1986; Kurzhanski and Vályi, 1996; Kurzhanski and Varaiya, 2014; Ananiev, 2011; Ananyev, 2015]. The results of the work may be used both for quality improvement of measuring systems, and for creation of counteraction systems of observation.

This work is organized as follows. Section 2 is devoted to the background of guaranteed estimation. In section 3 our problem is formulated. In section 4 we consider the most simple case of constant matrices in the system. A special attention is paid to conclusions under steady-state solutions of the Riccati equation. In section 5 we pass to the common case. Concepts of strategy, and counterstrategy are reminded. The HJBI equation is written down, and the possibility of its solution in generalized sense is discussed. In the last section, problems of numerical solution are considered and some examples are given.

2 Guaranteed Estimation

In this work, we consider the linear non-stationary equations

$$\dot{x}(t) = A(t)x(t) + B(t)v(t), \tag{1}$$

$$y(t) = G(t)x(t) + c(t)v(t), \;\; t \in [0,T], \tag{2}$$

containing an uncertain function $v(\cdot)$, where $A(\cdot)$, $B(\cdot)$, $G(\cdot)$, $c(\cdot)$ are bounded Borelian matrices; $x(t) \in R^n$, $y(t) \in R^m$, $v(t) \in R^k$. Suppose that the uncertain function $v(\cdot)$ in (1) and (2) is constrained by the inequality

$$\|v(\cdot)\|^2 = \int_0^T |v(t)|^2 dt \le 1, \tag{3}$$

where $|\cdot|$ is the Euclidean norm. Besides, the matrix $c(\cdot)$ must satisfy the condition

$$c(t)c'(t) \ge \delta I_m, \; \forall t \in [0,T], \tag{4}$$

where $\delta > 0$ and $I_m \in R^{m\times m}$ is the identity matrix. Hereafter the symbol $'$ means the transposition. According to general theory of guaranteed estimation [Kurzhanski and Varaiya, 2014] let us give

Definition 1. *The collection $\mathcal{X}_T(y)$ of state vectors $\{x(T)\}$ is said to be the* **informational set** *if for any $x \in \mathcal{X}_T(y)$ there exists a function $v(\cdot)$ satisfying* (3) *and such that equality* (2) *holds.*

Denote by $\mathcal{C}(t)$ the matrix $(c(t)c'(t))^{-1}$. Under assumption (4) we have the equalities $v(t) = c'(t)w(t) + \mathcal{C}_1(t)f(t)$ and $\|v(\cdot)\|^2 = \|c'(\cdot)w(\cdot)\|^2 + \|\mathcal{C}_1(\cdot)f(\cdot)\|^2$, where $\mathcal{C}_1(t) = I_k - c'(t)\mathcal{C}(t)c(t)$ is the orthogonal projection on the subspace $\ker c(t)$. Using (2), we may rewrite inequality (3) as

$$\int_0^T \Big\{ |y(t) - G(t)x(t)|^2_{\mathcal{C}(t)} + |f(t)|^2_{\mathcal{C}_1(t)} \Big\} dt \le 1. \tag{5}$$

From now on the symbol $|x|^2_Q$ means $x'Qx$.

It is easily seen that $x \in \mathcal{X}_T(y)$ iff there exists a function $f(\cdot)$ satisfying (5) and subjecting to the equation

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)\mathcal{C}_1(t)f(t) \\ &+B(t)c'(t)\mathcal{C}(t)(y(t) - G(t)x(t)) \end{aligned} \tag{6}$$

with final condition $x(T) = x$. On the other hand, such a function exists iff the minimum of the left-hand side of inequality (5) is less or equal 1. Thus, using standard optimization reasonings, we come to the conclusion.

Lemma 1. *The informational set has the form $\mathcal{X}_T(y) = \big\{x \in R^n : |x|^2_{P(T)} - 2d'(T)x + q(T) \le 1\big\}$, where parameters may be found from equations*

$$\begin{aligned} \dot{P}(t) &= -P(t)A(t) - A'(t)P(t) - P(t)B(t) \\ &\times B'(t)P(t) + (G(t) + c(t)B'(t)P(t))' \\ &\times \mathcal{C}(t)(G(t) + c(t)B'(t)P(t)), \;\; P(0) = 0; \end{aligned} \tag{7}$$

$$\begin{aligned} \dot{d}(t) &= -\big(A(t) + B(t)B'(t)P(t)\big)' d(t) \\ &+(G(t) + c(t)B'(t)P(t))'\mathcal{C}(t)(y(t) \\ &+c(t)B'(t)d(t)), \;\; d(0) = 0; \end{aligned} \tag{8}$$

$$\begin{aligned} \dot{q}(t) &= \big|y(t)\big|^2_{\mathcal{C}(t)} - \big|B'(t)d(t)\big|^2_{\mathcal{C}_1(t)}, \\ q(0) &= 0. \end{aligned} \tag{9}$$

If the matrix $P(t)$ is invertible on $(0,T]$, we can introduce the values $\hat{x}(t) = P^{-1}(t)d(t)$ and $h(t) = q(t) - \big|d(t)\big|^2_{P^{-1}(t)}$, which satisfy the equations

$$\begin{aligned} \dot{\hat{x}}(t) &= A(t)\hat{x}(t) + \big(c(t)B'(t) + G(t)P^{-1}(t)\big)' \\ &\times \mathcal{C}(t)(y(t) - G(t)\hat{x}(t)), \end{aligned} \tag{10}$$

$$\dot{h}(t) = \big|y(t) - G(t)\hat{x}(t)\big|^2_{\mathcal{C}(t)}. \tag{11}$$

The value $\hat{x}(T)$ is the center of bounded ellipsoid $\mathcal{X}_T(y)$.

3 Problem Formulation

Consider our observation process as a differential game for Riccati equation (7). This may be described as follows. Let $B(t) = g(t)b(t)$ and uncertain matrices $b(\cdot)$, $c(\cdot)$ in (1), (2), (4) are subjected to the constraints

$$b(t) \in \mathbf{B}, \;\; c(t) \in \mathbf{C}, \tag{12}$$

where $\mathbf{B}$, $\mathbf{C}$ are compact sets in the spaces $R^{k\times k}$ and $R^{m\times m}$, respectively. Condition (4) holds as before. At any instant t, the matrices b, c in (12) can be chosen by a 2-nd player (opponent) who tries to make the worse quality of observation process. On the other hand, the matrices

$$g(t) \in \mathbf{G}_1, \;\; G(t) \in \mathbf{G}, \tag{13}$$

where $\mathbf{G}_1$, $\mathbf{G}$ are compact sets in the spaces $R^{n\times k}$ and $R^{m\times n}$, respectively, can be chosen by a 1-st player (observer) who tries to make the best quality of observation process. Both players evaluate the quality of observation by the terminal payoff

$$\gamma(T) = |P^{-1}(T)|, \tag{14}$$

where $|P| = \max_{|x|\le 1} |Px|$ is the Euclidean norm of matrix P. The 1-st player tries to minimize the payoff, and his opponent tries to maximize. Let us explain the choice of payoff (14). This value is proportional to the diameter of final informational set

$\mathcal{X}_T(y)$ if $P(T) > 0$. Indeed, the support function of $\mathcal{X}_T(y)$ is equal to $\rho(l|\mathcal{X}_T(y)) = \max_{x\in\mathcal{X}_T(y)} l'x = l'\hat{x}(T)+\sqrt{1-h(T)}|l|_{P^{-1}(T)}$. Therefore, the diameter $\max_{x,y\in\mathcal{X}_T(y)}|x-y|$ equals $2\sqrt{(1-h(T))|P^{-1}(T)|}$. As a rule, the value $h(T)$ is selected by the 2-nd player who supposes $h(T) = 0$ to maximize the diameter. Note that $\gamma(T) = \infty$ if the matrix $P(T)$ is singular.

4 Optimization of Riccati Equation with Time-Invariant Parameters

Let all the matrices in relations (1), (2), (4), and (12), (13) be time-invariant. Consider the low value $\gamma_* = \max_{b,c}\min_{g,G}\gamma(T)$ of the game and its upper value $\gamma^* = \min_{g,G}\max_{b,c}\gamma(T)$. Always we have $\gamma^* \geq \gamma_*$, and the strong inequality $\gamma^* > \gamma_*$ may be realised. From now on, we use the standard notation from Matlab, where $[A_1,\ldots,A_k]$ means the row-concatenation of matrices of appropriate dimensions (sometimes, the comma is replaced by the blank), and $[A_1;\ldots;A_k]$ means the column-concatenation.

Example 1. Consider a two-dimensional system with $A = 0$, $G = I_2$, $c = [O_2, I_2]$, where $I_2 \in R^{2\times 2}$ is the identity matrix, and O_2 is the zero matrix. Besides, the set $\mathbf{B} = \{[[\sqrt{a},0;0,\sqrt{1-a}],O_2] : 0 \leq a \leq 1\}$ and the set $\mathbf{G}_1 = \{[z,1-z] : 0 \leq z \leq 1\}$. The solution of Riccati is equal to the diagonal matrix $[p_1(t),0;0,p_2(t)]$, where $p_1(t) = tanh(\sqrt{2az}t)/(\sqrt{2az})$, $p_2(t) = tanh(\sqrt{1-a}(1-z)t)/(\sqrt{1-a}(1-z))$. For $T = 5$ we have $\gamma^* = 0.5068$, $\gamma_* = 0.3751$.

Let us solve our game in the class 'counterstrategy/strategy', when the 1-st player may use any functions $g(b,c)$, $G(b,c)$. In this case, the game has a saddle point (see [Krasovskii and Subbotin, 1988; Fleming, Soner, 2006]) and the value of game

$$\overline{\gamma} = \max_{b,c}\min_{g,G}\gamma(T) = \min_{g(\cdot,\cdot),G(\cdot,\cdot)}\max_{b,c}\gamma(T) \qquad (15)$$

that is equal to γ_*. There is a simpler case of the game under

Assumption 1. *Let constraints* (12), (13) *satisfy the conditions: there are matrices* $b^* \in \mathbf{B}$, $c^* \in \mathbf{C}$, *such that* $bb' \geq b^*b^{*'}$, $cc' \geq c^*c^{*'}$, $\forall b \in \mathbf{B}$, $\forall c \in \mathbf{C}$, *and* $gbc' = 0$, $\forall g \in \mathbf{G}_1$, $\forall b \in \mathbf{B}$, $\forall c \in \mathbf{C}$. *Hereafter the inequality* $A \geq B$ *means* $|x|^2_A \geq |x|^2_B$, *for all* x, *where* A, B *are square simmetrical matrices.*

Theorem 1. *Under assumption* 1 *the optimization is fulfilled on* g, G, *and the value of the game equals* $\overline{\gamma} = \gamma_* = \min_{g,G}|P^{-1}(T)|$, *where matrices* b^*, c^* *are substituted in equation* (7).

Now consider stationary solutions of Riccati equation (7). Such solutions arise under very long time of observation. We make

Assumption 2. *The system* $\dot{x} = \mathbf{A}x + gb\mathcal{C}_1 f$, $y = Gx + cv$, *where* $\mathbf{A} = A - gbc'\mathcal{C}G$ (see (6)), *is completely observable and completely controllable, i.e.* $\mathrm{rank}[gb\mathcal{C}_1, \mathbf{A}gb\mathcal{C}_1,\ldots,\mathbf{A}^{n-1}gb\mathcal{C}_1] = n$, $\mathrm{rank}[G; G\mathbf{A};\ldots;G\mathbf{A}^{n-1}] = n$, $cc' > 0$, $\forall g \in \mathbf{G}_1$, $\forall b \in \mathbf{B}$, $\forall G \in \mathbf{G}$, $\forall c \in \mathbf{C}$.

It is known [Liptser and Shiryayev, 2001] that under assumption 2 there exists a unique positive-definite solution of stationary Riccati equation

$$-\mathbf{A}'P - P\mathbf{A} + G'\mathcal{C}G - Pgb\mathcal{C}_1 b'g'P = 0. \qquad (16)$$

Condition (16) may be considered as an equality condition for minimax problem (15). Consider

Example 2. Let $A = [0,1;0,0]$. The sets $\mathbf{G}_1 = \{[1,z;z,1] : 0 \leq z \leq 1\}$, $\mathbf{B} = \{[[\sqrt{1-a},0;0,2\sqrt{a}],O_2] : 0.1 \leq a \leq 1\}$, $\mathbf{G} = \{[1-r\ 0;0\ 4r] : 0 \leq r \leq 0.9\}$, $\mathbf{C} = \{[O_2,[\sqrt{1-k}\ 0;0\ \sqrt{k}]] : 0.1 \leq k \leq 0.9\}$. The assumption 2 holds, but assumption 1 does not hold. It is required to find the value of the game and optimal counterstrategies $r_*(a,k), z_*(a,k)$ delivering the minimum in (15) and optimal strategies a^*, k^*. The value of the game is approximately equals 4.3585. It is reached at $a^* = 0.66$, $k^* = 0.884$. The optimal functions r_*, z_* are shown on Fig. 1 and 2.

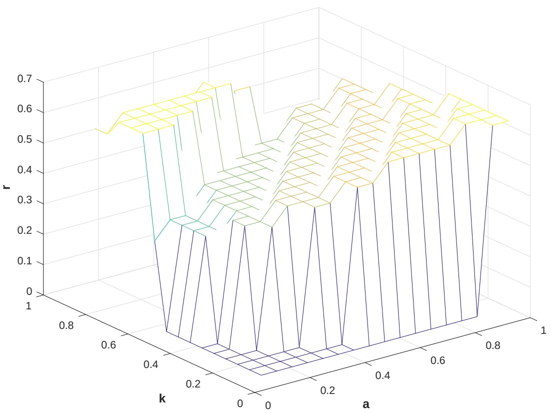

Figure 1. Optimal counterstrategy r_*.

The program for this example uses a grid on uncertain parameters with the step $\delta = 0.05$.

5 Common Case

Unfortunately, Riccati equation (7) does not satisfy the Lipschitz condition. Nevertheless, we can overcome this difficulty at least in two ways. At first, we transform the Riccati equation by fractional decomposition and write $P(t) = M(t)N^{-1}(t)$, where $M(\cdot)$, $N(\cdot)$ are matrix differentiable functions. Using the relation $\dot{N}^{-1} = -N^{-1}\dot{N}N^{-1}$ and substituting $M(t)N^{-1}(t)$ into (7) (see also (16)), we

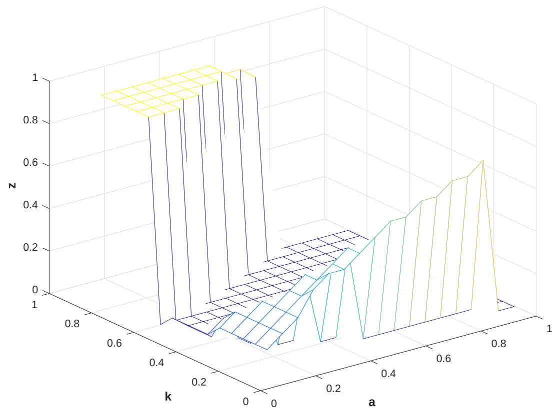

Figure 2. Optimal counterstrategy z_*.

obtain the equality $\dot{M}(t) - M(t)N^{-1}(t)\dot{N}(t) = -\mathbf{A}'(t)M(t) + G'(t)\mathcal{C}(t)G(t)N(t) - M(t)N^{-1}(t) \times\big(g(t)b(t)\mathcal{C}_1(t)b'(t)g'(t)M(t) + \mathbf{A}(t)N(t)\big)$, which may be satisfied by two solutions of the linear matrix equations

$$\begin{aligned} \dot{M}(t) &= -\mathbf{A}'(t)M(t) + G'(t)\mathcal{C}(t)G(t)N(t), \\ \dot{N}(t) &= g(t)b(t)\mathcal{C}_1(t)b'(t)g'(t)M(t) \\ &\quad + \mathbf{A}(t)N(t). \end{aligned} \tag{17}$$

The initial conditions for equations (17) may be chosen as $M_0 = 0$, $N_0 = I_n$.

From now on, we accept the analog of assumption 2 for non-stationary systems.

Assumption 3. *The matrix* $P_B(\tau, t) = \int_\tau^t \mathbf{X}(t,r)g(r) \times b(r)\mathcal{C}_1(r)b'(r)g'(r)\mathbf{X}'(t,r)dr > 0,\ \forall \tau, t,\ 0 \le \tau < t \le T$, *for all measurable functions* $b(r) \in \mathbf{B}$, $c(r) \in \mathbf{C}$, $G(r) \in \mathbf{G}$, *and* $g(r) \in \mathbf{G}_1$. *Hereafter* $\mathbf{X}(t,r)$ *is the fundamental matrix, for which* $\partial \mathbf{X}(t,r)/\partial t = \mathbf{A}(t)\mathbf{X}(t,r)$, $\mathbf{X}(r,r) = I_n$. *Besides, the matrix* $P_G(\tau,t) = \int_\tau^t \mathbf{X}'(r,t)G'(r)\mathcal{C}(r)G(r)\mathbf{X}(r,t)dr > 0$, $\forall \tau, t, 0 \le \tau < t \le T$, *for all measurable parameter functions.*

Under assumption 3, it follows from [Liptser and Shiryayev, 2001] that the matrix $P(t)$ is nonsingular for any $t \in (0,T]$. Moreover, due to the compactness, the nonsingularity will be uniform on any interval $[t_0, T]$, $t_0 > 0$, with respect to all measurable parameter functions. The same may be told about matrices $M(t)$ and $N(t)$.

Thus, our differential game is reduced to the game with linear matrix equations (17). Let us introduce the designations $\mathbf{u} = \{g, G\}$ and $\mathbf{v} = \{b, c\}$. These control parameters are contained in compact sets: $\mathbf{u} \in \mathbb{U} = \mathbf{G}_1 \times \mathbf{G}$, $\mathbf{v} \in \mathbb{V} = \mathbf{B} \times \mathbf{C}$. The payoff $\omega(M(T), N(T)) = \gamma(T) = |N(T)M^{-1}(T)|$ of the game is a continuous function of the final state. The initial state is known. Any functions $\mathbf{u}(t, M, N, \mathbf{v}) \in \mathbb{U}$ of t, the state $\{M, N\}$, and the parameter $\mathbf{v}$ satisfying the constraints, will be consider for the strategies of the 1-st player who tries to minimize $\omega(M(T), N(T))$. The strategies of the 2-nd player, who tries to maximize $\omega(M(T), N(T))$, are any functions $\mathbf{v}(t, M, N) \in \mathbb{V}$. The controls of the 1-st player are said to be counterstrategies, [Krasovskii and Subbotin, 1988]. The solution of the (17) is defined step-by-step with the help of piecewise-constant controls as in [Krasovskii and Subbotin, 1988], [Subbotin, 1999, p. 7]. In the same works, a concept of the value of the game in the class ‘counterstrategy/strategy’ is explained in detail. The game has the saddle point and the value $\mathbf{c}(t_0, M_0, N_0)$, where $t_0 \in [0,T]$, if the game begins from the position (t_0, M_0, N_0).

In our problem, one needs to find a saddle point (a value of the game)

$$\overline{\omega} = \mathbf{c}(0, M_0, N_0) \tag{18}$$

and corresponding optimal strategies $\mathbf{u}_*$, $\mathbf{v}^*$. For problem's solution one need to build a function $\mathbf{c}(t, M, N)$ giving the value of the game under different initial positions (t, M, N). Under assumptions 3 we may suppose that there exist constants α, β, such that $0 < \alpha \le P(t)$, $P^{-1}(t) \le \beta$ for all $t \in [t_0, T]$, $t_0 > 0$, and for all control parameters. At the final instant the boundary condition $\mathbf{c}(T, M, N) = \omega(M, N)$ must hold.

As in section 4, the game become simpler under the following

Assumption 4. *Let the compact sets* $\mathbf{B}(t)$, $\mathbf{C}(t)$ *may depend on time and let there exist the matrices* $b^*(t) \in \mathbf{B}(t)$, $c^*(t) \in \mathbf{C}(t)$, *such that* $b(t)b'(t) \ge b^*(t)b^{*'}(t)$, $c(t)c'(t) \ge c^*(t)c^{*'}(t)$, $\forall b(t) \in \mathbf{B}(t)$, $\forall c(t) \in \mathbf{C}(t)$. *Besides, the relation* $g(t)b(t)c'(t) = 0$ *must hold.*

Under assumption 4 the game is reduced to a problem of optimal control.

5.1 HJBI Equation

For our case, in [Subbotin, 1999, Theorem 9.1] it was proved that the value of the game equals $\overline{\omega} = \mathbf{c}(0, M_0, N_0)$, where the function $\mathbf{c} : [0,T] \times R^{n\times n} \times R^{n\times n} \to R$ satisfies (in corresponding minimax formalization) the equation

$$\partial \mathbf{c}(t, M, N)/\partial t + H(t, M, N, D\mathbf{c}) = 0 \tag{19}$$

with boundary condition $\mathbf{c}(T, M, N) = \omega(M, N)$. In equation (19) the symbol $D\mathbf{c}$ means the generalized gradient of function $\mathbf{c}(t, M, N)$ with respect to variables M, N, and the Hamiltonian H is defined by the following way

$$\begin{aligned} H(t, M, N, \Phi, \Psi) = \max_{\mathbf{v}\in\mathbb{V}} \min_{\mathbf{u}\in\mathbb{U}} h(t, \mathbf{u}, \mathbf{v}, M, N, \\ \Phi, \Psi), \end{aligned} \tag{20}$$

where $h(t,\mathbf{u},\mathbf{v},M,N,\Phi,\Psi) = \langle \Phi, -\mathbf{A}'(t)M + G'\mathcal{C}GN \rangle + \langle \Psi, gb\mathcal{C}_1 b'g'M + \mathbf{A}(t)N \rangle$. Here the inner product $\langle A,B\rangle$ means trace$A'B$. If the function $\mathbf{c}(t,M,N)$ has been built, the optimal strategies of 1-st and 2-nd players are defined as selectors of inclusions

$$\mathbf{u}_*(t,M,N,\mathbf{v}) \in \operatorname*{Argmin}_{\mathbf{u}\in\mathbb{U}} h(t,\mathbf{u},\mathbf{v},M,N, D\mathbf{c}(t,M,N)),\ \ \mathbf{v}^*(t,M,N) \in \operatorname*{Argmax}_{\mathbf{v}\in\mathbb{V}} \min_{\mathbf{u}\in\mathbb{U}} h(t, \mathbf{u},\mathbf{v},M,N,D\mathbf{c}(t,M,N)).$$

It is known that the solution of (19) in minimax sense coincides with viscosity solution (see [Subbotin, 1999; Fleming, Soner, 2006]). Note that both the solutions are unique.

6 A Numerical Solution

A numerical procedure can be built on the base of [Souganidis, 1985; Souganidis, 1999; Taras'ev et. al., 2006]. Here we do not perform a decomposition and consider initial equation (7) on the interval $[t_0,T]$, $t_0>0$. Denote by $F(t,P,\mathbf{u},\mathbf{v})$ the right-hand side of this equation. Let us establish some properties of solutions of equation (7). By $\mathcal{K}^n_{\alpha,\beta}$ we denote the segment of nonnegative-defined and simmetrical matrices $\{A : 0 \le \alpha I_n \le A \le \beta I_n\} = \{A : \alpha \le |A| \le \beta\}$. The matrix $A(t)$ will be considered Lipschitzean in t.

Lemma 2. *Let assumption* 3 *hold. Then solutions of equation* (7)*, the mappings* $F(t,P,\mathbf{u},\mathbf{v})$ *and* $\omega(P) = |P^{-1}|$*, possess the following properties.*

R1. *For any instant* $t_0 \in (0,T)$ *there exist positive constants* α, β *that do not depend on control parameters and such that* $P(t) \in \mathcal{K}^n_{\alpha,\beta}$, $\forall t \in [t_0,T]$.

R2. *The function* $F(t,P,\mathbf{u},\mathbf{v})$ *may be continued on the set* $[0,T]\times R^{n\times n}\times\mathbb{U}\times\mathbb{V}$ *in such a way that it is bounded and the uniform Lipschitz condition* $|F(t_1,P_1,\mathbf{u},\mathbf{v}) - F(t_2,P_2,\mathbf{u},\mathbf{v})| \le C_1\big(|P_1 - P_2| + |t_1-t_2|\big)$ *holds.*

R3. *The function* $\omega(P)$, $P\in\mathcal{K}^n_{\alpha,\beta}$*, may be continued on the space* $R^{n\times n}$ *in such a way that it is bounded and the Lipschitz condition* $|\omega(P_1)-\omega(P_2)| \le C_2|P_1-P_2|$ *holds.*

The lemma may be proved with the help of the Kirszbraun theorem (see [Federer, 1969, Theorem 2.10.43]) about the continuation of Lipschitzean maps.

Henceforth, we believe that the mappings F and ω are continued due to lemma 2. The Hamiltonian is now defined as $H(t,P,S) = \max_{\mathbf{v}\in\mathbb{V}}\min_{\mathbf{u}\in\mathbb{U}} h(t,\mathbf{u}, \mathbf{v},P,S)$, where $h(t,\mathbf{u},\mathbf{v},P,S) = \langle S, F(t,P,\mathbf{u},\mathbf{v})\rangle$. The function $\mathbf{c} : [0,T]\times R^{n\times n} \to R$ (the value of the game) satisfies in corresponding formalization the HJBI equation

$$\partial\mathbf{c}(t,P)/\partial t + H(t,P,D\mathbf{c}) = 0. \tag{21}$$

Here the symbol $D\mathbf{c} \in R^{n\times n}$ means the gradient of $\mathbf{c}(t,P)$ and is the matrix. For approximation of $\mathbf{c}(t,P)$, we consider the partition $\Delta = \{0 = t_0 < t_1 < \cdots < t_{N(\Delta)+1} = T\}$ of the interval $[0,T]$. The diameter $\max_i |t_{i+1} - t_i|$ of the partition is denoted by $|\Delta|$. Define the function $\mathbf{c}_\Delta : [0,T]\times R^{n\times n} \to R$ as

$$\begin{gathered}\mathbf{c}_\Delta(T,P) = \omega(P) \ \text{ on } \ R^{n\times n},\\ \mathbf{c}_\Delta(t,P) = \max_{\mathbf{v}\in\mathbb{V}}\min_{\mathbf{u}\in\mathbb{U}}\big\{\mathbf{c}_\Delta(t_{i+1},P\\ +(t_{i+1}-t)F(t_{i+1},P,\mathbf{u},\mathbf{v})\big\},\\ \text{if } \ t\in[t_i,t_{i+1}), \ \text{ and } \ i\in 0:N(\Delta).\end{gathered} \tag{22}$$

Using [Souganidis, 1999, Theorem 4.4], we obtain

Theorem 2. *Under* $|\Delta| \to 0$*, the function* (22) *converges to* $\mathbf{c}(t,P)$ *locally uniformly on* $[0,T]\times R^{n\times n}$*. The function* $\mathbf{c}(t,P)$ *is the unique solution of equation* (21) *in minimax or viscosity sense. Besides, there exists a constant* $K(C_2, ||\omega||, ||\omega_P||)$*, such that* $|\mathbf{c}_\Delta(t,P) - \mathbf{c}(t,P)| \le K|\Delta|^{1/2}$ *for all* $(t,P)\in[0,T]\times R^{n\times n}$*. Here* $||\cdot||$ *is the* sup*-norm of corresponding function.*

We can suggest the following numerical algorithm.

1. Choose a finite set (a grid) $\mathcal{N} = \{kP_j\}$, $k \in 0 : N_1$, $j \in 1 : N_2$, where $\{P_j\}$ is a collection of positive-definite matrices of small norm. The set must be contained in the segment $\mathcal{K}^n_{0,\beta}$, which uniformly covers the attainability domains of the Riccati equation.
2. Select a partition $\Delta = \{0 = t_0 < t_1 < \cdots < t_{N+1} = T\}$ of $[0,T]$.
3. Form and remember the function $\omega_N(P) = \max_{\mathbf{v}\in\mathbb{V}}\min_{\mathbf{u}\in\mathbb{U}}\omega\big(P + (t_{N+1}-t_N)F(t_{N+1},P, \mathbf{u},\mathbf{v})\big)$ and corresponding optimal controls $\mathbf{u}^*_N$ and $\mathbf{v}^*_N(\mathbf{u})$.
4. On subsequent steps the grid function is formed: $\omega_i(P) = \max_{\mathbf{v}\in\mathbb{V}}\min_{\mathbf{u}\in\mathbb{U}}\omega_{i+1}\big(P + (t_{i+1} - t_i)F(t_{i+1},P,\mathbf{u},\mathbf{v})\big)$ and corresponding optimal controls $\mathbf{u}^*_i$ and $\mathbf{v}^*_i(\mathbf{u})$. If the value $P + (t_{i+1} - t_i)F(t_{i+1},P,\mathbf{u},\mathbf{v})$ does not lie in the grid, then this value is changed for the nearest element from $\mathcal{N}$.
5. The value $\omega_0(P)$ gives an approximate value of the game.

6.1 Examples

1. Consider the system of example 2 and suppose that $P_0 = 0$, $T = 10$. Equation (21) is of the form

$$\partial\mathbf{c}(t,P)/\partial t + \langle D\mathbf{c}(t,P), AP + PA'\rangle + \max_{a,k}\min_{r,z}\langle D\mathbf{c}(t,P), G^2\mathcal{C} - Pg^2bb'P\rangle.$$

By method based on above algorithm, we get the value $\omega_0(0) = 4.34$ that close to the value in example 2. Note that the solutions of the Riccati equation are fast stabilized here to the steady-state solution under all control parameters.

2. Let us optimize the estimation for the two-dimensional oscillating system

$$\dot{x}(t) = Ax(t) + g(t)b(t)v(t),$$
$$y(t) = G(t)x(t) + c(t)v(t),$$

on the interval $[0, T]$, where $T = 2\pi$, $A = [0, 1; -1, 0]$. The control parameters are: $G(t) = [r(t)\ 0; 0\ 1 - r(t)]$ with $0 \le r(t) \le 1$; $b(t) = \big[[0, 0; 0, a(t)], O_2\big]$, $g(t) = [0, 0; 0, z(t)]$, where $a(t),\ z(t) \in [-1, 1]$; $c(t) = [O_2, c_1(t)]$ with $c_1(t)c_1'(t) \ge I_2$. Here the case when the assumption 4 holds, i.e. the 2-nd player chooses $c_1(t)c_1'(t) = I_2$ and $a(t) \equiv 0$. Therefore, we obtain the Riccati equation in the form $\dot{P}(t) = -A'P(t) - P(t)A + G^2(t)$. It has the explicit solution, and we have $P^{-1}(T) = \left(\int_0^T \mathbf{X}(T-t)G^2(t)\mathbf{X}'(T-t)dt\right)^{-1}$, where the matrix $\mathbf{X}(t) = [\cos t, \sin t; -\sin t, \cos t]$. Here the functional $|P^{-1}(T)|$ is concave in variable function $r(\cdot)$. Hence, the approximate optimal solution is a piecewise constant function with values in $\{0,\ 1\}$. In the class of constant functions the minimal value of the functional equals π^{-1}.

7 Conclusion

The problem of observations' control for non-stationary linear systems is considered. The quality of observation is measured by the diameter of informational set at the end of time interval. The problem is reduced to a differential game for the Riccati equations, where the first part of parameters is chosen by the 1-st player (an observer) and the second part is chosen by the 2-nd player (an opponent) who tries to worsen the quality of observation. In the common case, there are a saddle point in the class 'counterstrategy/strategy'. The value of the game may be found by integration of corresponding HJBI equation, the solution of which is understood in a generalized sense. The optimal strategies are also defined due to this solution. The numerical approximation is specified and the estimation of the rate of convergence is given for the approximating scheme. Particular cases of the equations with constant coefficients are considered, and the solutions for steady-state regimes of the Riccati equation are given. The examples are considered as well.

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